ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 24, Number 2, Spring 1994

# A BAIRE CATEGORY THEOREM FOR THE DOMAINS OF ITERATES OF A LINEAR OPERATOR

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ABSTRACT. We show that for a densely-defined linear operator $T$ in a Banach space $X$, with nonempty resolvent set, the intersection of the domains of all its successive iterates must be dense in $X$. In particular, this is true for a self-adjoint operator on a complex Hilbert space. Moreover, if the range of $T$ is dense, then the intersection of the ranges of all its successive iterates must also be dense in $X$. We generalize these results to an extension of Baire's category theorem involving sequences of natural open subsets of the domains of the iterates of $T$. We also describe some examples which show that for closed operators with empty resolvent set or nonclosed operators with nonempty generalized resolvent set, the intersection of the domains of all their successive iterates may not be dense in $X$.

**0. Introduction.** We show (see Theorem 1.3) that for a densely-defined linear operator $T$ in a Banach space $X$, with nonempty resolvent set, the intersection of the domains of all its successive iterates must be dense in $X$. Moreover, if $T$ has dense range, then the intersection of the ranges of all its successive iterates must also be dense in $X$. A more general statement covering both of these facts is proven.

When applied to complex Hilbert space $\mathcal{H}$, we get that for any self-adjoint (or maximally symmetric) linear operator $T$, the intersection of the domains of all its successive iterates must be dense in $\mathcal{H}$.

We remark that the above results ensure that there exists a common dense domain for every polynomial in $T$ over the scalar field of $X$.

In Theorem 1.6 we extend Theorem 1.3 in a direction that simultaneously generalizes Baire's category theorem in $X$. Section 2 provides examples which show that for closed operators with empty resolvent

Received by the editors on October 20, 1991, and in revised form on February 1, 1993.
*Key words and phrases.* Densely-defined linear operator, Banach space, resolvent set, open set, dense set, Hilbert space, self-adjoint operator, closed, closable and nonclosable operators, generalized resolvent set.

set or nonclosed operators with nonempty generalized resolvent set, the conclusion of Theorem 1.3 may fail (see Propositions 2.4 and 2.5).

When $X$ is a Banach space, let $\mathcal{L}(X)$ be the set of all bounded linear operators $A : X \to X$. Let $I$ denote the identity operator on $X$. Let $\mathcal{D}(T)$ be a subspace of $X$. For a linear operator $T : \mathcal{D}(T) \to X$ we define $\rho(T)$, the resolvent set of $T$, to be the set of all scalars $\mu$ such that $T - \mu I$ is a one-to-one mapping of $\mathcal{D}(T)$ onto $X$, and $(T - \mu I)^{-1} \in \mathcal{L}(X)$. This definition follows, for example, Rudin [**13**, 13.26]. Note that $\rho(T)$ is always an open subset of the scalar field and $\lambda \mapsto (T - \lambda I)^{-1} : \rho(T) \to \mathcal{L}(X)$ is analytic. (See, for example, [**5**, Lemma XII.1.3 and **4** Lemma VII.3.2]). Moreover, by the closed graph theorem, when $T$ is a closed operator $\rho(T)$ coincides with the set of all scalars $\mu$ such that $T - \mu I$ maps one-to-one and onto $X$.

Also consider the generalized resolvent set of $T$, $\gamma(T)$, which consists of all scalars $\mu$ such that $T - \mu I$ is one-to-one with dense range $\mathcal{V}_\mu = (T - \mu I)\mathcal{D}(T)$ in $X$, for which $(T - \mu I)^{-1}$ is bounded on $\mathcal{V}_\mu$. If $T$ is a closed operator, then $\rho(T)$ and $\gamma(T)$ coincide. To see this, one must check that $\gamma(T) \subseteq \rho(T)$ when $T$ is closed. We sketch this argument. Fix $\lambda \in \gamma(T)$. We must show that $\mathcal{V}_\lambda = X$. Fix a sequence $(y_n)_{n=1}^\infty$ in $\mathcal{V}_\lambda$ converging in the norm of $X$ to some $z \in X$. Using the boundedness of $(T - \lambda I)^{-1}$, the completeness of $X$ and the fact that $T - \lambda I$ is a closed operator, it readily follows that $z \in \mathcal{V}_\lambda$.

