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REMAINDERS, SINGULAR SETS AND THE CANTOR SET

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ABSTRACT. Let X be a completely regular Hausdorff space which is not locally compact. Characterizations are given for when X has a compactification αX for which $\operatorname{Cl}_{\alpha X}(\alpha X - X)$ is the Cantor set C. This occurs if and only if C is the singular set of a continuous function.

For such spaces, there is also a compactification αX for which $\operatorname{Cl}_{\alpha X}(\alpha X - X)$ is the closed unit interval in case X has a residue which is countable.

1. Introduction. All topological spaces considered here are completely regular and Hausdorff. A remainder of a Hausdorff compactification αX of a space X is the set $\alpha X - X$. Substantial investigation has been devoted to the question of which spaces Y can serve as remainders for a space X. (See [**3, 5, 7** and **8**], for example.) Y. Unlü [**12**] and the present authors [**6**] have characterized when the Cantor set C is a remainder of a locally compact space X. Clearly, C cannot be a remainder of a nonlocally compact space.

In this paper we characterize when, for nonlocally compact X, there is a compactification αX of X for which the closure of $\alpha X - X$ in αX is C. For any X we let $R(X)$ denote the set of all points in X which do not possess a compact neighborhood. Then $\operatorname{Cl}_{\alpha X}(\alpha X - X) = (\alpha X - X) \cup R(X)$. Thus, for spaces satisfying $C = \operatorname{Cl}_{\alpha X}(\alpha X - X)$, it follows that $R(X)$ is a subset of C. When X is almost locally compact, that is, when $X - R(X)$ is dense in X, (see [**10**]), and $L(X)$ is the locally compact part of X, we observe that then αX is also a compactification of $L(X)$ for which $\alpha X - L(X) = C$ so that all compact metric spaces are remainders of $L(X)$ (see [**6**]).

If Y is compact and f is a continuous mapping from X into Y, the singular set of f is the set of all points p in Y for which $\operatorname{Cl}_X f^{-1}(N_p)$

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is noncompact, for all neighborhoods N_p of p. The singular set of f is denoted by $S(f)$. A number of authors have shown that, for locally compact X, certain remainders $\alpha X - X$ arise as the singular sets of such functions. However, not every compactification of X is such a "singular" compactification. R. Chandler and F. Tzung have extended the results to the nonlocally compact case. In particular, the following is Theorem 3 of [**4**]:

Theorem 1.1 (Chandler-Tzung). *Let f be a continuous function from X into Y, where Y is compact. If f is one-to-one on $R(X)$ and perfect at each point of $R(X)$, then $S(f) - f(R(X))$ is a remainder of X.*

Here we show that if $\mathrm{Cl}_{\alpha X}(\alpha X - X) = C$, then there is a continuous function f from X into I, the (closed) unit interval, for which $S(f) = C$. For such spaces we show that if, in addition, $R(X)$ is finite or countably infinite, then there is a compactification γX of X for which $\mathrm{Cl}_{\gamma X}(\gamma X - X)$ is I and $I - R(X)$ is a remainder of X. This supplements the results of [**11**] and [**4**].

2. The main theorem. Let βX denote the Stone-Cěch compactification of X. A set U in X is π-open whenever U is an open set with compact boundary (see [**5**]). If U is π-open and $V = X - \mathrm{Cl}_X U$, then $\mathrm{Cl}_{\beta X} U \cap (\beta X - X)$ and $\mathrm{Cl}_{\beta X} V \cap (\beta X - X)$ is a partition of $\beta X - X$ into disjoint open sets, since βX is a perfect compactification of X (see [**7**]). We say that a collection $\{\mathcal{G}_n \mid n \in N\}$ of families of X is dyadic if $\mathcal{G}_n = \{G_i^n \mid i = 1, 2, \dots, 2^n\}$ satisfies $\mathrm{Cl}_X G_i^n \cap \mathrm{Cl}_X G_j^n = \varnothing$ whenever $i \neq j$, and $\varnothing \neq \mathrm{Cl}_X(G_{2i-1}^{n+1} \cup G_{2i}^{n+1}) \subseteq G_i^n$, $i = 1, \dots, 2^n$, for all $n \in N$. In [**4**], a continuous map $f : X \to Y$ is defined to be perfect at a point $x \in X$ if f is closed at x and $f^{-1}\ (f(x))$ is compact.

