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STRONGLY REGULAR FUSIONS OF TENSOR PRODUCTS OF STRONGLY REGULAR GRAPHS

A.D. SANKEY

1. Introduction. A strongly regular graph (srg) is a regular graph, neither complete nor null, with two distinct vertices x and y having λ or μ common neighbors according to whether x and y are adjacent or not. (This and other definitions in this section will be repeated below.) An srg is an association scheme with three relations. We write f_0 for the identity relation, f_1 for adjacency, and f_2 for nonadjacency. By the tensor product, or Kronecker product, of two srg's with relations f_i and g_i, respectively, we mean the association scheme whose relations are $f_i \otimes g_j$, $0 \le i, j \le 2$. A *fusion* of the tensor product is a subscheme in the sense of Bannai [**1**]. We say a fusion is strongly regular if it is a rank 3 association scheme (hence its relations are those of an srg). A general problem for coherent configurations (cc's) is to decide under what circumstances the tensor product has coherent fusions. This work addresses that question for the specific case of srg's, which are equivalent to homogeneous rank 3 cc's [**4**]. We give the parameters of all pairs of strongly regular graphs whose tensor product admits primitive strongly regular fusions, and list the resulting feasible srg parameters. An imprimitive srg is a complete n-partite graph or its complement. We include these as possible factors in the tensor products, but require primitivity for successful fusions.

In Section 2 we give definitions and background material. Section 3 is a discussion of rank 3 srg's, followed by an example. In Section 4 we outline the method of working with intersection matrices and their eigenvalues. Section 5 is a summary of results, with remarks on some of the interesting cases that arise.

2. Preliminaries.

Definition. A *strongly regular graph* is a regular graph, neither complete nor null, with the property that the number of vertices

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adjacent to each of two vertices x and y depends only on whether x and y are adjacent.

Definition. An srg Γ is *primitive* if both Γ and its complement $\overline{\Gamma}$ are connected.

An imprimitive srg is a complete n-partite graph or its complement. Background material on srg's can be found in the survey articles of Hubaut [**5**] and of Brouwer and van Lint [**2**].

Associated with any srg Γ is the triple of matrices $A_0 = I$, A_1 the adjacency matrix of the graph, and A_2 the adjacency matrix of its complement $\overline{\Gamma}$, where I is the identity matrix of appropriate size. Henceforth, we identify the matrix A_i with the relation f_i. Let Γ_1 and Γ_2 be two srg's with associated triples $\{A_0, A_1, A_2\}$ and $\{B_0, B_1, B_2\}$. By the *tensor product* of Γ_1 and Γ_2 we mean the association scheme whose relations are the nine tensor products $C_{3i+j} = A_i \otimes B_j$, $0 \le i,\ j \le 2$. (We will sometimes abuse notation and refer to the set of matrices as the tensor product.) A fusion of the tensor product is formed by choosing a nonempty subset S of $\{1, 2, \dots, 8\}$. This determines a partition of the indexing set into three subsets $\{0\}$, S, and $\overline{S} = \{1, 2, \dots, 8\} \backslash S$. Set $C_S = \sum_{\sigma \in S} C_\sigma$ and $C_{\overline{S}} = \sum_{\sigma \in \overline{S}} C_\sigma$. By a fusion of the tensor product we shall mean the set $\{I = C_0, C_S, C_{\overline{S}}\}$. Let Γ be the graph whose adjacency matrix is C_S. Then $C_{\overline{S}}$ is by definition the adjacency matrix of the complement $\overline{\Gamma}$, so we may assume that $|S| \le 4$. If Γ is strongly regular, this fusion is the triple associated with Γ, and we say the fusion is strongly regular.

Parameters. The parameters of an srg Γ are (n, k, λ, μ) where n is the number of vertices, k is the valency, and λ (μ) is the number of common neighbors to two given adjacent (nonadjacent) vertices. The parameters of $\overline{\Gamma}$ are

$$(n, n-k-1, n-2k+\mu-2, n-2k+\lambda).$$

Since one of Γ and $\overline{\Gamma}$ has valency no greater than $(n-1)/2$, we may assume Γ is that one, so that $k \le (n-1)/2$. In the case of equality the parameters for Γ and $\overline{\Gamma}$ are identical, so there is no ambiguity in the choice of parameters for Γ.