If $T$ is not closed, then $\rho(T)$ is empty (see Section 1). In this case $\gamma(T)$ may be nonempty; e.g., $T = I|_{c_{00}}$ and $X = l_2$, where $l_2 := \{x = (x_n)_{n=1}^\infty$: such that each $x_n$ is a scalar and $||x||_2 := (\sum_{n=1}^\infty |x_n|^2)^{1/2} < \infty\}$ and $c_{00}$ is the subspace of all finitely nonzero sequences in $l_2$. Here $\gamma(T)$ consists of every scalar except 1. There are also examples of densely-defined, nonclosable operators $T$ in Hilbert space with $\gamma(T) = \rho(T) = \varnothing$. Hence, the fact that $\gamma(T) = \rho(T)$ does not imply that $T$ is closed (or even closable). See Reed and Simon [**12**], Problem 1, p. 312] for such an example. Moreover, there are densely-defined, non-closable operators in Hilbert space with $\gamma(T) \neq \varnothing$. Susumu Okada made me aware of the work of Gindler [**7**]. Such an operator $T$ is the inverse of the operator $A$ defined in [**7**] on page 529. An examination of the proof in [**5**] and [**4**] mentioned above reveals that $\gamma(T)$ is also open and $\lambda \mapsto Q_\lambda$ is analytic on $\gamma(T)$; where $Q_\lambda \in \mathcal{L}(X)$ is the extension of $(T - \lambda I)^{-1}$ to all of $X$.

We now establish some more notation. **N** denotes the set of positive integers, **Z** denotes the integers, while **R** and **C** denote the real and complex numbers, respectively. Let $(X, ||\cdot||_X)$ be a Banach space. We will often write $||\cdot||$ instead of $||\cdot||_X$. Let $\mathcal{D}(T)$ be a subspace of $X$ and $T : \mathcal{D}(T) \to X$ be a linear operator. For each $n \in \mathbf{N}$, we define the domain of $T^n$, $\mathcal{D}(T^n)$, by

$$\mathcal{D}(T^n) := \{x \in \mathcal{D}(T) : Tx, T^2x, \dots, T^{n-1}x \in \mathcal{D}(T)\}.$$

$\mathcal{D}(T^0)$ means the domain of $I$, which is $X$. $\Pi(T)$ denotes the set of all operators $\alpha T + \beta I$, where $\alpha$ and $\beta$ are scalars; while $\Sigma(T)$ is the set of all $S \in \Pi(T)$ for which $S(\mathcal{D}(T))$ is dense in $(X, ||\cdot||_X)$.

I thank the referee for pointing out that some of the techniques used herein are similar to those used when discussing hypercyclic vectors. See, for example, [**1**, Section III.5] and [**8**, Section 1]. Also see [**3**] and [**9**] for recent work concerning hypercyclic vectors, related to [**8**].

**1. The intersection of the domains of iterates of a linear operator.** We will use the following result from Bourbaki [**2**, Theorem 2.3.1]. (Also see Esterle [**6**, Corollary 2.2]).

**Theorem 1.1.** *Let $\{(X_n, d_n)\}_{n=0}^{\infty}$ be a sequence of complete metric spaces and $\{T_n\}_{n=1}^{\infty}$ be a sequence of continuous mappings such that each $T_n$ maps $X_n$ into $X_{n-1}$ and each $T_n$ has dense range. Then*

$$\bigcap_{n=1}^{\infty} T_1 T_2 \cdots T_n X_n \quad \text{is dense in } (X_0, d_0).$$

The following simple lemma shows that the operators under consideration are always closed. We include a proof, for completeness.

**Lemma 1.2.** *Let $T$ be a linear operator from a dense domain $\mathcal{D}(T)$ in a Banach space $(X, ||\cdot||_X)$ into $(X, ||\cdot||_X)$. If $\rho(T) \neq \varnothing$, then $T$ is closed.*

*Proof.* Fix $\mu \in \rho(T)$. Then $(T - \mu I)^{-1}$ is a bounded linear operator on $X$. So it is closed. Thus $T - \mu I$ is closed, and hence $T$ is closed. $\square$

We come now to our first result, the proof of which is briefly postponed.

