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SCHRÖDINGER OPERATORS AND INDEX BOUNDS FOR MINIMAL SUBMANIFOLDS

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1. Introduction. Let M^m be a complete m-dimensional manifold, possibly with boundary, that is minimally immersed in a compact n-manifold N^n. In the present paper we study the operator $\Delta + q$ on M where Δ is the Laplace-Beltrami operator and q is a real-valued function on M. We also consider the more general case of an operator $\nabla^2 + Q$ on a Riemannian vector bundle over M. Here ∇^2 is the Bochner-Laplacian of the vector bundle and Q is a symmetric endomorphism. Our objective is to estimate the number of bound states, or nonpositive eigenvalues, of such operators involving $L^{m/2}$-norms of q and constants depending only on m and the ambient manifold N. If $\partial M \neq \varnothing$, we require that the eigenfunctions satisfy Dirichlet boundary conditions, and if M is not compact, we define the number of bound states as the limit of the number of bound states for the Dirichlet problems for an increasing and exhausting sequence of compact domains in M.

Previously known estimates, see for instance [**1**] and [**6**] all depend on the geometry of M, through bounds on, for example, injectivity radius and Ricci curvature. In our setting, we have replaced this by a dependence on the ambient manifold N, which might be more natural when one studies minimal submanifolds of a fixed ambient manifold.

An application of our estimates gives index bounds for minimal submanifolds of dimension at least three, see Theorem 3. Recall that the index of a minimal submanifold is the number of negative eigenvalues of the second variation operator. Another application provides upper bounds for the Betti numbers of minimal submanifolds, see Theorem 4.

Finally, when the dimension of m is 2, we show, utilizing the Gauss

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map, that if M is a minimal surface in $\mathbf{R}^n$, then

$$\operatorname{index}(M) \leq C(n) \int_M (-K),$$

where K is the Gaussian curvature of M. Using the Gauss-Gonnet theorem, one then obtains, if $\partial M \neq \varnothing$, that the index of M is bounded by the negative of the Euler characteristic of M and the total curvature of the boundary of the immersion of M, see Theorem 5 and Corollary 1. In Theorem 6, we obtain similar bounds for minimal surfaces in S^n. These results answer affirmatively a question posed in [**5**], and extend to arbitrary codimension and to surfaces with boundary, the index bounds for minimal surfaces in $\mathbf{R}^3$ in [**12**], and for surfaces in $\mathbf{R}^4$ in [**5**].

Our methods borrow a lot from the arguments of P. Li and S.-T. Yau in [**8**]. For the case of vector bundles over minimal submanifolds, we need to derive modifications of some results in [**6**]. We would also like to thank R. Schoen for helpful discussions.

2. An estimate on the number of bound states. Let M^m, $m \geq 3$, be a minimally immersed m-dimensional manifold in the compact manifold N. Consider the operator

$$L = \Delta + q$$

on M^m, where Δ is the Laplace-Beltrami operator and q is a real-valued function on M. We study the associated eigenvalue problem

$$Lf + \lambda f = 0,$$

where, if $\partial M \neq \varnothing$, we insist that

$$f|_{\partial M} = 0.$$

If M is compact, we define the number of bound states of L as the number of nonpositive eigenvalues of L. For noncompact M, we define the number of bound states as the limit of the number of bound states for an increasing and exhausting sequence of compact domains in M. This limit always exists (although it might be $+\infty$) since the number of bound states increase for an increasing sequence of domains and the limit is easily seen to be independent of the sequence chosen.

Theorem 1. *Let M^m, $m \geq 3$, be a complete minimally immersed submanifold in the compact manifold N. The number of bound states β of the operator*

$$\Delta + q,$$

on M satisfies

$$\beta \leq C(m, N) \int_M (\max(q, 1))^{m/2},$$

where $C(m, N)$ is a constant depending only on m and N.

Remark 1. If q is nonnegative and positive somewhere and $\partial M = \varnothing$, then $\beta \geq 1$ for minimal submanifolds M without boundary and we obtain

$$\operatorname{vol}(M) \geq \frac{1}{C(m, N)}.$$

We can also conclude for the same reason that the estimate of Theorem 1 does not hold with the integrand $(\max(q, 1))^{m/2}$ replaced by $q^{m/2}$.