Adjacency algebras. The linear span $\mathcal{A} := \langle A_0, A_1, A_2\rangle$ over $\mathbf{C}$ is closed under ordinary matrix multiplication and entry-wise multiplication. In fact, $\mathcal{A}$ is a commutative, semi-simple associative subalgebra of the $n\times n$ matrices $M_n(\mathbf{C})$. We call $\mathcal{A}$ the *adjacency algebra* or *Bose-Mesner algebra* associated with Γ (see [**3**]). Multiplication in $\mathcal{A}$ is given by the equations below.

$$\begin{aligned} A_1^2 &= kI + \lambda A_1 + \mu A_2 \\ A_1A_2 &= (k-\lambda-1)A_1 + (k-\mu)A_2 \\ A_2^2 &= lI + (n-2k+\lambda)A_1 + (n-2k+\mu-2)A_2. \end{aligned}$$

Eigenvalues. The adjacency matrix A_1 of Γ has eigenvalues k, r, and s with $k \geq r \geq 0 > s$ [**3**]. These eigenvalues determine the parameters k, λ, and μ of Γ, and the graph is primitive if and only if k, r, and s are distinct and nonzero. Since $A_2 = J - I - A_1$, where J is the all ones matrix, the eigenvalues of A_2 are $l = n-k-1, -(s+1), -(r+1)$.

Type. We say an srg Γ has type I if its parameters have the form $(4\mu+1, 2\mu, \mu-1, \mu)$, and type II otherwise. All srg's with noninteger eigenvalues have type I.

Lattice graphs. Given $g-2$ orthogonal Latin squares of order n, the *square lattice graph* $L_g(n)$ has the n^2 cells as vertices, with two adjacent if and only if they are in the same row or column or contain the same letter [**5**].

Partial geometries. A *partial geometry* $pg(r,k,t)$ is a set of points together with a set of lines such that every pair of points lies on at most one line, every line contains k points ($k \geq 2$), every point is on r lines ($r \geq 2$), and given a line L and a point p not on L, there are exactly t lines through p which meet L ($k \geq t \geq 1$). The point graph of $pg(r,k,t)$ has the points as vertices, with two adjacent if and only if they lie on a common line. The point graph of a partial geometry $pg(r,k,t)$ is strongly regular with parameters

$$\left(\frac{k((r-1)(k-1)+t)}{t}, r(k-1), k-2+(r-1)(t-1), rt\right).$$

We say an srg Γ is *geometrizable* if it is the point graph of a partial geometry.

3. Rank 3 SRG's. Many of the known srg's are so-called rank 3 graphs. Suppose G is a finite group acting transitively on a set Ω. The *rank* of the action is the number of orbitals, or the number of orbits for the induced action on $\Omega \times \Omega$. (Equivalently, the number of orbits for the stabilizer of a point.) Form the *orbital graph* associated with a nondiagonal orbital $\mathcal{O}$ by putting vertices x and y adjacent if and only if $(x, y) \in \mathcal{O}$. If G acts rank 3 and has even order, then the orbitals are symmetric and the orbital graphs are complementary srg's [**3**].

Example. Let $V_{2d}(2)$ be a vector space of dimension $2d$ over $GF(2)$, $d \geq 2$. $V_{2d}(2)$ acts transitively (by translation) on the 2^{2d} vectors. The orthogonal group $GO^{\varepsilon}(2)$ acts on nonzero vectors just as it acts on points of the projective geometry $PG_{2d-1}(2)$, fixing the origin. The stabilizer in $V_{2d}(2)\cdot GO^{\varepsilon}(2)$ of the origin has orbits $\{0\}$, the set of points on a quadric, and the set of points off the quadric. Thus, we have a rank three action, and we get two complementary srg's with the points of $V_{2d}(2)$ as vertices.

Let Γ_1 and Γ_2 be two such graphs, for the actions on $V_{2a}(2)$ and $V_{2b}(2)$, respectively. Let $H = V_{2a}(2) \cdot GO^{\varepsilon}(2) \times V_{2b}(2) \cdot GO^{\nu}(2)$. H acts rank 9 on $V_{2(a+b)}(2)$, with orbital graphs given by the nondiagonal relations of the tensor product of Γ_1 and Γ_2. Set $G = V_{2(a+b)}(2) \cdot GO^{\varepsilon\nu}(2)$, $\varepsilon, \nu = \pm 1$. We know G acts rank 3 on $V_{2(a+b)}(2)$. Since $H \leq G$, the orbits under the action of G are unions of orbits for the action of H. Thus, $\Gamma_1 \otimes \Gamma_2$ has a fusion isomorphic with the orbital graph of G. For example, when $\varepsilon = \nu = +1$, take $S = \{1, 3, 5, 7\}$ and $A_S = I \otimes B_1 + A_1 \otimes I + A_1 \otimes B_2 + A_2 \otimes B_1$ is the adjacency matrix of Γ. (See (8)–(10) in the theorem of Section 5.)