**Theorem 1.3.** *Let $T$ be a linear operator from a dense subspace $\mathcal{D}(T)$ of a Banach space $(X, ||\cdot||_X)$ into $X$. Suppose that the resolvent set $\rho(T)$ of $T$ is nonempty.*

*If $\{S_n\}_{n=1}^{\infty}$ is any sequence of operators in $\Sigma(T)$ then*

$$\bigcap_{n=1}^{\infty} S_1 S_2 \cdots S_n \mathcal{D}(T^n) \quad \text{is dense in } (X, ||\cdot||_X).$$

*In particular,*

$$\bigcap_{n=1}^{\infty} \mathcal{D}(T^n) \quad \text{is dense in } (X, ||\cdot||_X).$$

*Moreover, if $T$ has dense range, then*

$$\bigcap_{n=1}^{\infty} T^n \mathcal{D}(T^n) \quad \text{is dense in } (X, ||\cdot||_X).$$

Let us consider a simple example, to which Theorem 1.3 applies. Let $X =$ complex-valued $L_p(\mathbf{R})$, $1 \le p < \infty$; i.e., $X$ is the Banach space of all (equivalence classes of) Lebesgue-measurable functions $f : \mathbf{R} \to \mathbf{C}$ such that $|f|^p$ is integrable, with the usual norm. Define $\mathbf{x}(t) := t$, for all $t \in \mathbf{R}$ and $\mathcal{D}(T) := \{f \in X : \mathbf{x} \cdot f \in X\}$. Further define $T : \mathcal{D}(T) \to X$ by setting $Tf := \mathbf{x} \cdot f$. It is straightforward to check that $T$ is an unbounded linear operator in $X$ with dense domain and dense range. Moreover, $i \in \rho(T)$.

In the case where $X$ is a Hilbert space, Theorem 1.3 has an immediate corollary.

**Corollary 1.4.** *Let $T$ be a self-adjoint linear operator in a complex Hilbert space $\mathcal{H}$. Then*

$$\bigcap_{n=1}^{\infty} \mathcal{D}(T^n) \quad \text{is dense in } \mathcal{H};$$

*and if $T$ also has dense range then*

$$\bigcap_{n=1}^{\infty} T^n \mathcal{D}(T^n) \quad \textit{is dense in } \mathcal{H}.$$

*Proof of Corollary* 1.4. It is enough to note that $\mathcal{D}(T)$ is dense, by definition, and $i \in \rho(T)$. □

We remark that if $T$ is a closed, densely-defined, maximally symmetric linear operator in a complex Hilbert space $\mathcal{H}$, then the above corollary still holds and every self-adjoint operator is maximally symmetric. On the other hand, a closed, densely-defined, symmetric operator may be maximally symmetric, yet not self-adjoint. (See Rudin [**13**, Chapter 13], for example, for more information about maximally symmetric operators.)

*Proof of Theorem* 1.3. Define $X_0 := X$ and $||\cdot||_0 := ||\cdot||_X$. Let $X_n := \mathcal{D}(T^n)$ for all $n \in \mathbf{N}$. Let $\mu \in \rho(T)$ so that $(T-\mu I)^{-1}$ exists in $\mathcal{L}(X)$. Define $T_\mu := T - \mu I$, and for all $n \in \mathbf{N}$, let

$$||x||_n := ||x|| + ||T_\mu x|| + \cdots + ||T_\mu^n x||, \qquad \text{for all } x \in X_n.$$

Each $(X_n, ||\cdot||_n)$ is a normed linear space. Moreover, as $\rho(T) \neq \varnothing$, $T$ is a closed operator by Lemma 1.2, and consequently, each $(X_n, ||\cdot||_n)$ is a Banach space.

Consider a fixed $n \in \mathbf{N}$. Both $T$ and $I$ map $(\mathcal{D}(T^n), ||\cdot||_n) \to (\mathcal{D}(T^{n-1}), ||\cdot||_{n-1})$. Fix $x \in \mathcal{D}(T^n)$. Then it is easy to see that $||Ix||_{n-1} \leq ||x||_n$, while

$$\begin{aligned} ||Tx||_{n-1} &\leq ||T_\mu x||_{n-1} + |\mu|\,||x||_{n-1} \\ &= ||T_\mu x|| + ||T_\mu(T_\mu x)|| + \cdots + ||T_\mu^{n-1}(T_\mu x)|| + |\mu|\,||x||_{n-1} \\ &\leq ||x||_n + |\mu|\,||x||_n = (1+|\mu|)||x||_n. \end{aligned}$$

So $T$ and $I$ are continuous linear maps. Further, for all scalars $\alpha$ and $\beta$, $S := \alpha T + \beta I$ maps $(\mathcal{D}(T^n), ||\cdot||_n)$ continuously into $(\mathcal{D}(T^{n-1}), ||\cdot||_{n-1})$.

