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ON A CONJECTURE OF R.L. GRAHAM

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1. Introduction. Let a and b be two positive integers. We shall call the ratio $a/(a,b)$ the *reduced ratio* of a and b. In 1970, R.L. Graham [**5**] conjectured that *the maximum of the reduced ratios of pairs of integers in a finite set of positive integers is at least the cardinality of the set.*

Let $M(n)$ be the least common multiple of $1, 2, \ldots, n$. We shall call $\mathcal{S}$ a *standard set* if either

$$\mathcal{S} = \{1, 2, \ldots, n\}, \qquad S = \{M(n)/1, M(n)/2, \ldots, M(n)/n\},$$

or

$$\mathcal{S} = \{2, 3, 4, 6\}.$$

We say a finite set of positive integers is *primitive* if the greatest common divisor of the members of the set is 1. It is easy to see that if Graham's conjecture is valid for any primitive set, then it is valid in general. One may further conjecture (see [**8**]), that *if the maximum of the reduced ratios of pairs of integers in a primitive set is at most its cardinal number, then the set is standard*. We call this the strong Graham conjecture. It is clear that the strong Graham conjecture implies Graham's conjecture.

Graham's conjecture has gotten much attention since it was proposed and there are many partial results. See [**4**] for a survey up to 1980.

Szegedy [**7**] and Zaharescu [**9**] independently proved Graham's conjecture for all sufficiently large cardinalities. In fact, Szegedy established the strong Graham conjecture for such cardinalities. Both Szegedy's and Zaharescu's proofs rely on 'Hoheisel type' results in prime number theory. Namely, they use the theorem that there is some constant $c > 0$, such that the number of primes in the interval $[x, x+x^{1-c}]$ is asymptotically $x^{1-c}/\log x$ as $x \to \infty$. This is a deep result in analytic number theory, and it would be very difficult to use the Szegedy or Zaharescu proofs to give an explicit bound for "sufficiently large." To be sure,

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in [**3**] it is shown that Graham's conjecture is true for all cardinalities exceeding 10^{70} provided the Riemann hypothesis is assumed.

In this paper, we give still another proof of Graham's conjecture (and the strong Graham conjecture) for sufficiently large cardinalities. However, the deepest tool we use is the prime number theorem with only a moderately weak error term; namely, we use that

$$\pi(x) = li(x) + o\left(\frac{x}{\log^9 x}\right), \tag{1}$$

for $x \to \infty$, where $\pi(x)$ is the number of primes up to x and $li(x)$ is the logarithmic integral function $\int_2^x dt/\log t$. In fact, using the explicit prime number theorem in [**6**], we are able to prove the following result.

Theorem 1. *For all integers $n \geq 10^{50,000}$, if $\mathcal{S}$ is a primitive set of positive integers of cardinality n, and if the maximal reduced ratio of pairs of numbers from $\mathcal{S}$ is at most n, then $\mathcal{S}$ is a standard set.*

Our proof roughly follows the same lines as that of Szegedy [**7**].

After this paper was submitted for publication, we learned that Balasubramanian and Soundararajan, using other tools, have been able to prove the strong Graham conjecture for all n, see [**1**].

2. Preliminary considerations. In this section we recall some results that are relevant to the problem. Throughout the paper we shall let $\mathcal{S} = \{a_1, \dots, a_n\}$ denote a primitive set of positive integers of cardinality n.

Lemma 1. *If p is a prime and a and b are positive integers with $a \equiv b \not\equiv 0 \bmod p$ and $a < b$, then $b/(a,b) > p$.*

This simple result is a key starting point for thoughts on Graham's conjecture. For example, from this lemma one can prove, as did Szemerédi, that Graham's conjecture is true for sets of prime cardinality (see [**8**]). The proof of Lemma 1 is immediate, since from the hypothesis we have $b/(a,b) \equiv a/(a,b) \bmod p$, so that $b/(a,b) \geq p + a/(a,b) \geq p+1$.

Boyle [**2**] proved in 1978 that if there is a prime $q > (n-1)/2$ dividing some a_i, then Graham's conjecture holds for $\mathcal{S}$. The following result was proved by Szegedy in [**7**].