Theorem 2.1. *Let X be nonlocally compact. Then the following are equivalent*:

(A) *X has a compactification αX for which $\mathrm{Cl}_{\alpha X}(\alpha X - X) = C$, the Cantor set, and $\alpha X - X = C - R(X)$ is a remainder of X.*

(B) *There is a continuous map f from X into I, where f is perfect* *at each point of $R(X)$, one-to-one on $R(X)$, and $S(f) = C$.*

(C) *X has a dyadic collection $\{\mathcal{G}_n \mid n \in N\}$, where each $\mathcal{G}_n = \{G_i^n \mid i = 1, \dots, 2^n\}$ is a family of π-open subsets of X satisfying*

(i) *$K_n = X - \cup\{G_i^n \mid i = 1, \dots, 2^n\}$ is compact;*

(ii) *$K_n \cup G_i^n$ is noncompact for $i = 1, \dots, 2^n$; and for each $p \in R(X)$ and $n \in N$, there exists $i(p)$ such that*

(iii) *$p \in K(p) = \cap\{G_{i(p)}^n \mid n \in N\}$, $K(p)$ is compact and $\{G_{i(p)}^n \mid n \in N\}$ is a base of neighborhoods of $K(p)$, and*

(iv) *$K(p) \neq K(q)$ for $p \neq q$ in $R(X)$.*

Proof. (A) *implies* (B). Let f_0 be a continuous extension to αX of the inclusion mapping of C into I, and let f be the restriction of f_0 to X. Obviously, f is one-to-one on $R(X)$. We show that $S(f) = C$ and that f is perfect at each point of $R(X)$.

If $p \in I - C$, let N_p be a compact I-neighborhood of p for which $C \cap N_p = \varnothing$. Then $f^{-1}(N_p) = f_0^{-1}(N_p)$ is a compact subset of X so that $p \notin S(f)$. For $p \in C$, let M_p be any I-neighborhood of p. Then $\mathrm{Cl}_{\alpha X}(\alpha X - X) \cap f_0^{-1}(M_p)$ is a neighborhood of p in $\mathrm{Cl}_{\alpha X}(\alpha X - X) = C$. If $p \in \alpha X - X$, then $\mathrm{Cl}_X f^{-1}(M_p)$ cannot be compact or else $f_0^{-1}(M_p) \cap [\alpha X - \mathrm{Cl}_X f^{-1}(M_p)]$ is an αX-neighborhood of p which does not meet X, a contradiction. For $p \in X$, then $\mathrm{Cl}_X f^{-1}(M_p)$ cannot be compact by the definition of $R(X)$. Thus, in both cases, $p \in S(f)$ and hence $S(f) = C$.

Next consider $f^{-1}(f(p))$, for $p \in R(X)$. Since f_0 is one-to-one on $\mathrm{Cl}_{\alpha X}(\alpha X - X)$, $f^{-1}(f(p)) = f_0^{-1}(f_0(p))$ is compact. Now let U be any open neighborhood of $f^{-1}(f(p))$. There exists αX-open U_α such that $U = U_\alpha \cap X$. Then $\alpha X - U_\alpha$ is compact and misses $f^{-1}(f(p)) = f_0^{-1}(f_0(p))$. Thus $T = f_0(\alpha X - U_\alpha)$ is a compact set disjoint from $f(p) = f_0(p)$. Choose an I-open neighborhood V of $f(p)$ for which $V \cap T = \varnothing$. Then $x \in f^{-1}(V)$ implies $x \in U$ so that $f^{-1}(f(p)) \subseteq f^{-1}(V) \subseteq U$. Hence f is perfect at each point of $R(X)$ so that (B) holds.

(B) *implies* (A). The proof of Theorem 3 of [**4**] shows that Theorem 1.1 of [**9**] can be applied. Then (A) is immediate from these results and their proofs.

(A) *implies* (C). Assume that $\mathrm{Cl}_{\alpha X}(\alpha X - X) = C$. For each $n \in N$ choose a family $T_n = \{F_1^n, \dots, F_{2^n}^n\}$ of pairwise disjoint nonempty compact C-open subsets of C which cover C and satisfy $F_{2i-1}^{n+1} \cup F_{2i}^{n+1} = F_i^n$, for $i = 1, \dots, 2^n$, and $\sup\{|x-y| \,|\, x, y \in F_i^n\} \leq 3^{-n}$, for all i, n. Let $i(p)$ denote the index for which a point $p \in F_{i(p)}^n$. It is immediate that $\cap\{F_{i(p)}^n \mid n \in N\} = \{p\}$, for each $p \in C$.