Let us now show that $S(\mathcal{D}(T^n))$ is dense in $(\mathcal{D}(T^{n-1}), ||\cdot||_{n-1})$ when $S(\mathcal{D}(T))$ is dense in $(X, ||\cdot||)$. Fix $y \in \mathcal{D}(T^{n-1})$ and then fix $\varepsilon > 0$. Since $S$ commutes with $T_\mu$, we have that for any $x \in \mathcal{D}(T^n)$,

$$\begin{aligned}
||y - Sx||_{n-1} &= ||y - Sx|| + ||T_\mu(y - Sx)|| + \cdots \\
&\quad + ||T_\mu^{n-2}(y - Sx)|| + ||T_\mu^{n-1}(y - Sx)|| \\
&= ||(T_\mu^{-1})^{n-1}(T_\mu^{n-1}y - ST_\mu^{n-1}x)|| \\
&\quad + ||(T_\mu^{-1})^{n-2}(T_\mu^{n-1}y - ST_\mu^{n-1}x)|| \\
&\quad + \cdots + ||T_\mu^{-1}(T_\mu^{n-1}y - ST_\mu^{n-1}x)|| \\
&\quad + ||T_\mu^{n-1}y - ST_\mu^{n-1}x||, \\
&\leq (||T_\mu^{-1}||^{n-1} + ||T_\mu^{-1}||^{n-2} + \cdots + ||T_\mu^{-1}|| + 1) \\
&\quad \cdot ||T_\mu^{n-1}y - ST_\mu^{n-1}x|| \\
&= K_n ||T_\mu^{n-1}y - ST_\mu^{n-1}x||,
\end{aligned}$$

where $K_n := 1 + ||T_\mu^{-1}|| + \cdots + ||T_\mu^{-1}||^{n-1}$ is a finite constant. Now consider $T_\mu^{n-1}y \in X$. $S(\mathcal{D}(T))$ is dense in $X$. So there exists $z \in \mathcal{D}(T)$ such that $||T_\mu^{n-1}y - Sz|| < \varepsilon/K_n$. But $T_\mu^{-1}$ maps $X$ onto $\mathcal{D}(T)$, so if we define $x_0 = (T_\mu^{-1})^{n-1}z$, then $x_0$ belongs to $\mathcal{D}(T^n)$. Moreover,

$$\begin{aligned}
||y - Sx_0||_{n-1} &\leq K_n ||T_\mu^{n-1}y - ST_\mu^{n-1}x_0|| \\
&= K_n ||T_\mu^{n-1}y - Sz|| < K_n \cdot \frac{\varepsilon}{K_n} = \varepsilon.
\end{aligned}$$

Hence, if $\{S_n\}_{n=1}^\infty$ is any sequence of operators in $\Sigma(T)$, then each $S_n$ maps $(\mathcal{D}(T^n), ||\cdot||_n)$ continuously and densely into $(\mathcal{D}(T^{n-1}), ||\cdot||_{n-1})$. The result now follows by applying Theorem 1.1. □

In [**10**, Theorem 1.2] we proved a generalization of Theorem 1.1 above. The referee provided us with a more succinct proof that we realized allows for a slightly broader generalization of Theorem 1.1, which is Theorem 1.5 below (Theorems 1.5 and 1.1 are, in fact, equivalent because the proof of Theorem 1.5 uses Theorem 1.1). Theorem 1.5 includes Baire's category theorem for complete metric spaces and also a result of Beauzamy [**1**, Proposition 1.B.1]. We will use Theorem 1.5 to derive a generalization of Baire's category theorem in our setting of domains of iterates of a linear operator in a Banach space (Theorem 1.6).

**Theorem 1.5.** *Let $\{(X_n, d_n)\}_{n=0}^{\infty}$ be a sequence of complete metric spaces and $\{T_n\}_{n=1}^{\infty}$ a sequence of continuous mappings such that each $T_n$ maps $X_n$ into $X_{n-1}$. Let $\{\Theta_n\}_{n=0}^{\infty}$ be a sequence of sets such that $\Theta_0$ is a dense open set in $(X_0, d_0)$, while for each $n \geq 1$, $\Theta_n$ is an open set in $(X_n, d_n)$ and $T_n\Theta_n$ is dense in $(X_{n-1}, d_{n-1})$. Then the set $\Gamma$ is dense in $(X_0, d_0)$, where*