Proof. The first part of our proof is identical to the argument of P. Li and S.-T. Yau for Theorem 2 in [**8**]. It is included here for the sake of completeness only. Assume M is orientable, otherwise we pass to the oriented double cover. Define

$$p(x) = \max(q(x), 1),$$

and let $H(x, y, t)$ be the kernel for the operator

$$\frac{1}{p(x)}\Delta - \frac{\partial}{\partial t},$$

on D. Here $D = M$ if M is compact and otherwise D is a compact domain in M. If $\partial D \neq \varnothing$, we require $H(x, y, t)$ to satisfy Dirichlet boundary conditions. Let $\{\mu_i\}_{i=0}^{\infty}$ be the eigenvalues of $1/p(x)\Delta$ on D, again for the Dirichlet problem if $\partial D \neq \varnothing$. Define

$$h(t) = \sum_{i=0}^{\infty} e^{-2\mu_i t}.$$

Then

$$h(t) = \iint H^2(x,y,t)p(x)p(y)\,dV(y)\,dV(x),$$

where for this argument we adopt the convention that the integrations are taken over D, unless otherwise specified. Differentiation with respect to t gives

$$\begin{aligned}\frac{d}{dt} &= 2\iint H(x,y,t)p(x)p(y)\frac{\partial H}{\partial t}(x,y,t)\,dV(y)\,dV(x)\\ &= 2\int p(x)\int H(x,y,t)(\Delta_y H(x,y,t))\,dV(y)\,dV(x)\\ &= -2\int P(x)\int |\nabla_y H(x,y,t)|^2\,dV(y)\,dV(x).\end{aligned}$$

Repeated applications of the Hölder inequality yields

$$\begin{aligned} h(t) &= \int p(x)\int H^2(x,y,t)p(y)\,dV(y)\,dV(x)\\ &\le \int p(x)\Big(\int H^{2m/(m-2)}(x,y,t)\,dV(y)\Big)^{(m-2)/(m+2)}\\ &\qquad \cdot\Big(\int H(x,y,t)p^{(m+2)/4}(y)\,dV(y)\Big)^{4/(m+2)} dV(x)\\ &\le \Big(\int p(x)\Big(\int H^{2m/(m-2)}(x,y,t)\,dV(y)\Big)^{(m-2)/m} dV(x)\Big)^{m/(m+2)}\\ &\qquad \cdot\Big(\int p(x)\Big(\int H(x,y,t)p^{(m+2)/4}(y)\,dV(y)\Big)^2 dV(x)\Big)^{2/(m+2)}. \end{aligned} \tag{1}$$

Now define

$$P(x,t) = \int H(x,y,t)p^{(m+2)/4}(y)\,dV(y),$$

and note that

$$\Big(\frac{1}{p(x)}\Delta_x - \frac{\partial}{\partial t}\Big)P(x,t) = 0,$$

and

$$P(x,0) = p^{(m-2)/4}(x).$$

We compute

$$\frac{d}{dt}\int P^2(x,t)p(x)\,dV(x) = 2\int P(x,t)\frac{\partial P}{\partial t}(x,t)p(x)\,dV(x)$$
$$= 2\int P(x,t)\Delta_x P(x,t)\,dV(x)$$
$$= -2\int |\nabla_x P(x,t)|^2\,dV(x) \leq 0,$$

and, hence,

$$\int P^2(x,t)p(x)\,dV(x) \leq \int P^2(x,0)p(x)\,dV(x)$$
$$= \int p^{m/2}(x)\,dV(x).$$

Inequality (1) can therefore be written

$$(2)\quad h^{(m+2)/m}(t)\left(\int p^{m/2}(x)\,dV(x)\right)^{-2/m} \leq \int p(x)\left(\int H^{2m/(m-2)}(x,y,t)\,dV(y)\right)^{(m-2)/m} dV(x).$$

The Nash embedding theorem guarantees the existence of an isometric embedding of the ambient space N into some Euclidean space. We can thus regard M and N as submanifolds of this Euclidean space. According to the Sobolev inequality in [**9**], we have for compactly supported functions g on M, with $\nabla g \in L^1(M)$,

$$\left(\int_M |g|^{m/m-1}\,dV\right)^{m-1/m} \leq C(m)\int_M (|\nabla g| + |g|\,|\mathcal{H}|)\,dV$$

where $C(m)$ is a constant depending only on m, and $\mathcal{H}$ is the mean curvature vector of M as a submanifold of Euclidean space. Since M is a minimal submanifold of N, $\mathcal{H}$ is less than or equal to the length, $|A|$, of the second fundamental form of N a submanifold of Euclidean space. By replacing $|A|$ by its maximum value on N (which is finite since N

is compact), and replacing g by $f^{2(m-1)/m-2}$, and finally squaring the inequality obtained, we arrive at

$$(3)\qquad \left(\int_M |f|^{2m/m-2}\,dV\right)^{m-2/m} \le c_1 \int_M |\nabla f|^2\,dV + c_2 \int |f|^2\,dV,$$

where c_1 is a constant depending only on m, whereas c_2 also depends on $\max_N |A|$. Applying (3), with $f(y) = H(x,y,t)$, to inequality (2), we obtain