4. Intersection matrices. We make use of the regular representation of $\mathcal{A}$,

$$\sigma : A_j \mapsto M_j = (p_{ij}^k)_{0 \leq i,k \leq 2}, \qquad 0 \leq j \leq 2$$

where p_{ij}^k is defined by $A_i A_j = \sum_{k=0}^{2} p_{ij}^k A_k$. In this way we encode the essential information about $\mathcal{A}$ in the set of 3×3 *intersection matrices*

M_j, $0 \le j \le 2$. Observe that M_j has the same eigenvalues as A_j. The intersection matrices commute pairwise because the adjacency matrices do. Thus, we can simultaneously diagonalize M_1 and M_2, which gives us

$$M_1 \sim M_1' = \operatorname{diag}(k, r, s)$$
$$M_2 \sim M_2' = \operatorname{diag}(l, -1-r, -1-s).$$

Given two srg's Γ_1 and Γ_2, we want to decide which fusions of their tensor product give rise to primitive srg's. This amounts to finding all fusions for which C_S has three distinct nonzero eigenvalues. Hence, we work with tensor products of intersection matrices. Let $\mathcal{A} = \langle A_0 = I, A_1, A_2\rangle$ and $\mathcal{B} = \langle B_0 = I, B_1, B_2\rangle$ be the adjacency algebras associated with Γ_1 and Γ_2 respectively. Let $\{M_0' = I, M_1', M_2'\}$ and $\{N_0' = I, N_1', N_2'\}$ be the corresponding triples of diagonalized intersection matrices. Form the tensor products

$$D_{3i+j} = M_i' \otimes N_j', \qquad 0 \le i, j \le 2.$$

The diagonal entries of D_i are the values of the nine linear characters of $\mathcal{A} \otimes \mathcal{B}$ on C_i. The character table for $\mathcal{A} \otimes \mathcal{B}$ is given below. The values on $I = C_0$ are left out.

C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
k_2	l_2	k_1	k_1k_2	k_1l_2	l_1	l_1k_2	l_1l_2
r_2	$-1-r_2$	k_1	k_1r_2	$-k_1(1+r_2)$	l_1	l_1r_2	$-l_1(1+r_2)$
s_2	$-1-s_2$	k_1	k_1s_2	$-k_1(1+s_2)$	l_1	l_1s_2	$-l_1(1+s_2)$
k_2	l_2	r_1	r_1k_2	r_1l_2	$-1-r_1$	$-k_2(1+r_1)$	$-l_2(1+r_1)$
r_2	$-1-r_2$	r_1	r_1r_2	$-r_1(1+r_2)$	$-1-r_1$	$-r_2(1+r_1)$	$(1+r_2)(1+r_1)$
s_2	$-1-s_2$	r_1	r_1s_2	$-r_1(1+s_2)$	$-1-r_1$	$-s_2(1+r_1)$	$(1+s_2)(1+r_1)$
k_2	l_2	s_1	s_1k_2	s_1l_2	$-1-s_1$	$-k_2(1+s_1)$	$-l_2(1+s_1)$
r_2	$-1-r_2$	s_1	s_1r_2	$-s_1(1+r_2)$	$-1-s_1$	$-r_2(1+s_1)$	$(1+r_2)(1+s_1)$
s_2	$-1-s_2$	s_1	s_1s_2	$-s_1(1+s_2)$	$-1-s_1$	$-s_2(1+s_1)$	$(1+s_2)(1+s_1)$.

The problem reduces to finding all $S \subseteq \{1, \dots, 8\}$ for which $D_S = \sum_{\sigma \in S} D_\sigma$ has three distinct nonzero entries. There are 119 cases to check, since $|S| = 2$, 3 or 4. We treat one case in the example below.