Lemma 2. *If there is a prime $q > n/2$ dividing some a_i, and if the maximal reduced ratio from $\mathcal{S}$ is at most n, then $\mathcal{S}$ is a standard set.*

For completeness, we sketch a proof of Lemma 2. The result is trivial if $q > n$, so assume that $q \le n$. Without loss of generality, we may assume that q divides each of $a_1, \dots, a_s$ and that q does not divide $a_{s+1}, \dots, a_n$, for some integer s with $1 \le s \le n-1$. Let B be the least common multiple of $a_1/q, \dots, a_s/q$ and let A be the greatest common divisor of $a_{s+1}, \dots, a_n$. Our first observation is that $B|A$. Indeed, this will follow if we show that for each $i = 1, \dots, s$ and each $j = s+1, \dots, n$, we have $a_i/q|a_j$. But

$$n \ge \frac{a_i}{(a_i, a_j)} = q\frac{a_i/q}{(a_i/q, a_j)},$$

so that if a_i/q does not divide a_j, then $n \ge 2q$, a contradiction.

Now let $a_i = \min\{a_1, \dots, a_s\}$ and let $a_j = \max\{a_{s+1}, \dots, a_n\}$. Since $B/(a_1/q), \dots, B/(a_s/q)$ are distinct integers, we have $B/(a_i/q) \ge s$. Since $a_{s+1}/A, \dots, a_n/A$ are distinct integers, we have $a_j/A \ge n-s$. Thus

$$n \ge \frac{a_j}{(a_i, a_j)} = \frac{a_j}{a_i/q} = \frac{a_j}{A} \cdot \frac{A}{B} \cdot \frac{B}{a_i/q} \ge s(n-s)\frac{A}{B}.$$

The cases $n \le 4$ may be handled by simple arguments, so assume $n \ge 5$. Since A/B is an integer, we have either $s = 1$ or $s = n-1$, since if not, $s(n-s) \ge 2(n-2) > n$. We also have $A = B$, since $s(n-s) = n-1$. If $s = 1$, we have $B = a_1/q$, so since $\mathcal{S}$ is a primitive set, we must have $B = A = 1$ and $a_1 = q$. If $a_j = n-1$, then $\{a_2, \dots, a_n\} = \{1, \dots, n-1\}$, so that $a_1 = q = n$ and $\mathcal{S}$ is a standard set. Similarly, $\mathcal{S}$ is a standard set if $a_j = n$. If $a_j > n$, the lemma also holds. This concludes the case $s = 1$. But if $s = n-1$, we may replace $\{a_1, \dots, a_n\}$ with the set $\{C/a_1, \dots, C/a_n\}$, where C is the least common multiple of the set and so end up again in the case $s = 1$. Thus $\mathcal{S}$ is standard, which concludes our sketch of the proof of the lemma.

From now on, we shall always assume that the primitive set $\mathcal{S}$ satisfies the hypothesis of Theorem 1. That is, for each i, j, we have $a_i/(a_i, a_j) \le n$.

3. Wanted pairs. From (1) it follows that if n is sufficiently large, there is a prime P satisfying

$$2n - n/L \le P \le 2n - n/(2L), \tag{2}$$

where $L = \log^8 n$. For n at least 4 we have $L > 4$, so that $P > 7n/4$.

Let us call an ordered pair $\langle a_i, a_j \rangle$ of elements from $\mathcal{S}$ a *wanted pair* if $a_i + a_j \equiv 0 \bmod P$. Let N denote the number of members of $\mathcal{S}$ that are not in any wanted pair. By Lemmas 1 and 2, the negatives of these N members of $\mathcal{S}$, together with $\mathcal{S}$, form a set of $n + N$ distinct nonzero residues mod P. Thus, $n + N \le P - 1$. But the number of wanted pairs is $n - N$, so there are at least $2n - P + 1$ wanted pairs.

We shall now count these wanted pairs another way.

Definition. For each integer m, let $\mathcal{F}(m)$ denote the set of wanted pairs $\langle a_i, a_j \rangle$ with $a_i/(a_i, a_j) = m$. Further, let $F(m)$ denote the cardinality of $\mathcal{F}(m)$.

Thus the total number of wanted pairs is $\sum_{m=1}^{\infty} F(m)$.