Now for each $n \in N$ inductively choose families $\mathcal{H}_n = \{H_i^n \mid i = 1, \dots, 2^n\}$ of αX-open sets which satisfy $F_i^n \subseteq H_i^n$, $\mathrm{Cl}_{\alpha X}(H_{2i-1}^{n+1} \cup H_{2i}^{n+1}) \subseteq H_i^n$ and $\{\mathrm{Cl}_{\alpha X} H_i^n \mid i = 1, \dots, 2^n\}$ is a pairwise disjoint family. Then $K_n = \alpha X - \cup\{H_i^n \mid i = 1, \dots, 2^n\}$ is compact. For all $n \in N$, set $\mathcal{G}_n = \{G_i^n \mid i = 1, \dots, 2^n\}$, where $G_i^n = H_i^n \cap X$, $i = 1, \dots, 2^n$. Since X is dense in αX it follows from the choice of H_i^n's that $\mathrm{Cl}_{\alpha X} G_i^n = \mathrm{Cl}_{\alpha X} H_i^n = \mathrm{Cl}_X G_i^n \cup F_i^n$, for all i and n. Also, $\mathrm{Cl}_X G_{i(p)}^{n+1} \subseteq G_{i(p)}^n$, for all $n \in N$, so that $\cap\{\mathrm{Cl}_X G_{i(p)}^n \mid n \in N\} = \cap\{G_{i(p)}^n \mid n \in N\}$. Hence for $p \in R(X)$, we have $(*)$: $K(p) = \cap\{G_{i(p)}^n \mid n \in N\} = \cap\{\mathrm{Cl}_X G_{i(p)}^n \mid n \in N\} = \cap\{\mathrm{Cl}_X G_{i(p)}^n \cup F_{i(p)}^n \mid n \in N\} = \cap\{\mathrm{Cl}_{\alpha X} G_{i(p)}^n \mid n \in N\}$.

Thus $K(p)$ is compact, as desired.

Next, note that the choice of H_i^n's insures that $\mathrm{Cl}_X G_i^n \cap \mathrm{Cl}_X G_j^n = \varnothing$, for $i \neq j$, and $\mathrm{Cl}_X(G_{2i-1}^{n+1} \cup G_{2i}^{n+1}) \subseteq G_i^n$. Also, $\mathrm{Cl}_X G_i^n \subseteq G_i^n \cup K_n$, so each G_i^n is π-open. Thus the families $\mathcal{G}_n$ form a dyadic collection of π-open sets. Note that $R(X) \cap \mathrm{Cl}_X G_i^n \subseteq G_i^n$. Since each $\mathrm{Cl}_{\alpha X} G_i^n$ contains a point $x \in C$, if $x \notin X$ clearly $\mathrm{Cl}_X G_i^n$ is not compact, and if $x \in X$, then $x \in R(X)$ hence $\mathrm{Cl}_X G_i^n$ cannot be compact. Hence $G_i^n \cup K_n$ is not compact and $C(ii)$ holds.

Now let U be any X-open set containing $K(p)$. Then $\mathrm{Cl}_{\alpha X}(X-U) \cap K(p) = \varnothing$. Using $(*)$, it follows that $\mathrm{Cl}_{\alpha X} G_{i(p)}^n \cap \mathrm{Cl}_{\alpha X}(X - U) = \varnothing$ for some n. Hence, $G_{i(p)}^n \subseteq U$, so that (iii) holds.

Finally, since for $p \neq q$ in $R(X)$ we have $|p - q| > n^{-1}$, for some $n \in N$, evidently $F_{i(p)}^n \cap F_{i(q)}^n = \varnothing$. Thus, $G_{i(p)}^n \cap G_{i(q)}^n = \varnothing$ so that $K(p) \neq K(q)$. Now C(iv) holds and (C) is verified.