Example. Take $S = \{1, 2, 4, 8\}$. Then the diagonal entries of D_S are

$$\begin{aligned} a &= k_2 + l_2 + k_1 k_2 + l_1 l_2 \\ b &= -1 - l_1 + r_2(k_1 - l_1) \\ c &= -1 - l_1 + s_2(k_1 - l_1) \\ d &= k_2 + r_1(k_2 - l_2) \\ e &= r_1 + r_2 + 2r_1 r_2 \\ f &= r_1 + s_2 + 2r_1 s_2 \\ g &= k_2 + s_1(k_2 - l_2) \\ h &= r_2 + s_1 + 2r_2 s_1 \\ i &= s_1 + s_2 + 2s_1 s_2. \end{aligned}$$

Assume there are exactly three distinct entries above. They are $k = a$, $r = e > 0$ and $s = b < 0$. If $h - i = 0$, then $s_1 = -1/2$, so Γ_1 has type I parameters, which means that $k_1 = l_1$. But then $b = -1 - l_1 = h = -1/2$ which is impossible since l_1 is an integer. Hence $h \neq i$. This fact, together with $e - h \neq 0$ imply $i > 0$. Now, from $e - f = i - h$, $e - h = i - f$, we have

$$\begin{aligned} s_1 &= -1 - r_1 \\ s_2 &= -1 - r_2. \end{aligned}$$

Clearly $r = -1 - s$. Suppose that $k_1 \neq l_1$. Then since $b \neq c$, we must have $-b - 1 = c$. But this implies $k_1 = -l_1 - 1 < 0$. Thus $k_1 = l_1$, and a similar argument shows $k_2 = l_2$. Now $r = d = k_2$, $s = -l_1 - 1$, so $-k_2 - 1 = -l_1 - 1$, which implies $k_2 = k_1$, $n_2 = n_1$. We now have $G_1 = G_2 = P_1(t)$, $G = P_1(2t(2t + 1))$ where $t = k_1 + k_1^2$.

In working with intersection matrices results are obtained on the level of parameters. That is, for any two sets of feasible srg parameters, we can find all fusions which yield primitive strongly regular parameters with no knowledge of the existence of the srg's in question. The set $\{D_0, D_S, D_{\overline{S}}\}$ corresponds to the fusion $\{C_0, C_S, C_{\overline{S}}\}$ of the tensor product of two *graphs* when graphs with those parameters are known to exist. Our results are stated in terms of parameters, although examples of graphs with the parameters listed are generally known.

5. Summary of results. In the proposition and theorem below we list all fusions which yield feasible primitive strongly regular parameters. These statements are followed by a set of remarks addressing existence in certain cases. The computations were done case by case, as in the example above. The results were confirmed by computer for $n_i \leq 100$. Only basic number theory was required for the hand calculations. Many cases were trivially ruled out. For instance, the appearance of a zero eigenvalue contradicts the assumption of primitivity.

Let G_i denote the parameter set $(n_i, k_i, \lambda_i, \mu_i)$ with eigenvalues r_i, s_i for $i = 1, 2$. G denotes the parameter set (n, k, λ, μ) for the indicated fusion of $G_1 \otimes G_2$. The following three families of srg parameters occur in the statements of results. (See [**3, 5**] for background.)

I. $P(t) = (4t+1, 2t, t-1, t)$, $t \geq 1$. (The type I parameters.) For existence, $4t+1$ must be a sum of two squares of integers. The Paley graphs, with $4t+1 = q$ an odd prime power are of this type.

II. $P_2(t) = ((2t+1)^2, 2t^2, t(t-1)+1, t(t-1))$, $t \geq 2$. Srg's corresponding to partial geometries $pg(t, 2t+1, t-1)$ are of this type. These are the dual parameters to orthogonal arrays $OA(2t+1, t)$.

III. $P_+(t) = (4t^2, t(2t-1), t(t-1), t(t-1))$, $t \geq 1$ and $P_-(t) = (4t^2, (t-1)(2t+1), (t+1)(t-2), t(t-1))$, $t \geq 2$. Rank 3 actions of $V_{2d}(2) \cdot 0^{\varepsilon}_{2d}(2)$ on $V_{2d}(2)$, $d \geq 2$, afford $P_-(2^{d-1})$, $\varepsilon = -1$, and the complement of $P_+(2^{d-1})$, $(\varepsilon = 1)$ $d \geq 2$.