Which numbers m have $F(m) > 0$? From our assumption that $\mathcal{S}$ satisfies the hypothesis of the strong Graham conjecture, it follows that if $\langle a_i, a_j \rangle \in \mathcal{F}(m)$, then

$$\frac{a_i}{(a_i, a_j)} + \frac{a_j}{(a_i, a_j)} = P,$$

since the sum is $0 \bmod P$, $P > 7n/4$ and the larger summand is at least one half of the sum. Thus these two reduced ratios are both in the interval $[P - n, n]$. It follows that $F(m) = 0$ unless $P - n \le m \le n$. Thus, from the above observations, we have

$$\sum_{m=P-n}^{n} F(m) \ge 2n - P + 1. \tag{3}$$

The main idea of the proof is to show that the contribution to this sum of the terms with $F(m) > 1$ is small. Since the sum is at least the number of positive terms, it follows from (3) that very few terms are 0, in fact almost all terms are positive. We will show that so many terms are positive that there is at least one *prime* value of m with $F(m) > 0$. Then some a_i is divisible by a prime, namely m, that exceeds $n/2$, so that the theorem follows from Lemma 2.

Suppose

$$\mathcal{F}(m) = \{\langle b_1, c_1\rangle, \dots, \langle b_{F(m)}, c_{F(m)}\rangle\}.$$

It is easy to see that $\langle b_i, c_i\rangle \in \mathcal{F}(m)$ if and only if $\langle c_i, b_i\rangle \in \mathcal{F}(P-m)$, so that

$$F(m) = F(P-m). \tag{4}$$

For $i, j \in \{1, \dots, F(m)\}$, let

$$X_{ij} = \frac{b_i}{(b_i, c_j)}, \qquad Y_{ij} = \frac{c_i}{(b_j, c_i)}. \tag{5}$$

Then $X_{ij} \le n$ and $Y_{ij} \le n$. Note that

$$\frac{X_{ij}X_{ji}}{Y_{ij}Y_{ji}} = \frac{b_i b_j}{c_i c_j} = \frac{m^2}{(P-m)^2}.$$

Since the last fraction is in reduced form, it follows that $X_{ij}X_{ji} = Zm^2$ for some positive integer Z. We claim that $Z = 1$. Indeed, since $F(m) > 0$, we have $m \in [P-n, n]$, so that $m > 3n/4$. Thus, if $Z \ge 2$, one of X_{ij} or X_{ji} will be at least $\sqrt{2}m > n$, a contradiction. Hence, $Z = 1$ as claimed and

$$X_{ij}X_{ji} = m^2, \qquad Y_{ij}Y_{ji} = (P-m)^2. \tag{6}$$

Consider the map

$$\phi : \mathcal{F}(m) \to \mathbf{N}^2,$$

where $\phi(\langle b_i, c_i\rangle) = \langle X_{1i}, Y_{1i}\rangle$. What we have proved implies that $\mathrm{Im}\,(\phi)$ is contained in

$$\{\langle X, Y\rangle \in \mathbf{N}^2 : X | m^2, Y | (P-m)^2, \max\{X, Y, m^2/X, (P-m)^2/Y\} \le n\}. \tag{7}$$

We claim that ϕ is a one-to-one map. Indeed, (6), $X_{1i} = X_{1j}$ and $Y_{1i} = Y_{1j}$ imply that $X_{i1} = m^2/X_{1i} = m^2/X_{1j} = X_{j1}$ and $Y_{i1} = (P-m)^2/Y_{1i} = (P-m)^2/Y_{1j} = Y_{j1}$. Also, (5), $X_{1i} = X_{1j}$ and $Y_{1i} = Y_{1j}$ imply that $(b_1, c_i) = (b_1, c_j)$ and $(b_i, c_1) = (b_j, c_1)$. Thus, (5), $X_{i1} = X_{j1}$ and $Y_{i1} = Y_{j1}$ imply that $b_i = b_j$ and $c_i = c_j$. We have proved our claim.

Definition. For any positive integer m, let $\mathcal{G}(m)$ be the set of positive divisors X of m^2 with $X \le n$ and $m^2/X \le n$. Let $G(m)$ be the cardinality of $\mathcal{G}(m)$.

Note that the cardinality of the set (7) is $G(m)G(P-m)$. We thus have proved the following result.

Lemma 3. *For any integer $m > 0$, we have $F(m) \le G(m)G(P-m)$.*

We now prove the following result.