$$\begin{aligned}
h^{m+2/m}(t)&\left(\int p^{m/2}(x)\,dV(x)\right)^{-2/m} \\
&\le \int p(x)\left(\int H^{2m/(m+2)}(x,y,t)\,dV(y)\right)^{m-2/m} dV(x) \\
&\le \int p(x)\left(c_1 \int |\nabla_y H(x,y,t)|^2\,dV(y)\right. \\
&\qquad\qquad \left. + c_2 \int H^2(x,y,t)\,dV(y)\right) dV(x) \\
&= -\frac{1}{2}c_1\frac{dh}{dt} + c_2 \int p(x) \int H^2(x,y,t)\,dV(y)\,dV(x) \\
&\le -\frac{1}{2}c_1\frac{dh}{dt} + c_2 h(t),
\end{aligned}$$

where in the last inequality above we used the fact that $p(y) \ge 1$ for all $y \in M$. Hence, we have obtained the differential inequality

$$\left(\int p^{m/2}(x)\,dV(x)\right)^{-2/m} \le \frac{(-c_1/2)(dh/dt)}{h^{m+2/m}} + c_2 h^{-2/m}.$$

Setting $\varphi = h^{-2/m}$, we can write this inequality as

$$\frac{4}{mc_1}\left(\int p^{m/2}(x)\,dV(x)\right)^{-2/m} \le \frac{d\varphi}{dt} + \frac{4c_2}{mc_1}\varphi.$$

Multiplication by the integrating factor $e^{4c_2t/(mc_1)}$ yields

$$(\varphi e^{4c_2t/(mc_1)}) \ge \frac{4}{mc_1}\left(\int p^{m/2}(x)\,dV(x)\right)^{-2/m} e^{4c_2t/(mc_1)}.$$

Integrating from 0 to t, using the fact that $\varphi(0) = 0$ since $h(t) \to \infty$ as $t \to 0+$, we have

$$\varphi e^{4c_2 t/(mc_1)} \geq \frac{1}{c_2}(e^{4c_2 t/(mc_1)} - 1)\bigg(\int p^{m/2}(x)\,dV(x)\bigg)^{-2/m},$$

or, in terms of h,

$$h(t) \leq \frac{c_2^{m/2}}{(1 - e^{-4c_2 t/(mc_1)})^{m/2}} \int p^{m/2}(x)\,dV(x).$$

We note, as in [**8**], that the quadratic form corresponding to the operator $L = \Delta + q$ on D satisfies

$$\frac{\int |\nabla\phi|^2\,dV - \int p\phi^2\,dV}{\int \phi^2\,dV} = \frac{\int p\phi^2\,dV}{\int \phi^2\,dV}\bigg(\frac{\int |\nabla\phi|^2\,dV}{\int p\phi^2\,dV} - 1\bigg).$$

Now $\int |\nabla\phi|^2\,dV / \int p\phi^2\,dV$ is the quadratic form associated to the operator $1/p\Delta$ on D. Hence, the number of nonpositive eigenvalues of L on D, which we denote β_D, is equal to the number of eigenvalues of $1/p\Delta$ on D that are less than or equal to one. We therefore have

$$\begin{aligned}\beta_D \cdot e^{-2t} &\leq \sum_{i=0}^{\infty} e^{-2\mu_i t} = h(t)\\ &\leq \frac{c_2^{m/2}}{(1 - e^{-4c_2 t/(mc_1)})^{m/2}} \int p^{m/2}(x)\,dV(x)\\ &\leq \frac{c_2^{m/2}}{(1 - e^{-4c_2/t(mc_1)})^{m/2}} \int_M p^{m/2}(x)\,dV(x),\end{aligned}$$

so that

$$\beta_D \leq \min_{t>0}\left(\frac{c_2^{m/2} e^{2t}}{(1 - e^{-4c_2 t/(mc_1)})^{m/2}}\right) \int_M p^{m/2}(x)\,dV(x),$$

which proves the theorem since the right-hand side of the inequality above does not depend on the domain D.

3. Vector bundles over minimal submanifolds. A Riemann vector bundle $W \to M$ is a smooth vector bundle with a metric and a connection ∇ preserving that metric. The Bochner Laplacian ∇^2 is defined by $\nabla^2 = \operatorname{Tr}(\nabla^* \circ \nabla)$, where ∇^* is the formal adjoint of ∇. Suppose Q is a symmetric endormorphism of W and set $L = \nabla^2 + Q$. Consider this operator on $W \to D$, where $D = M$ if M is compact and otherwise D is a compact domain in M. If $\partial D \neq \varnothing$, we apply the operator L only to sections of W that vanish on ∂D. Then L defines a unique self-adjoint operator on $L^2(W \to D)$.