Proposition. *Let Γ_i be a strongly regular graph $(i = 1, 2)$ such that $n_1 = n_2 = n$ and $2k_i \leq n$. Then the tensor product has a unique strongly regular fusion with $S = \{1, 2, 3, 6\}$, and the associated graph Γ is isomorphic with the lattice graph $L_2(n)$.*

Proof. It is sufficient to quote the result that $L_2(n)$, $n > 4$, is uniquely determined by its parameters [**6**]. We give a direct proof with the hope that it is more enlightening. $S = \{1, 2, 3, 6\}$ implies that Γ has adjacency matrix

$$C_S = I \otimes B_1 + I \otimes B_2 + A_1 \otimes I + A_2 \otimes I.$$

If we take $\{x_1, x_2, \ldots, x_n\}$ as vertices of Γ_1 and $\{y_1, y_2, \ldots, y_n\}$ as vertices of Γ_2, then $\{(x_i, y_i) \mid 1 \leq i, j \leq n\}$ is the vertex set of Γ. Now, for $j \neq k$, $\{y_j, y_k\}$ is an edge in Γ_2 or in its complement. So, from $I \otimes B_1 + I \otimes B_2$, we have (x_i, y_j) and (x_i, y_k) adjacent for all

$i = 1, \dots, n$ and for $j \neq k$. Thus, for each vertex x_i of Γ_i, we have a copy of the complete graph on n vertices. Likewise, from $A_1 \otimes I + A_2 \otimes I$ we have (x_i, y_k) and (x_j, y_k) adjacent for all k and for all $i \neq j$. Hence Γ is $L_2(n)$. $\square$

Theorem. *Let G_i, $i = 1, 2$, be the parameter set of a strongly regular graph with $2k_i \leq n_i$. Then the tensor product admits a strongly regular fusion with $S \neq \{1, 2, 3, 6\}$ if and only if one of the following holds*:

(1) $G_1 = G_2 = L_2(3) = (9, 4, 1, 2)$ *and* $G = (81, 24, 9, 6)$. ($S = \{1, 3, 8\}, \{1, 5, 6\}, \{2, 3, 7\}, \{2, 4, 6\}$).

(2) $G_1 = G_2 = (16, 5, 0, 2)$ (*parameters of the unique Clebsch graph*) *and* $G = (256, 45, 16, 6)$. ($S = \{2, 4, 6\}$).

(3) $G_1 = G_2 = P_+(1) = (4, 1, 0, 0)$ *and* $G = P_+(2) = (16, 6, 2, 2)$. $S = \{1, 2, 3, 7\}$).

(4) $G_1 = G_2 = P_+(1)$ *and* $G = P_-(2) = (16, 5, 0, 2)$. ($S = \{2, 4, 6\}, \{4, 5, 6\}, \{4, 5, 7\}$).

(5) $G_1 = G_2 = P_1(t)$ *and* $G = P_2(2t)$. ($S = \{4, 8\}, \{5, 7\}$).

(6) $G_1 = G_2 = P_1(t)$ *and* $G = P_1(2t(2t+1))$. ($S = \{1, 2, 4, 8\}, \{1, 2, 5, 7\}$).

(7) $G_1 = (2t, 1, 0, 0)$, $G_2 = (25, t-1, t-2, 0)$, *and* $G = L_2(2t)$, $t \geq 2$. ($S = \{1, 3, 5, 6\}$).

(8) $G_i = P_-(t_i)$ *and* $G = P_+(2t_1t_2)$. ($S = \{2, 5, 6, 7\}$).

(9) $G_i = P_+(t_i)$, $t_2 > 1$ *and* $G = P_+(2t_1t_2)$. ($S = \{1, 3, 5, 7\}$).

(10) $G_1 = P_-(t_1)$, $G_2 = P_+(t_2)$ *and* $G = P_-(2t_1t_2)$. ($S = \{2, 3, 5, 7\}$).

(11) $G_1 = P_+(1)$, $G_2 = P_+(t)$, $t \geq 1$ *and* $G = P_-(2t)$. ($S = \{2, 4, 7\}$).

(12) $G_1 = P_-(t)$, $G_2 = P_+(1)$ *and* $G = P(2t, 1) =$ *complement of* $P_+(2t)$, $t \geq 2$. ($S = \{3, 7, 8\}$).