Lemma 4. *We have*

$$\sum_{m:F(m)>0} F(m) \le \sum_{m:F(m)>0} G(m)^2.$$

Proof. From Lemma 3 and Cauchy's inequality we have

$$\begin{aligned}\sum_{m:F(m)>0} F(m) &\le \sum_{m:F(m)>0} G(m)G(P-m)\\ &\le \Bigg(\sum_{m:F(m)>0} G(m)^2 \sum_{m:F(m)>0} G(P-m)^2\Bigg)^{1/2}.\end{aligned}$$

By (4), $F(m) > 0$ if and only if $F(P-m) > 0$. Thus the sums in the parentheses are equal and the lemma follows. $\square$

We now turn our attention to estimating $G(m)$. It is clear from the definition that $G(m) \le d(m^2)$ for any positive integer m, where d is the divisor function. Let $d_3(m)$ denote the number of ordered triples a, b, c of positive integers with $abc = m$. It is evident that $d(m^2) \leq d_3(m)$ for all positive integers m, since both functions are multiplicative and, for any prime power p^a, we have $d(p^{2a}) = 2a + 1 \leq \binom{a+2}{2} = d_3(p^a)$. Thus, we have

$$G(m) \leq d_3(m) \tag{8}$$

for every positive integer m. In fact, we can say a little more.

Lemma 5. *For $m \in [P - n, n]$, we have*

$$G(m) \leq \#\{\langle t, k, l\rangle \in \mathbf{N}^3 : tkl = m, (k, l) = 1, \\ k, l \in [\sqrt{n/t}(1 - 1/L), \sqrt{n/t}]\}.$$

Proof. Let $m \in [P - n, n]$. For each $X \in \mathcal{G}(m)$, let k, l be coprime with

$$\frac{X}{m} = \frac{m}{m^2/X} = \frac{k}{l}.$$

Thus $k|m$, $l|m$, so there is some integer t with $m = tkl$. This mapping from $\mathcal{G}(m)$ to ordered triples, t, k, l, is one-to-one since $X = mk/l = tk^2$. It remains to show that

$$k, l \in [\sqrt{n/t}(1 - 1/L), \sqrt{n/t}\,].$$

The upper bound for k, l is immediate since $n \geq X = tk^2$ and $n \geq m^2/X = tl^2$. For the lower bound, note that $m \geq n(1 - 1/L)$, so that

$$k = \frac{m}{tl} \geq \frac{n}{tl}\left(1 - \frac{1}{L}\right) \geq \sqrt{\frac{n}{t}}\left(1 - \frac{1}{L}\right)$$

by the upper bound for l. We similarly have the same result for l. This concludes the proof of the lemma. □

Remark. From (2) the triple $\langle m, 1, 1\rangle$ is always in the set described in Lemma 5. Thus, if $G(m) > 1$, this set must contain at least one other triple $\langle t, k, l\rangle$. Any triple $\langle t, k, l\rangle$ in this set satisfies

$$\min\{k, l\} \geq \left(1 - \frac{1}{L}\right)\max\{k, l\},$$

so that if $\langle t,k,l\rangle \neq \langle m,1,1\rangle$, then $\max\{k,l\} \geq L$ and $\min\{k,l\} \geq L-1$. Indeed, otherwise k and l would be coprime integers in an interval of length less than 1, an impossibility unless they are both 1.

4. The arithmetic function d_3. In this section we are going to prove several inequalities concerning the function d_3 defined in the preceding section. Preliminary to that we shall first prove some simple results about the divisor function d.

Lemma 6. *For $x \geq 1$, we have*

$$\sum_{h\leq x} d(h) \leq x(\log x + 1) \quad \text{and} \quad \sum_{h\leq x} \frac{d(h)}{h} < \frac{1}{2}(\log x + 2)^2.$$

Proof. We have

$$\begin{aligned}\sum_{h\leq x} d(h) &= \sum_{h\leq x}\sum_{j\mid h} 1 = \sum_{j\leq x}\left[\frac{x}{j}\right] \leq x\sum_{j\leq x}\frac{1}{j} \\ &\leq x\left(1 + \int_1^x \frac{dt}{t}\right) = x(\log x + 1),\end{aligned}$$

which proves the first assertion of the lemma. To see the second assertion, we use the first assertion and partial summation, getting

$$\begin{aligned}\sum_{h\leq x}\frac{d(h)}{h} &= \frac{1}{x}\sum_{h\leq x} d(h) + \int_1^x \frac{1}{t^2}\sum_{h\leq t} d(h)\,dt \\ &\leq \log x + 1 + \int_1^x \frac{\log t + 1}{t}\,dt \\ &= \frac{1}{2}\log^2 x + 2\log x + 1 < \frac{1}{2}(\log x + 2)^2.\end{aligned}$$