$$\Gamma := \Theta_0 \cap \bigcap_{n=1}^{\infty} T_1 T_2 \cdots T_n \Theta_n.$$

*Proof of Theorem* 1.5. Define $A_0 := \Theta_0$ and, inductively, define $A_n := \Theta_n \cap T_n^{-1}(A_{n-1})$ for all $n \in \mathbf{N}$. For every $n \in \mathbf{N}$, $A_n$ is an open set in $(X_n, d_n)$ and $T_n(A_n) \subseteq A_{n-1}$; while $A_0$ is dense in $(X_0, d_0)$. Moreover, the closure of $T_n(A_n)$ coincides with the closure of $A_{n-1}$ in $(X_{n-1}, d_{n-1})$ for all $n \in \mathbf{N}$. For each $n \geq 0$, we can endow $A_n$ with a topologically equivalent metric $q_n$, such that $(A_n, q_n)$ is complete. Further, it is easy to see that each $T_n$ is continuous from $(A_n, q_n)$ to $(A_{n-1}, q_{n-1})$ and that the range of $T_n$, restricted to $A_n$, is dense in $(A_{n-1}, q_{n-1})$. By Theorem 1.1,

$$\Delta := \bigcap_{n=1}^{\infty} T_1 \cdots T_n A_n$$

is dense in $(A_0, q_0)$. So $\Delta$ is dense in $(\Theta_0, d_0)$ and hence also dense in $(X_0, d_0)$. But $A_n \subseteq \Theta_n$ for all $n \in \mathbf{N}$. So $\Gamma$, as defined above, contains $\Delta$; and this completes the proof. □

Consider now a Banach space $(X, ||\cdot||_X)$ with a dense subspace $\mathcal{D}(T)$ and a linear operator $T : \mathcal{D}(T) \to X$ for which $\rho(T)$ is nonempty. Let $\mu \in \rho(T)$. It is simple to check that $T_\mu = T - \mu I$ is a Banach space isomorphism of $(\mathcal{D}(T), ||\cdot||_\mu)$ onto $(X, ||\cdot||_X)$, where

$$||x||_\mu := ||x||_X + ||T_\mu x||_X, \qquad \text{for all } x \in \mathcal{D}(T). \tag{1}$$

Thus $\Theta$ is an open set in $(\mathcal{D}(T), ||\cdot||_\mu)$ if and only if $U = T_\mu\Theta$ is an open set in $(X, ||\cdot||_X)$. Now all the norms defined on $\mathcal{D}(T)$ as in (1) above by members $\mu$ and $\nu$ of $\rho(T)$ are equivalent. So, for all $\mu, \nu \in \rho(T)$, $\Theta$

is an open set in $(\mathcal{D}(T), ||\cdot||_\mu)$ if and only if $U = T_\nu\Theta$ is an open set in $(X, ||\cdot||_X)$.

Moreover, we may replace the phrase 'an open set' everywhere in the previous paragraph by 'a dense set' everywhere, and all the statements remain true. Iterating the above arguments, we see that all the Banach spaces $\mathcal{D}(T^n)$, with a norm $||\cdot||_n$ defined as in the proof of Theorem 1.3, are isomorphic to $(X, ||\cdot||_X)$ and each other, under the obvious mappings. Fix $n \in \mathbf{N}$. Let $\mu_1, \dots, \mu_n \in \rho(T)$, and $U \subseteq X$ and $\Theta \subseteq \mathcal{D}(T^n)$ be related by

$$\Theta := (T-\mu_1)^{-1}(T-\mu_2)^{-1}\cdots(T-\mu_n)^{-1}U.$$

Then $\Theta$ is an open set in $(\mathcal{D}(T^n), ||\cdot||_n)$ if and only if $U$ is an open set in $(X, ||\cdot||_X)$. Further, for any $S \in \Pi(T)$, and $U \subseteq \mathcal{D}(T)$ and $\Theta \subseteq \mathcal{D}(T^{n+1})$ related as above, $S(U)$ is dense in $(X, ||\cdot||)_X$ if and only if $S(\Theta)$ is dense in $(\mathcal{D}(T^n), ||\cdot||_n)$.

The above discussion pertains to our second result below, which follows readily from Theorem 1.5 and the proof of Theorem 1.3. We therefore omit the proof.