(C) *implies* (A). For each $n \in N$ and $i = 1, \dots, 2^n$, let $H_i^n = \beta X - (\mathrm{Cl}_{\beta X}(X - G_i^n))$ and set $A_n = H_1^n \cup H_3^n \cup \dots \cup H_{2^n-1}^n$ and $B_n = H_2^n \cup H_4^n \cup \dots \cup H_{2^n}^n$. Observe that $A_n \cup B_n$ covers $\mathrm{Cl}_{\beta X}(\beta X - X)$ and, since $H_i^n \cap X = G_i^n$, we have $A_n \cap B_n = \varnothing$.

For each $n \in N$, we define a continuous map f_n from $\mathrm{Cl}_{\beta X}(\beta X - X)$ into the two-point discrete space $\{0,1\}$ by $f_n(A_n) = 0$ and $f_n(B_n) = 1$. Now define a function f from $\mathrm{Cl}_{\beta X}(\beta X - X)$ into $\{0,1\}^N$ by $f = \prod_n f_n$. We will show that (1) f is onto; (2) $f(\beta X - X) = C - f(R(X))$; and (3) the restriction of f to $R(X)$ is one-to-one. These conditions allow the application of Theorem 1.1 of [**9**]. For (1), let $y = (y_n) \in \{0,1\}^N = C$. For any positive integer m and $n \le m$, choose $C_n = A_n$ when $y_n = 0$ and $C_n = B_n$ when $y_n = 1$. From C(ii) it follows that $\cap\{C_n \mid n = 1, \dots, m\}$ contains a point $x \in \mathrm{Cl}_{\beta X}(\beta X - X) \cap H_i^m$, for some i. Clearly, $f(x)$ agrees with y in the first m-coordinates. Thus, the image of $\mathrm{Cl}_{\beta X}(\beta X - X)$ is compact and dense in C so that (1) holds.

Next, take $x \in \beta X - X$ and $p \in R(X)$. Then there are neighborhoods N_1 and N_2 of x and $K(p)$, respectively, in βX for which $\mathrm{Cl}_{\beta X} N_1 \cap \mathrm{Cl}_{\beta X} N_2 = \varnothing$. But $N_2 \cap X$ is an X-neighborhood of $K(p)$ so by C(iii) there is a $G^n_{i(p)}$ for which $p \in G^n_{i(p)} \subseteq N_2 \cap X$. Then $x \in H^n_j$, for some $j \neq i(p)$, and it follows from the assumption that the collection of $\mathcal{G}_n$'s is dyadic that $f_k(p) \neq f_k(x)$, for some $k \le n$. Hence $f(p) \neq f(x)$ so that $f(x) \in C - f(R(X))$. Thus it is immediate that $f(\beta X - X) = C - f(R(X))$ which is (2).

For (3) let p and q be distinct points of $R(X)$. By C(iii) and C(iv) it follows that $G^n_{i(p)} \neq G^n_{i(q)}$ for some n, and as in the proof of (2), $f_k(p) \neq f_k(q)$ for some $k \le n$. Hence $f(p) \neq f(q)$ and (3) holds.

Now (1), (2) and (3) insure that we can apply Theorem 2.1 of [**9**] and its proof from which (A) follows.

This completes the proof. $\square$

The proof that (A) implies (B) of (2.1) may be applied whenever $\mathrm{Cl}_{\alpha X}(\alpha X - X)$ is a subset of R, the real numbers. Thus we can state

Corollary 2.2. *Let αX be any compactification of X for which $\mathrm{Cl}_{\alpha X}(\alpha X - X)$ is a subset of R. Then there is a mapping f of X onto a compact set D of real numbers for which $S(f) = \mathrm{Cl}_{\alpha X}(\alpha X - X)$, and f is one-to-one on $R(X)$, and perfect at each point of $R(X)$.*

Thus, Corollary 2.2 shows that any αX for which $\mathrm{Cl}_{\alpha X}(\alpha X - X)$

is a subset of R can be obtained as in Theorem 1.1. Suppose that X satisfies (A) of Theorem 2.1. If Y is a continuous image of C under a mapping f which is one-to-one on $R(X)$ and satisfies $f(C - R(X)) \cap f(R(X)) = \phi$, then it follows from 1.1 of [**9**] (and its proof) that there is a compactification γX of X for which $\mathrm{Cl}_{\gamma X}(\gamma X - X) = Y$ and $Y - f(R(X))$ is a remainder of X. However, all continuous images Y of C need not satisfy $\mathrm{Cl}_{\gamma X}(\gamma X - X) = Y$ for some compactification γX, even when Y contains a copy of $R(X)$. For example, let $Y = \{a, b\}$ be the discrete two point space. Then Y is a continuous image of C. If X is a space satisfying (A) of 2.1 and if $R(X)$ is a singleton, then $R(X)$ is trivially homeomorphic with $\{a\}$, but $Y - \{a\}$ cannot be a remainder of X.