This completes the proof of the lemma. □

Lemma 7. *For $1 \leq x^{2/3} \leq y \leq x$, we have*

$$\sum_{x-y\leq h\leq x} d_3(h) < y(\log x + 3)^2.$$

Proof. If h is factored into 3 positive integers, then the product of the smaller two of these factors must be $\le h^{2/3}$. Thus $d_3(h) \le 3\sum_{j|h, j\le h^{2/3}} d(j)$. Thus, from Lemma 6,

$$\begin{aligned}\sum_{x-y\le h\le x} d_3(h) &\le 3 \sum_{x-y\le h\le x}\ \sum_{j|h, j\le h^{2/3}} d(j) \le 3 \sum_{j\le x^{2/3}} d(j) \sum_{x-y\le h\le x, j|h} 1\\ &= 3\sum_{j\le x^{2/3}} d(j) \sum_{(x-y)/j\le k\le x/j} 1 \le 3 \sum_{j\le x^{2/3}} d(j)\left(\frac{y}{j}+1\right)\\ &< \frac{3}{2} y(\log x^{2/3}+2)^2 + 3x^{2/3}(\log x^{2/3}+1)\\ &\le \frac{2}{3} y(\log x+3)^2 + y(2\log x+3).\end{aligned}$$

The lemma easily follows. □

Lemma 8. *For $x \ge 1$, we have*

$$\sum_{h\le x} \frac{d_3(h)}{h} < \frac{1}{6}(\log x+3)^3 \quad \textit{and} \quad \sum_{h\ge x}\frac{d_3(h)^2}{h^2} < \frac{1}{6^5 x}\sum_{k=0}^{9}\frac{9!}{k!}(\log x+6)^k.$$

Proof. First note that from Lemma 6, we have

$$\begin{aligned}\sum_{h\le x} d_3(h) &= \sum_{h\le x}\sum_{j|h} d(j) = \sum_{j\le x} d(j)\left[\frac{x}{j}\right]\\ &\le x\sum_{j\le x}\frac{d(j)}{j} < \frac{1}{2}x(\log x+2)^2.\end{aligned}$$

Thus, by partial summation, we have

$$\begin{aligned}\sum_{h\le x}\frac{d_3(h)}{h} &= \frac{1}{x}\sum_{h\le x} d_3(h) + \int_1^x \frac{1}{t^2}\sum_{h\le t} d_3(h)\,dt\\ &< \frac{1}{2}(\log x+2)^2 + \int_1^x \frac{1}{2t}(\log t+2)^2\,dt\\ &= \frac{1}{2}(\log x+2)^2 + \frac{1}{6}(\log x+2)^3 < \frac{1}{6}(\log x+3)^3.\end{aligned}$$

This is the first assertion in the lemma. Using this result, we have

$$\begin{aligned}\sum_{h\le x}\frac{d_3(h)^2}{h} &= \sum_{h\le x}\frac{d_3(h)}{h}\sum_{abc=h}1\\ &= \sum_{abc\le x}\frac{d_3(abc)}{abc} \le \sum_{abc\le x}\frac{d_3(a)}{a}\frac{d_3(b)}{b}\frac{d_3(c)}{c}\\ &\le 6\sum_{a\le x^{1/3}}\sum_{b\le x^{1/2}}\sum_{c\le x}\frac{d_3(a)}{a}\frac{d_3(b)}{b}\frac{d_3(c)}{c}\\ &< \frac{1}{6^2}(\log x^{1/3}+3)^3(\log x^{1/2}+3)^3(\log x+3)^3\\ &= \frac{1}{6^5}(\log x+9)^3(\log x+6)^3(\log x+3)^3 < \frac{1}{6^5}(\log x+6)^9.\end{aligned}$$

We now use partial summation with this last inequality to get

$$\begin{aligned}\sum_{h\ge x}\frac{d_3(h)^2}{h^2} &= \int_x^\infty \frac{1}{t^2}\sum_{x\le h\le t}\frac{d_3(h)^2}{h}\,dt\\ &< \frac{1}{6^5}\int_x^\infty \frac{(\log t+6)^9}{t^2}\,dt\\ &= \frac{1}{6^5 x}\sum_{k=0}^{9}\frac{9!}{k!}(\log x+6)^k.\end{aligned}$$