We would like to estimate the number of nonpositive eigenvalues of L on $W \to M$. In the case of M not compact, we define the number of nonpositive eigenvalues, or bound states, as the limit of the number of bound states for an increasing and exhausting sequence of compact domains D in M, and if $\partial M \neq \varnothing$ we impose Dirichlet boundary conditions. As before, we assume that M^m is complete and minimally immersed in a compact manifold N. In addition, we assume that Q is bounded above in the sense that

$$\langle Q(E), E\rangle_x \leq q(x)\langle E, E\rangle, \qquad x \in M,$$

for all admissible sections E of $W \to M$, where q is some function on M. Then we have the following estimate on the number of bound states of L.

Theorem 2. *Let β be the numbers of bound states of the operator $L = \nabla^2 + Q$, and let the manifolds M and N and the bundle W be as above. Then*

$$\beta \leq C(m, N)(\operatorname{rank} W) \int_M (\max(q, 1))^{m/2},$$

where q is defined above and $C(m, N)$ is a constant depending only on m and the manifold N.

As above, assume M is nonorientable; otherwise, pass to the oriented double cover. To prove the theorem, we first need to introduce the kernel of the operator

$$\frac{1}{p}\nabla^2 - \frac{\partial}{\partial t},$$

on $W \to D$, with D as above and $p = \max(q, 1)$. As before, if $\partial D \neq \varnothing$, we impose Dirichlet boundary conditions. We denote this kernel by

$\overline{H}(x,y,t)$, and the kernel for $1/p\Delta - \partial/\partial t$ on D, we denote by $H(x,y,t)$, where Δ is the Laplace-Beltrami operator. Let $|\overline{H}|$ be the pointwise norm of the endomorphism $\overline{H}$. Then we have the following lemma.

Lemma 1. *With $|\overline{H}|$ as above, we have in the sense of distributions,*

$$\left(\frac{\partial}{\partial t} - \frac{1}{p}\Delta\right)|\overline{H}| \leq 0.$$

Proof of Lemma 1. The proof of this lemma is almost identical to that of Lemma 4.1 in [**6**], where the case $p \equiv 1$ is considered, so we will not include it here. □

From this lemma, we obtain the following relationship between the pointwise norms of the heat kernels $\overline{H}$ and H.

Lemma 2. *Let $\overline{H}$ and H be as above. Then*

$$|\overline{H}(x,y,t)| \leq H(x,y,t),$$

for all $t \geq 0$.

Proof. In the case of $p \equiv 1$, this result appears in [**6**]. Our proof is completely analogous to that of H. Donnelly and P. Li, but we include it here for the sake of completeness. We have

$$\begin{aligned}
|\overline{H}&(x,y,t)| - H(x,y,t) \\
&= \int_0^t \frac{\partial}{\partial s} \int_D |\overline{H}(x,z,s)| H(z,y,t-s) p(z)\, dV(z)\, ds \\
&= \int_0^t \int_D \left(\frac{\partial}{\partial s}|\overline{H}(x,z,s)|\right) H(z,y,t-s) p(z)\, dV(z)\, ds \\
&\quad + \int_0^t \int_D |\overline{H}(x,z,s)| \frac{\partial}{\partial s}(H(z,y,t-s)) p(z)\, dV(z)\, ds \\
&= \int_0^t \int_D \left(\frac{\partial}{\partial s}|\overline{H}(x,z,s)|\right) H(z,y,t-s) p(z)\, dV(z)\, ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \int_D |\overline{H}(x,z,s)|(-\Delta_z H(z,y,t-s))\,dV(z)\,ds\\
&= \int_0^t \int_D \left(\frac{\partial}{\partial s} - \frac{1}{p(z)}\Delta_z\right)|\overline{H}(x,z,s)|H(z,y,t-s)\,dV(z)\,ds\\
&\leq 0,
\end{aligned}
$$

since by the previous lemma,

$$\left(\frac{\partial}{\partial s} - \frac{1}{p(z)}\Delta_z\right)|\overline{H}(x,z,s)| \leq 0.$$

This completes the proof of the lemma since

$$|\overline{H}(x,y,0)| = H(x,y,0). \qquad \square$$

We now combine Lemmata 1 and 2 to prove Theorem 2. $\square$

Proof of Theorem 2. Since the quadratic form associated to $\nabla^2 + Q$ on $W \to D$ satisfies

$$\int_D \langle -\nabla^2 E - Q(E), E\rangle\,dV \geq \int_D \langle -\nabla^2 E - q\cdot E, E\rangle\,dV,$$

for sections E of $W \to D$, we see that the number of nonpositive eigenvalues of $\nabla^2 + Q$ on $W \to D$ is bounded by the number of nonpositive eigenvalues of $\nabla^2 + q$. This number in turn is bounded by the number of nonpositive eigenvalues of $\nabla^2 + p$, with $p = \max(q, 1)$. As in the proof of Theorem 1, we note that the number of nonpositive eigenvalues of $\nabla^2 + p$ is equal to the number of eigenvalues of $1/p\nabla^2$ that are less than or equal to one. Hence, applying Lemma 2,