3. I **as** $\mathrm{Cl}_{\alpha X}(\alpha X - X)$**.** Rogers [**11**], Magill [**8**] and others (see [**3**], for example) have studied the question of when continua are remainders of locally compact spaces. Clearly, no continuum can be a remainder of a nonlocally compact space, but in what follows we provide conditions insuring that some γX satisfies $\mathrm{Cl}_{\gamma X}(\gamma X - X) = I$. Under these conditions, I is then a remainder of $L(X)$ whenever X is almost locally compact.

Theorem 3.1. *Suppose X satisfies condition* (A) *of Theorem* 2.1. *If $R(X)$ is finite or countably infinite, then there is a compactification γX for which $\mathrm{Cl}_{\gamma X}(\gamma X - X) = I$ and $I - R(X)$ is a remainder of X.*

Proof. Assume that αX exists such that $\mathrm{Cl}_{\alpha X}(\alpha X - X) = C$ as in 2.1 (A). First we show that there is a homeomorphism of C such that when the canonical mapping g of C onto I is applied, g satisfies $g(x) \neq g(y)$ for all $x \in R(X)$ and all $y \in C$, where $x \neq y$.

For each $y = (y_n)$ in C, where (y_n) is the ternary representation of y, we associate a homeomorphism f_y of C onto C as follows: For $(x_n) \in C$, set $f_y(x_n) = (z_n)$, where $z_n = x_n$ when $y_n = 0$ and when $y_n = 2$, set $z_n = 0$ if $x_n = 2$ and $z_n = 2$ if $x_n = 0$. Now for $a = (a_n) \in R(X)$ we say that f_y is "constant on a tail" of a if, for $f_y(a_n) = (b_n)$, there is $n_y \in N$ such that $b_n = 0$ for all $n \geq n_y$ or $b_n = 2$ for all $n \geq n_y$. For each such "tail" of a there are only finitely many f_y's which are constant on that tail. Since the collection of f_y's is uncountable but $R(X)$ is finite or countably infinite, there is some f_y which is not constant on any tail of a, for all $a \in R(X)$. Let f_{y_0} be such a map.

Next, set $g(x) = \sum_1^\infty x_n/2^{n+1}$, for $x = (x_n) \in C$. Then g is a continuous surjection of C onto I. Thus, the composition $g \circ f_{y_0}$ maps C continuously onto I and is one-to-one on $R(X)$. Moreover, $g \circ f_{y_0}(R(X)) \cap g \circ f_{y_0}(C - R(X)) = \phi$ because of the selection of f_{y_0}.

Hence (by the remarks following 2.2) $I = \mathrm{Cl}_{\gamma X}(\gamma X - X)$ for some compactification γX of X and $I - R(X)$ is a remainder of X. This completes the proof. □

R. Chandler and F. Tzung have shown in [**4**] that $(0,1]$ is a remainder of X when X is realcompact and $R(X) = \{p\}$ is a singleton contained in a set of countable character. However, Theorem 3.1 can yield $(0,1]$ as a remainder of X when X is not real compact. Clearly, the Chandler-Tzung theorem also applies when 3.1 does not.

Example 3.2. Let D be any countable set in C. Then $C - D$ is dense in C. Take $X = I \times I - \{(C - D) \times \{0\}\}$. Evidently X satisfies the conditions of Theorem 3.1 so that $I - \widehat{D}$ is a remainder of X when $\widehat{D}$ is a copy of $D = R(X)$. We note that D can be a singleton here in which case $I - \{p\}$ is a remainder of X for some $p \in I$.

Also, if $[0, \omega_1)$ is the space of all countable ordinals, where ω_1 is the first uncountable ordinal, then $Y = [0, \omega_1] \times I - (\{\omega_1\} \times (C - D))$ is a pseudocompact space which satisfies the conditions of (3.1) but which is not covered in [**4**].

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