This completes the proof of the lemma. □

5. Conclusion of the proof of Theorem 1. In this section we shall show that for all large n,

$$\sum_{m:F(m)>0}(F(m)-1) < \frac{n}{3L\log n}. \tag{9}$$

Note that from (1) and (2), the number of primes in $[P-n,n]$ is at least

$$\pi(n)-\pi\left(n-\frac{n}{2L}\right) > \frac{n}{3L\log n} \tag{10}$$

for all large n. Recall that $F(m) > 0$ implies that $m \in [P-n, n]$. Thus, from (3) and (9) we have that

$$\begin{aligned}\sum_{m:F(m)>0} 1 &= \sum_{m=P-n}^{n} F(m) - \sum_{m:F(m)>0} (F(m)-1) \\ &> 2n - P + 1 - \frac{n}{3L\log n}.\end{aligned}$$

But from (10), the number of composites in the interval $[P-n, n]$ is less than $2n - P + 1 - n/(3L\log n)$, so that there must be some prime value of m with $F(m) > 0$. It follows that some a_i is divisible by a prime exceeding $n/2$, namely m, and so Lemma 2 is applicable. Thus, the strong Graham conjecture holds for those values of n for which (9) and (10) both hold.

From Lemma 4, we have that

(11)
$$\begin{aligned}\sum_{m:F(m)>0} (F(m)-1) &\le \sum_{m:F(m)>0} (G(m)^2 - 1) \\ &\le \sum_{\substack{m\in[P-n,n]\\ G(m)>1}} (G(m)^2-1) \le \sum_{\substack{m\in[P-n,n]\\ G(m)>1}} G(m)^2.\end{aligned}$$

Let

$$\begin{aligned}\mathcal{G} = \{\langle t,k,l\rangle \in \mathbf{N}^3 : tkl \in [P-n, n],& \\ \sqrt{n/t}(1-1/L) \le l < k \le \sqrt{n/t}, L-1 \le l\}.&\end{aligned}$$

From Lemma 5 and the subsequent remark, if $m \in [P-n, n]$ and $G(m) > 1$, then

$$G(m) \le 1 + 2 \sum_{\substack{\langle t,k,l\rangle\in\mathcal{G}\\ tkl=m}} 1 \le 3 \sum_{\substack{\langle t,k,l\rangle\in\mathcal{G}\\ tkl=m}} 1.$$

Thus, from (8),

$$\begin{aligned}\sum_{\substack{m\in[P-n,n]\\ G(m)>1}} G(m)^2 &\le \sum_{\substack{m\in[P-n,n]\\ G(m)>1}} d_3(m)G(m)\\ &\le 3\sum_{\substack{m\in[P-n,n]\\ G(m)>1}} d_3(m)\sum_{\substack{\langle t,k,l\rangle\in\mathcal{G}\\ tkl=m}} 1\\ &\le 3\sum_{\langle t,k,l\rangle\in\mathcal{G}} d_3(tkl)\\ &\le 3\sum_{\langle t,k,l\rangle\in\mathcal{G}} d_3(t)d_3(k)d_3(l).\end{aligned} \tag{12}$$

We shall distinguish two cases: $t \le n/L^6$ and $t > n/L^6$. In the first case we have, using Lemmas 7 and 8,

$$\begin{aligned}S_1 :&= 3\sum_{\substack{\langle t,k,l\rangle\in\mathcal{G}\\ t\le n/L^6}} d_3(t)d_3(k)d_3(l)\\ &\le \frac{3}{2}\sum_{t\le n/L^6} d_3(t)\Bigg(\sum_{\sqrt{\frac{n}{t}}(1-\frac{1}{L})\le k\le\sqrt{\frac{n}{t}}} d_3(k)\Bigg)^2\\ &< \frac{3}{2}\sum_{t\le n/L^6} d_3(t)\left(\frac{\sqrt{n/t}}{L}(\log\sqrt{n/t}+3)^2\right)^2\\ &\le \frac{3n}{2L^2}(\log\sqrt{n}+3)^4\sum_{t\le n/L^6}\frac{d_3(t)}{t}\\ &< \frac{n}{4L^2}(\log\sqrt{n}+3)^4(\log(n/L^6)+3)^3\\ &= \frac{n}{64L^2}(\log n+6)^4(\log n-6\log L+3)^3.\end{aligned} \tag{13}$$