$$
\begin{aligned}
\beta e^{-t} &\leq \operatorname{Tr}\overline{H}(t) = \int_D \operatorname{Tr}\overline{H}(x,x,t)p(x)\,dV(x)\\
&\leq (\operatorname{rank} W)\cdot \int_D |\overline{H}(x,x,t)|p(x)|\,dV(x)\\
&\leq (\operatorname{rank} W) \int_D H(x,x,t)p(x)\,dV(x)\\
&= (\operatorname{rank} W)\cdot \operatorname{Tr} H(t).
\end{aligned}
$$

Noting that the function h in the proof of Theorem 1 satisfies

$$H(t) = h\left(\frac{1}{2}t\right),$$

we obtain from above, after multiplication by e^t,

$$\begin{aligned}\beta &\le (\operatorname{rank} W) \cdot \frac{c_2^{m/2} e^t}{(1 - e^{-2c_2 t/(mc_1)})^{m/2}} \int_D p^{m/2}(x)\, dV(x) \\ &\le (\operatorname{rank} W) \cdot \frac{c_2^{m/2} e^t}{(1 - e^{-2c_2 t/(mc_1)})^{m/2}} \int_M p^{m/2}(x)\, dV(x).\end{aligned}$$

Since the right-hand side is independent of D, we obtain, as in the proof of Theorem 1, the desired result by minimizing the right-hand side over $t > 0$. □

4. Applications to the second variation operator. Let M^m be as above, a complete m-dimensional manifold, possibly with boundary, which is minimally immersed in a compact ambient manifold N^n. The normal bundle $T^{\perp}(M) \to M$ of M is naturally a Riemannian vector bundle with metric and connection induced from the metric and connection of N. Explicitly, this connection is given by

$$\nabla_{e_i} E = (\nabla^N_{e_i} E)^{\perp},$$

where e_i is the tangent to M, E is a section of $T^{\perp}(M)$, ∇^N is the connection of N, and $\perp$ denotes taking the normal component. Let SM denote the space of symmetric linear transformations $TM \to TM$. The second fundamental form of M as a submanifold of N can be regarded as an endomorphism

$$A \in \operatorname{End}(T^{\perp}M, SM).$$

Set

$$\tilde{A} = A^t \circ A,$$

where the superscript t denotes taking transpose. If R denotes the curvature operator of N, we can define a partial Ricci transformation ric for $E \in T^{\perp}(M)$ by

$$\operatorname{ric}(E) = \sum_{i=1}^{m} (R(E, e_i)e_i)^{\perp},$$

where the sum is taken over an orthonormal basis $\{e_i\}_{i=1}^m$ of TM.

The second variation of the volume of M along the compactly supported section E of $T^\perp M$ is given by

$$\int_M \langle -LE, E\rangle,$$

where

$$LE = \nabla^2 E + \mathrm{ric}\,(E) + \tilde{A}E.$$

The index of a compact domain D in M is defined to be the number of negative eigenvalues for the Dirichlet problem on D for L. The index of M is defined to be the limit of the indices of an increasing and exhausting sequence of compact domains on M. The index therefore measures how far M is from being stable, with index zero corresponding to stability.

We can therefore apply the results of Section 3 with the endomorphism Q being $Q = \mathrm{ric} + \tilde{A}$. One has $\langle \tilde{A}E, E\rangle \le |A|^2|E|^2$, where $|A|^2$ is the square of the length of the second fundamental form. Now assume that the partial Ricci operator ric satisfies

$$\mathrm{ric}\,(E,E)_x \le \mathrm{ric}\,(x)|E|^2,$$

for all $x \in M$ and some function ric on M. From Theorem 2 we then directly have the following:

Theorem 3. *Let M^m, $m \ge 3$, be a complete manifold, possibly with boundary, which is minimally immersed in a compact manifold N. Then the index of M satisfies*

$$\mathrm{index}\,(M) \le C(m,N)\cdot(n-m)\int_M (\max(1, |A|^2 + \mathrm{ric}\,))^{m/2},$$

where $C(m,N)$ is a constant depending only on m and the ambient manifold N, and $|A|^2$ and ric are as defined above.