**Theorem 1.6.** *Let $T$ be a linear operator from a dense subspace $\mathcal{D}(T)$ of a Banach space $(X, ||\cdot||_X)$ into $X$. Suppose that the resolvent set $\rho(T)$ of $T$ is nonempty. Let $\{S_n\}_{n=1}^\infty$ be a sequence of operators in $\Pi(T)$. Further suppose that $\{\Theta_n\}_{n=0}^\infty$ is a sequence of sets such that $\Theta_0$ is a dense open set in $(X, ||\cdot||_X)$, and for all $n \in \mathbf{N}$, $\Theta_n$ is an open set in $(\mathcal{D}(T^n), ||\cdot||_n)$ while $S_n(\Theta_n)$ is dense in $(\mathcal{D}(T^{n-1}), ||\cdot||_{n-1})$. Then*

$$\Theta_0 \cap \bigcap_{n=1}^\infty S_1S_2\cdots S_n\Theta_n \quad \textit{is dense in } (X, ||\cdot||_X).$$

*In particular, if $\Theta_n$ is dense in $(\mathcal{D}(T^{n-1}), ||\cdot||_{n-1})$ for all $n \geq 1$, then*

$$\Theta_0 \cap \bigcap_{n=1}^\infty \Theta_n \quad \textit{is dense in } (X, ||\cdot||_X).$$

**2. Some examples.** Fix an infinite matrix $A = (a_{ij})_{i,j=1}^\infty$, where each $a_{ij}$ is a scalar. As usual, $i$ is the row index and $j$ is the column index. Let $B = (b_{ij})_{i,j=1}^{\infty} = (\bar{a}_{ji})_{i,j=1}^{\infty}$. We define $\mathcal{D}_A$ to be the set of all $x = (x_n)_{n=1}^{\infty}$ in $l_2$ for which each $\sum_{j=1}^{\infty} a_{ij}x_j$ converges in the scalar field to some $\varphi_i^A(x)$ and $(\varphi_i^A(x))_{i=1}^{\infty} \in l_2$. $\mathcal{D}_B$ is similarly defined. Let

$$T_A : \mathcal{D}_A \to l_2 : x \mapsto (\varphi_i^A(x))_{i=1}^{\infty}$$

and define $T_B : \mathcal{D}_B \to l_2$ similarly. Then $T_A$ and $T_B$ are linear operators in $l_2$. Our discussion of these operators follows, for example, Weidmann [**15**, Section 6.3]. Let us emphasize the following simple fact.

**Note 2.1.** The following are equivalent

(1) $c_{00} \subseteq \mathcal{D}_A$.

(2) $(a_{ij})_{i=1}^{\infty} = (\varphi_i^A(e_j))_{i=1}^{\infty} \in l_2$ for every $j \in \mathbf{N}$; where $e_j$ is the $j$-th unit vector in $l_2$.

Next, let $\mathcal{E}_A :=$ the domain of $T_A^*$, which is the set of all $z \in l_2$ with $\psi_z^A : x \mapsto \langle T_A x, z\rangle : \mathcal{D}_A \to \mathbf{K}$ bounded. Here $\langle \cdot, \cdot \rangle$ is the usual inner product on $l_2$. $\mathcal{E}_B$ is similarly defined.

The following two propositions are easy to check, and the second follows from the first (see, for example, Stone [**14**, Theorem 3.2] or Weidmann [**15**, Theorem 6.20]). We use the notation established above.

**Proposition 2.2.** *Suppose that each row $(a_{ij})_{j=1}^{\infty}$ of $A$ is in $l_2$. Then $\mathcal{D}_A = \mathcal{E}_B$, $T_A = T_B^*$ and $\mathcal{D}_B$ is dense in $l_2$. Indeed, $c_{00} \subseteq \mathcal{D}_B$.*

**Proposition 2.3.** *Suppose that each row $(a_{ij})_{j=1}^{\infty}$ of $A$ and each column $(a_{ij})_{i=1}^{\infty}$ of $A$ belong to $l_2$. Then $\mathcal{D}_A$ and $\mathcal{D}_B$ both contain $c_{00}$, $T_A = T_B^*$ and $T_B = T_A^*$. In particular, both $T_A$ and $T_B$ are densely-defined and closed.*

We can now state and demonstrate the following result, which shows that Theorem 1.3 may fail for a closed operator $T$ (which necessarily has $\rho(T) = \varnothing$).