For our next estimation we shall need $n \ge L^9$. For $L = \log^8 n$, this holds if $n \ge 10^{191}$. We now assume that n is at least this large. Thus, by Lemma 7, we have

$$\begin{aligned} S_2 := 3 &\sum_{\substack{\langle t,k,l\rangle \in \mathcal{G} \\ n/L^6 < t}} d_3(t)d_3(k)d_3(l) \\ &\leq 3 \sum_{L \leq k < L^3} \sum_{k(1-\frac{1}{L}) \leq l < k} d_3(k)d_3(l) \sum_{\frac{n}{kl}(1-\frac{1}{L}) < t \leq \frac{n}{kl}} d_3(t) \\ &< 3 \sum_{L \leq k < L^3} \sum_{k(1-\frac{1}{L}) \leq l < k} d_3(k)d_3(l) \frac{n/(kl)}{L} (\log(n/(kl)) + 3)^2 \\ &< \frac{3n}{L} (\log n - \log(L^2 - L) + 3)^2 \sum_{L \leq k < L^3} \sum_{k(1-\frac{1}{L}) \leq l < k} \frac{d_3(k)d_3(l)}{kl}. \end{aligned}$$

By Cauchy's inequality and Lemma 8,

$$\begin{aligned} &\sum_{L \leq k < L^3} \sum_{k(1-\frac{1}{L}) \leq l < k} \frac{d_3(k)d_3(l)}{kl} \\ &\qquad = \sum_{j < L^2} \sum_{jL < k < L^3} \frac{d_3(k)}{k} \frac{d_3(k-j)}{k-j} \\ &\qquad \leq \sum_{j < L^2} \left(\sum_{jL < k < L^3} \frac{d_3(k)^2}{k^2} \sum_{jL < k < L^3} \frac{d_3(k-j)^2}{(k-j)^2} \right)^{1/2} \\ &\qquad < \sum_{j < L^2} \sum_{k > jL - j} \frac{d_3(k)^2}{k^2} \\ &\qquad < \sum_{j < L^2} \frac{1}{6^5(jL - j)} \sum_{i=0}^{9} \frac{9!}{i!} (\log(jL) + 6)^i \\ &\qquad < \frac{1}{6^5(L-1)} \sum_{i=0}^{9} \frac{9!}{i!} (\log L^3 + 6)^i \sum_{j < L^2} \frac{1}{j}. \end{aligned}$$

For $L \geq (\log 10^{191})^8$, we have

$$\sum_{i=0}^{9} \frac{9!}{i!} (\log L^3 + 6)^i < 1.06 (\log L^3 + 6)^9,$$

so that from the above estimates we have

(14)

$$\begin{aligned} S_2 &< \frac{3n}{L}(\log n - \log(L^2-L)+3)^2 \frac{1.06(\log L^3+6)^9}{6^5(L-1)} \sum_{j<L^2} \frac{1}{j} \\ &< \frac{3.18n}{6^5 L(L-1)}(\log n - \log(L^2-L)+3)^2(\log L^3+6)^9(\log L^2+1) \\ &< \frac{17n}{L^2}(\log n - 2\log L + 3)^2(\log L+2)^{10}. \end{aligned}$$

We conclude from (11), (12), (13) and (14) that for $n \geq 10^{191}$ we have

$$\begin{aligned} \sum_{m:F(m)>0} (F(m)-1) &\leq S_1 + S_2 \\ &< \frac{n}{L^2}\bigg(\frac{1}{64}(\log n+6)^4(\log n - 6\log L+3)^3 \\ &\qquad + 17(\log n - 2\log L+3)^2(\log L+2)^{10}\bigg). \end{aligned} \tag{15}$$

The expression on the right of (15) is smaller than $n/(3L\log n)$ if and only if

$$\begin{aligned} &\frac{1}{64}(\log n+6)^4(\log n-6\log L+3)^3 \\ &\qquad + 17(\log n-2\log L+3)^2(\log L+2)^{10} < \frac{L}{3\log n}. \end{aligned} \tag{16}$$

With $L = \log^8 n$, we have (16) for $n \geq 10^{6,000}$. Thus, we have (9) for $n \geq 10^{6,000}$.

We now turn to (10). From Theorem 3 in [**6**], we have (10) with $L = \log^8 n$ and $n \geq 10^{50,000}$. This completes the proof of Theorem 1.

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