5. Estimates of Betti numbers. For the sake of brevity, we consider only the case of M^m, $m \ge 3$, with empty boundary, minimally immersed in a compact manifold N. Assume that M is compact and orientable; otherwise, pass to the oriented double cover. Then the singular cohomology group, $H^p(M)$, is isomorphic to the space of harmonic p-forms on M. Recall that a p-form w is said to be harmonic if

$$\Delta_H w = 0,$$

where Δ_H is the Hodge Laplacian on the bundle of smooth p-forms on M. The pth Betti number $\beta_p(M)$ is defined to be the dimension of $H^p(M)$. We can therefore estimate $\beta_p(M)$ by the number of nonpositive eigenvalues of Δ_H. The Weitzenböck formula says that

$$\Delta_H = -\nabla^2 + R_p,$$

where ∇^2 is the Bochner Laplacian and R_p is a bundle endomorphism depending on the curvature tensor of M. Now, assume that,

$$|R_p|(x) \le r_p(x),$$

for all $x \in M$, and some function r_p on M. Since the rank of the bundle of p-forms on M is $\binom{n}{p}$, we obtain from Theorem 2, above, the following result which directly corresponds to Theorem 6 of [**6**].

Theorem 4. *Let M^m, $\partial M = \varnothing$, $m \ge 3$ be a complete oriented minimally immersed submanifold of a compact manifold N. Then*

$$\beta_p(M) \le C(m, N) \binom{n}{p} \int_M (\max(1, r_p))^{m/2}\, dV,$$

where r_p is as defined above.

6. Index bounds for minimal surfaces in $\mathbf{R}^n$ and S^n. We will first consider the question of estimating the index for minimal surfaces in $\mathbf{R}^n$. Therefore let M^2 be a complete minimally immersed surface in $\mathbf{R}^n$, possibly with boundary. If M is orientable, the Gauss map

$$G : M \to G_{2,n},$$

is anti-holomorphic, where $G_{2,n}$ is the Grassmannian of oriented two-planes in $\mathbf{R}^n$, see [**2**]. The pull-back metric under G satisfies

$$G^*(ds^2_{G_{2,n}}) = (-K)\, ds^2_M,$$

where K is the Gaussian curvature of M, see [**3**]. Note that this pull-back metric might have isolated singular points. Let M_ε be the manifold with boundary obtained from M by cutting out disks of radius ε around the singular points. An upper bound for the index of M_ε, independent of ε, would then provide us with an upper bound for the index of M.

From Section 4 we know that the number of negative eigenvalues of the second variation operator for $M_\varepsilon \to \mathbf{R}^n$ is dominated by the number of negative eigenvalues of

$$L = \nabla^2 + |A|^2,$$

where we used the notation of that section. Since M is two-dimensional and minimal $|A|^2 = 2K$. Let $\{e_i\}_{i=1}^2$ be orthonormal tangent vectors of M, and let E be a compactly supported section of the normal bundle $T^\perp(M_\varepsilon)$. The quadratic form I associated to L is then given by

$$\begin{aligned} I(E,E) &= \int_{M_\varepsilon} (\langle \nabla E, \nabla E\rangle - \langle (-2K)E, E\rangle)\, dV \\ &= \int_{M_\varepsilon} \left(\sum_{i=1}^2 \langle (\nabla_{e_i} E, \nabla_{e_i} E\rangle - 2\langle E, E\rangle(-K) \right) dV \\ &= \int_{M_\varepsilon} \left(\sum_{i=1}^2 \langle (\nabla_{(-K)^{1/2} e_i} E, \nabla_{(-K)^{1/2} e_i} e\rangle - 2\langle E, E\rangle \right) (-K)\, d\overline{V} \\ &= \int_{M_\varepsilon} (\langle \overline{\nabla} E, \overline{\nabla} E\rangle - 2\langle E, E\rangle)\, d\overline{V}, \end{aligned}$$

where $\overline{\nabla}$ is the gradient on $T^\perp(M_\varepsilon)$ induced by the metric $(-K)\, ds_M^2$ on M_ε, and $d\overline{V}$ is the corresponding volume form. The index of M_ε can therefore be estimated by the number of negative eigenvalues of

$$\overline{\nabla}^2 + 2,$$

on $T^\perp(M_\varepsilon)$, where $\overline{\nabla}^2$ denotes the Bochner Laplacian corresponding to $\overline{\nabla}$. Let $\overline{H}(x,y,t)$ denote the kernel of $\overline{\nabla}^2 - \partial/\partial t$ on $T^\perp(M_\varepsilon)$, satisfying Dirichlet boundary conditions, and let $H(x,y,t)$ denote the heat kernel of M_ε with respect to the pull-back metric from $G_{2,n}$, also satisfying

Dirichlet boundary conditions. From the proof of Theorem 2, we obtain $p \equiv 1$,

$$\operatorname{Tr} \overline{H}(t) \leq (n-2) \operatorname{Tr} H(t), \qquad t \geq 0.$$