**Proposition 2.4.** *There exists a closed, densely-defined linear operator $T$ in $l_2$ such that $\mathcal{D}(T^2) = \{x \in \mathcal{D}(T) : Tx \in \mathcal{D}(T)\}$ is not dense in $l_2$.*

*Proof.* Note that $l_2$ is isometrically isomorphic to $l_2 \oplus l_2$ under the correspondence

$$(x_1, y_1, x_2, y_2, \dots) \leftrightarrow ((x_1, x_2, \dots), (y_1, y_2, \dots)).$$

Define $P = (p_{ij})_{i,j=1}^{\infty}$ and $Q = (q_{ij})_{i,j=1}^{\infty}$ by

$$p_{ij} := \frac{4^{i-1}}{2^j}, \qquad \text{if } j \geq i \text{ and } p_{ij} := 0,$$

if $j < i$; and

$$q_{ij} := \frac{1}{j2^i}, \qquad \text{for all } i, j \in \mathbf{N}.$$

We introduce $T : \mathcal{D}_Q \oplus \mathcal{D}_P \to l_2 \oplus l_2$ by setting

$$T(x, y) := (T_P y, T_Q x).$$

Clearly, by Proposition 2.3, $\mathcal{D}(T) := \mathcal{D}_Q \oplus \mathcal{D}_P$ is dense in $l_2 \oplus l_2$ and $T$ is a closed operator.

$$\begin{aligned}\mathcal{D}(T^2) &= \{(x, y) \in \mathcal{D}_Q \oplus \mathcal{D}_P : (T_P y, T_Q x) \in \mathcal{D}_Q \oplus \mathcal{D}_P\} \\ &\subseteq \{x \in \mathcal{D}_Q : T_Q x \in \mathcal{D}_P\} \oplus l_2.\end{aligned}$$

Now $\mathcal{D}_Q = \{x \in l_2 :$ each $y_i := (\sum_{j=1}^{\infty} x_j/j)1/2^i$ converges and $y = (y_i)_{i=1}^{\infty} \in l_2\}$. So $\mathcal{D}_Q = l_2$. Thus,

$$\begin{aligned}&\{x \in \mathcal{D}_Q : T_Q x \in \mathcal{D}_P\} \\ &\qquad = \left\{x \in l_2 : y = \left(\sum_{n=1}^{\infty} x_n \frac{1}{n}\right)\left(\frac{1}{2^j}\right)_{j=1}^{\infty} \in \mathcal{D}_P\right\} \\ &\qquad = \left\{x \in l_2 : \text{ each } z_i := \left(\sum_{n=1}^{\infty} x_n \frac{1}{n}\right) \sum_{j=i}^{\infty} \frac{4^{i-1}}{4^j}\right. \\ &\qquad\qquad\qquad \left. \text{converges and } z = (z_i)_{i=1}^{\infty} \in l_2\right\} \\ &\qquad = \left\{x \in l_2 : z = \left(\sum_{n=1}^{\infty} x_n \frac{1}{n}\right)\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3}, \dots\right) \in l_2\right\} \\ &\qquad = \left\{x \in l_2 : \sum_{n=1}^{\infty} x_n \frac{1}{n} = 0\right\}.\end{aligned}$$

Thus $\mathcal{D}(T^2) \subseteq \{x \in l_2 : \sum_{n=1}^{\infty} x_n/n = 0\} \oplus l_2$, which is the hyperplane $\{((1/n)_{n=1}^{\infty}, 0)\}^{\perp}$ in $l_2 \oplus l_2$. So $\mathcal{D}(T^2)$ is not dense in $l_2 \oplus l_2$. $\square$

We remark that the above example shows that the hypothesis '$\rho(T)$ is nonempty' in Theorem 1.3 cannot be weakened to '$T$ is closed'. On the other hand, there do exist closed operators $T$ in a Hilbert space with $\rho(T) = \varnothing$ and yet $\cap_{n=1}^{\infty}\mathcal{D}(T^n)$ is dense. Indeed, following Rudin [**13**, Example 13.4 and Chapter 13 Exercise 19] define $T_1, T_2$ and $T_3$ in $X =$ complex-valued $L_p[0,1]$, $1 \le p < \infty$, in the following way. Let $AC$ (respectively, $C^{\infty}$) denote the set of all absolutely continuous (respectively, infinitely differentiable) functions from $[0,1]$ into the complex numbers. Define $\mathcal{D}_1 := \{f \in AC : f' \in L_p\}$, $\mathcal{D}_2 := \{f \in \mathcal{D}_1 : f(0) = f(1)\}$ and $\mathcal{D}_3 := \{f \in \mathcal{D}_1 : f(0) = f(1) = 0\}$. We set $T_j f := if'$ for all $f \in \mathcal{D}_j$, $j = 1, 2, 3$.