Remarks. (1) In case 1 we have $A_1 = B_1$, $A_2 = B_2$. Let $S_1 = \{1, 3, 8\}$, $S_2 = \{1, 5, 6\}$, $S_3 = \{2, 3, 7\}$, and $S_4 = \{2, 4, 6\}$. We write

$$\begin{aligned}C_{S_1} &= I \otimes A_1 + A_1 \otimes I + A_2 \otimes A_2\\ C_{S_2} &= I \otimes A_1 + A_1 \otimes A_2 + A_2 \otimes I\\ C_{S_3} &= I \otimes A_2 + A_1 \otimes I + A_2 \otimes A_1\\ C_{S_4} &= I \otimes A_2 + A_1 \otimes A_1 + A_2 \otimes I.\end{aligned}$$

Since $L_2(3)$ is isomorphic to its complement, the four fusions are isomorphic. Note that Γ is not $L_3(9)$, since a vertex x_1 of $L_3(9)$ is joined to 24 distinct vertices which, together with x_1, break into three 9-cliques meeting in $\{x_1\}$. (See [**5**].) In particular, if x_2 is adjacent to x_1, then there is exactly one 9-clique containing the edge $\{x_1, x_2\}$. However, it is easy to see that G does not have this property. This case, along with cases 5 and 6, are the only ones involving type I parameters.

(2) Refer to case 2. We claim that Γ is not $L_3(16)$. The Clebsch graph has even subsets of $\{1,2,3,4,5\}$ as vertices, with two adjacent if and only if their symmetric difference has cardinality 4. Vertices of Γ are thus ordered pairs of even cardinality subsets of $\{1,2,3,4,5\}$. Let $x = (\varnothing, \varnothing)$, $y = (\varnothing, 12)$. Let T be the set of 16 vertices adjacent to both x and y. Suppose that Γ is $L_3(16)$. Then x and y are contained in a unique clique of size 16. Let $U \subseteq T$ be the subset of 14 points which together with x and y form a clique. Now suppose $z = (\varnothing, 13) \in U$. Then since z is not adjacent to $(\varnothing, 24)$ or $(\varnothing, 25)$ these two vertices are not in U. Thus $(\varnothing, 14)$, $(\varnothing, 23) \in U$, but $(\varnothing, 14) \not\sim (\varnothing, 23)$, a contradiction. Therefore, $(\varnothing, 13) \notin U$. Next, suppose that $(\varnothing, 14) \in U$. Then $(\varnothing, 23) \notin U$ and $(\varnothing, 25) \notin U$, contradicting $|U| = 14$. Thus $(\varnothing, 14) \notin U$, but now $(\varnothing, 15) \not\sim (\varnothing, 24)$ contradicts the fact that both must be in U. Therefore, x and y are contained in no 16-clique, so Γ is not $L_3(16)$.

(3) Cases 3,4,7,9 (with $G_1 = P_+(1)$), 10 (with $G_2 = P_+(1)$), and 11 are the only fusions involving imprimitive graphs.

(4) Refer to case 5. We have

$$\begin{aligned}C_{\{4,8\}} &= A_1 \otimes B_1 + A_2 \otimes B_2\\ C_{\{5,7\}} &= A_1 \otimes B_2 + A_2 \otimes B_1.\end{aligned}$$

If $G_1 = G_2$ is isomorphic to its complement, then the two fusions are isomorphic. Here G is not necessarily geometrizable. For example, if $G_i = P_1(6) = (25, 12, 5, 6)$, then Γ_i is the point graph of $pg(3, 5, 2)$

but Γ is not the point graph of $pg(12, 25, 11)$. That is, every point x_1 of $pg(12, 25, 11)$ is in 12 lines of size 25, and the 288 points on these lines account for all of the adjacencies to x_1. However, there are two 25-cliques in Γ, containing a given point x_1, which meet in more than one point.

(5) Refer to case 7. Γ_1 is t copies of the complete graph on two vertices, also known as a ladder graph. Γ_2 is two copies of the complete graph on t vertices. The fusion is similar to that of the proposition, with $I \otimes B_2$ replaced by $A_1 \otimes B_2$. These two fusions are isomorphic. Let σ be the automorphism which interchanges (x_1, y) and (x_2, y) for all vertices y in the second t-clique of Γ_2 and for x_1 adjacent to x_2 in Γ_1. Then σ interchanges $I \otimes B_2$ and $A_1 \otimes B_2$.

(6) Parameters in 9 and 10 are primitive except when $G_1 = P_+(1)$.

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University of Michigan, Department of Mathematics, Ann Arbor, Michigan, 48109-1003.

e-mail address: alyssa@math.lsa.umich.edu