Our previous bounds for traces of heat kernels used the assumption that the dimension of M is at least three. For surfaces in $\mathbf{R}^n$, we can use the Gauss map to obtain bounds for $\operatorname{Tr} H(t)$. First we note that since the Gauss map is holomorphic, $G(M_\varepsilon)$ is a minimal surface in $G_{2,n}$. We then embed $G_{2,n}$ isometrically into Euclidean space and, as above, use the Sobolev inequality in [**9**] to obtain an inequality of the form

$$\left(\int_{M_\varepsilon} f^2 \, d\overline{V} \right)^{1/2} \leq c_1 \int_{M_\varepsilon} (|\overline{\nabla} f| + c_2 |f|) \, d\overline{V}, \tag{5}$$

for $f \in C_0^1(M_\varepsilon)$, where c_1 and c_2 are constants depending only on n, and $\overline{\nabla}$ when acting on functions denotes the gradient with respect to the metric $G^*(ds^2_{G_{2,n}})$. The calculations below are similar to those of the first author and P. Li in [**4**]. We replace f in (5) by f^2 and adopt the convention that the integrals are taken over M_ε unless otherwise stated, and obtain

$$\left(\int f^4 \right)^{1/2} \leq c_1 \int |\overline{\nabla} f^2| + c_2 \int f^2 \leq c_1 \int |\overline{\nabla} f|^2 + (c_2+1) \int f^2. \tag{6}$$

Using interpolation, one estimates

$$||f||_2 \leq (||f||_4)^{2/3} (||f||_1)^{1/3},$$

i.e.,

$$\left(\int f^2 \right)^{3/2} \leq \left(\int f^4 \right)^{1/2} \int |f|,$$

which we substitute into (6):

$$\frac{(\int f^2)^{3/2}}{\int |f|} \leq c_1 \int |\overline{\nabla} f|^2 + (c_2+1) \int f^2. \tag{7}$$

We now would like to apply this inequality with $f(y) = H(x,y,t/2)$. With this choice of f,

$$\int |f| = 1, \tag{8}$$

and
$$H(x,x,t)=\int H^2(x,y,t/2)\,d\overline{V}(y)=\int f^2(y)\,d\overline{V}(y).$$
Taking the derivative with respect to t,
$$\begin{aligned}\frac{\partial}{\partial t}H(x,x,t)&=\int H(x,y,t/2)\frac{\partial H}{\partial t}(x,y,t/2)\,d\overline{V}(y)\\ &=\int(\overline{\Delta}_y H(x,y,t/2))H(x,y,t/2)\,d\overline{V}(y)\\ &=-\int|\overline{\nabla}_y H(x,y,t/2)|^2\,d\overline{V}(y),\end{aligned}\tag{9}$$
where $\overline{\Delta}$ is the Laplace-Beltrami operator on $(M,G^*(ds^2_{G_{2,n}}))$. Substituting our choice for f into (7), using (8) and (9), we obtain
$$(H(x,x,t))^{3/2}\le c_1\frac{\partial H}{\partial t}(x,x,t)+(c_2+1)H(x,x,t).$$
Letting $\varphi(t)=H^{1/2}(x,x,t)$, we can write this inequality as
$$\frac{2}{2c_1}\le\varphi'+\frac{c_2+1}{2c_1}\varphi.$$
Multiplying by the integrating factor $e^{(c_2+1)t/(2c_1)}$ and integrating from 0 to t using the fact that $\varphi(0)=0$, since $H(x,x,t)\to\infty$ as $t\to 0+$, we obtain
$$\varphi(t)\ge\frac{1}{c_2+1}(1-e^{-(c_2+1)t/(2c_1)}),$$
and, therefore,
$$H(x,x,t)\le\frac{(c_2+1)^2}{(1-e^{-(c_2+1)t/(2c_1)})^2}.$$
Combining this bound with our trace estimate above, we find that
$$\begin{aligned}\operatorname{Tr}\overline{H}(t)&\le(n-2)\operatorname{Tr}H(t)\\ &=(n-2)\int H(x,x,t)\,d\overline{V}(x)\\ &\le(n-2)\frac{(c_2+1)^2}{(1-e^{-(c_2+1)t/(2c_1)})^2}\int d\overline{V}\\ &=(n-2)\frac{(c_2+1)^2}{(1-e^{-(c_2+1)t/(2c_1)})^2}\int(-K)\,dV\\ &\le(n-2)\frac{(c_2+1)^2}{(1-e^{-(c_2+1)t/(2c_1)})^2}\int_M(-K)\,dV.\end{aligned}$$

Since the index of M_ε is bounded by the number of eigenvalues of $\overline{\nabla}^2$ that are strictly less than two, we have

$$(\operatorname{index}(M_\varepsilon)) \cdot e^{-2t} \leq \frac{(n-2)(c_2+1)^2}{(1-e^{-(c_2+1)t/(2c_1)})^2} \int_M (-K)\, dV,$$

or

$$\operatorname{index}(M_\varepsilon) \leq \frac{(n-2)(c_2+1)e^{2t}}{(1-e^{-(c_2+1)t/(2c_1)})^2} \int_M (-K)\, dV.$$

After minimizing the right-hand side over $t > 0$, we obtain

$$\operatorname{index}(M_\varepsilon) \leq C(n) \int_M (-K)\, dV,$$

where $C(n)$ is a constant depending on the n alone.