Straightforward calculations show that each $T_j : \mathcal{D}_j \to X$ is closed and densely-defined. Moreover, $\rho(T_1) = \rho(T_3) = \varnothing$, while $\rho(T_2) = \mathbf{C}\backslash\{2\pi k : k \in \mathbf{Z}\}$. Thus $\cap_{n=1}^{\infty}\mathcal{D}(T_2^n)$ is dense in $X$, by Theorem 1.3. Of course, $\mathcal{D}_2 \subseteq \mathcal{D}_1$ and $T_1|_{\mathcal{D}_2} = T_2$ so it follows that $\cap_{n=1}^{\infty}\mathcal{D}(T_1^n)$ is dense in $X$ also. Moreover,

$$\bigcap_{n=1}^{\infty} \mathcal{D}(T_3^n) = \{f \in C^{\infty} : f^{(n)}(0) = f^{(n)}(1) = 0 \text{ for all } n \ge 0\}.$$

Consequently, $\cap_{n=1}^{\infty}\mathcal{D}(T_3^n)$ is also dense in $X = L_p[0,1]$, $1 \le p < \infty$.

We next ask whether the hypothesis of Theorem 1.3 that '$\rho(T)$ is nonempty' can be weakened to 'the generalized resolvent set $\gamma(T)$ is nonempty'? Recall that a linear operator $T$ in $X$ is closable if the closure of its graph in $X \times X$ is also a graph. (See, for example, [**15**, Chapter 5] for more information on closable operators in Hilbert space). If $T$ is closable, with closure $S$, then $\rho(S) = \gamma(T)$. So, if $\gamma(T)$ is nonempty, then $\cap_{n=1}^{\infty}\mathcal{D}(S^n)$ is dense in $X$. But $\cap_{n=1}^{\infty}\mathcal{D}(T^n)$ need not be dense, as the following example shows.

**Proposition 2.5.** *There exists a densely-defined, linear operator $T$ in $l_2$ that is bounded on its domain and $\mathcal{D}(T^2)$ is not dense in $l_2$.*

*Proof.* Fix $\alpha$ and $\beta$ in $l_2$. Define the matrices $A = (a_{ij})_{i,j=1}^{\infty}$ and $B = (b_{ij})_{i,j=1}^{\infty}$ by $a_{ij} := \alpha_i$ and $b_{ij} := \beta_i/j$. Define $D := \{x \in l_2 :$

$\sum_{n=1}^{\infty} x_n = 0\}$. It is straightforward to check that $D$ is a dense subspace of $l_2$. The operator $T_A$ is identically zero when restricted from $\mathcal{D}_A$ to $D$, while $T_B$ is a bounded, linear operator on $\mathcal{D}_B = l_2$. Let us introduce the linear operator $T : D \oplus l_2 \to l_2 \oplus l_2$ by $T(x,y) := (T_B y, T_A x)$.

$$\begin{aligned}\mathcal{D}(T^2) &= \{(x,y) \in D \oplus l_2 : (T_B y, T_A x) \in D \oplus l_2\} \\ &= D \oplus \left\{ y \in l_2 : T_B y = \left( \sum_{n=1}^{\infty} \frac{y_n}{n} \right) (\beta_j)_{j=1}^{\infty} \in D \right\}.\end{aligned}$$

Now choose $\beta \in l_2$ by setting $\beta_j := j^{-2}$ for each $j \in \mathbf{N}$. Then $\sum_{j=1}^{\infty} \beta_j$ converges in $\mathbf{R}$ to $L = \pi^2/6$. So,

$$\mathcal{D}(T^2) = D \oplus \{y \in l_2 : \sum_{n=1}^{\infty} \frac{y_n}{n} \cdot L = 0\} = D \oplus H,$$

where $H := \{y \in l_2 : \sum_{j=1}^{\infty} y_j/j = 0\}$ is a hyperplane in $l_2$. Thus, $\mathcal{D}(T^2)$ is not dense in $l_2$. $\square$

In passing, we note that the example, from [**7**] discussed in the introduction, of a densely-defined, nonclosable operator $T$ with $\gamma(T) \neq \varnothing$, satisfies $\cap_{n=1}^{\infty} \mathcal{D}(T^n) = \mathcal{D}(T)$, since the range of $T$ is contained in $\mathcal{D}(T)$. We remark that the phenomenon of 'range $(T) \subseteq \mathcal{D}(T)$' for closed operators $T$ has been studied by Ôta [**11**].

**Acknowledgment.** I thank Alan McIntosh, Bernard Beavzamy, Richard Aron and Susumu Okada for helpful suggestions and communications; and Catherine for typing the manuscript. I am grateful for the support of a University Fellowship from Kent State University and a University of Pittsburgh Internal Research Grant.

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