If M is nonorientable, we can estimate the index of M by the index of its two-sheeted oriented cover. Note that the argument above applies if the immersion of M has isolated branch points, by defining M_ε to also omit the branch points. Since the bounds above are independent of ε, we have derive the following theorem.

Theorem 5. *Let M^2 be a branched complete minimally immersed surface in $\mathbf{R}^n$, possibly with $\partial M \neq \varnothing$. Then*

$$\operatorname{index}(M) \leq C(n) \int_M (-K).$$

Remark 2. Hence, in particular, if $\int(-K) < 1/c(n)$, M is stable, which is Theorem 1 in [**11**].

Corollary 1. *Let Γ be a collection of C^2 curves in $\mathbf{R}^n$. Any branched minimal immersion $M^2 \to \mathbf{R}^n$ with $\partial M^2 = \Gamma$ satisfies*

$$\operatorname{index}(M) \leq C(n)\left(\int_\Gamma K(s)\, dx - 2\pi\chi(M)\right),$$

where $K(s)$ is the curvature of Γ and $\chi(M)$ is the Euler characteristic of M.

Proof of the corollary. If M is orientable and the immersion of M is branched, the generalized Gauss-Bonnet formula, see [**10**, pp. 345 and 413], reads

$$\int_\Gamma K_g\,ds - 2\pi\sum_{\alpha=1}^{a}(m_\alpha - 1) - \pi\sum_{\beta=1}^{b}(M_\beta - 1) + \int_M K\,dV = 2\pi\chi(M),$$

where K_g is the geodesic curvature of Γ, and $(m_\alpha - 1)$ and $(M_\beta - 1)$ are the branch numbers of the branch points of the immersion of M in $\dot{M}$ and ∂M, respectively. With K being the curvature Γ, we therefore have the inequality

$$\int_M (-K)\,dV \leq \int_\Gamma K(s)\,ds - 2\pi\chi(M).$$

The corollary now follows from Theorem 4 where, as before, if M is nonorientable, we estimate the index of M by the index of its two-sheeted oriented cover. □

Theorem 6. *Let M^2 be a bounded immersed complete minimal surface in S^n, possibly with $\partial M \neq \varnothing$. Then*

$$\operatorname{index}(M) \leq C(n)\left(2\operatorname{area}(M) - \int_M K\right),$$

where $C(n)$ is a constant depending only on n. Also, if ∂M consists of a collection of Γ of C^2 curves in S^n,

$$\operatorname{index}(M) \leq C(n)\left(2\operatorname{area}(M) - 2\pi\chi(M) + \int_\Gamma K(s)\,ds\right),$$

where K is the curvature of Γ. In particular, if $\partial M = \varnothing$,

$$\operatorname{index}(M) \leq C(n)(2\operatorname{area}(M) - 2\pi\chi(M)).$$

Proof. From Section 4 it follows that the number of negative eigenvalues of the second variation operator of M as a minimal submanifold of S^n is bounded by the number of negative eigenvalues of

$$L = \nabla^2 + |A|^2 + 2,$$

where we used the notation of Section 4. Since M is minimal in S^n,

$$|A|^2 = (1 - K),$$

so the operator L can be written

$$\nabla^2 + 2(2 - K).$$

Now, assume that M is orientable; otherwise pass to the oriented double cover. Let

$$G : M \to G_{2,n+1},$$

be the Gauss map of M regarded as a surface in $\mathbf{R}^{n+1}$. Then G is conformal with

$$G^*(ds^2_{G_{2,n+1}}) = (2 - K)\, ds^2_M,$$

and $G(M)$ is minimal in $G_{2,n+1}$, see [**7**]. We therefore find that the index of M is bounded by the numbers of negative eigenvalues of

$$\overline{\nabla}^2 + 2,$$

where $\overline{\nabla}^2$ denotes the Bochner Laplacian on $T^\perp(M)$ induced by the pull-back metric from $G_{2,n+1}$. Note that in this case the pull-back metric is nonsingular. Following the argument for Theorem 5 and Corollary 1 and noting that

$$\operatorname{area}(G(M)) = \int_M (2 - K) = 2\operatorname{area}(M) - \int_M K,$$

we arrive at the statement of Theorem 6. □

Remark 3. The theorems above have immediate counterparts for the case of minimal surfaces in manifolds covered by $\mathbf{R}^n$ and S^n.

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