ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 24, Number 1, Winter 1994

TRAVELLING WAVE SOLUTIONS OF REACTION-DIFFUSION MODELS WITH DENSITY-DEPENDENT DIFFUSION

WALTER KELLEY

Dedicated to Paul Waltman on the occasion of his 60th birthday

1. Introduction. In a recent paper [**3**], we gave a technique for approximating travelling wave solutions $u(x-ct)$ of reaction-diffusion equations for large wave speed c. The approximation is asymptotic in the sense that it converges uniformly to the exact solution on the set of real numbers as $c \to \infty$.

Below we extend this technique to the case where the diffusion term is a function of u. Such equations are frequently used in mathematical biology to model dispersal of an animal population when there is increased diffusion due to population pressure (see [**1, 4, 5**]).

We will also show how to construct and verify "higher order" approximations. These approximations will improve the earlier ones when c is sufficiently large.

Specifically, consider the equation

$$u_t = D(u)_{xx} - f(u), \tag{1}$$

where D and f are smooth functions satisfying $D(0) = 0$, $D(u) > 0$, $D'(u) > 0$, $f(0) = f(1) = 0$, and $f(u) < 0$ for $0 < u < 1$. We seek travelling wave solutions of the form $u(z) = u(x-ct)$ and find from (1) that u satisfies

$$-c\frac{du}{dz} = \frac{d^2}{dz^2}D(u) - f(u).$$

Now let $\varepsilon = c^{-2}$, $w = \varepsilon^{1/2}z$. Then equation (1) transforms to

$$\varepsilon\frac{d^2}{dw^2}D(u) + \frac{du}{dw} - f(u) = 0. \tag{2}$$

Received by the editors on March 3, 1993.

The appropriate boundary conditions are

$$u(-\infty) = 1, \qquad u(0) = 1/2, \qquad u(\infty) = 0. \tag{3}$$

2. The basic approximation method. We propose to construct a trapping region for solutions of (2). First, write (2) as a first order system:

$$u' = v, \qquad v' = -\frac{D''}{D'}v^2 + \frac{f(u) - v}{\varepsilon D'}. \tag{4}$$

The trapping region will have the form

$$T \equiv \{(u, v) : f(u)(1 + \varepsilon c) < v < f(u)(1 + \varepsilon d), 0 < u < 1\},$$

where $c \geq d$ are constants to be determined. At the upper boundary $v = f(u)(1 + \varepsilon d)$, we require for $0 < u < 1$

$$0 < \left[v - \frac{D''}{D'}v^2 + \frac{f(u) - v}{\varepsilon D'} \right] \cdot \begin{bmatrix} (1 + \varepsilon d)f' \\ -1 \end{bmatrix},$$

or equivalently (since $f < 0$)

$$0 > (1 + \varepsilon d)^2 f' + \frac{D''}{D'}(1 + \varepsilon d)^2 f + \frac{d}{D'}.$$

The last inequality can also be written as

$$\frac{d}{(1 + \varepsilon d)^2} < -(D'f)'.$$

Thus it suffices to make

$$\frac{d}{(1 + \varepsilon d)^2} < \min\{-(D'f)'(u), 0 < u < 1\} \equiv m. \tag{5}$$

Note that (5) can be satisfied for any $\varepsilon > 0$ by choosing d somewhat larger than $-1/\varepsilon$.

Similarly, at the lower boundary $v = f(u)(1 + \varepsilon c)$ we require

$$\frac{c}{(1 + \varepsilon c)^2} > \max\{-(D'f)'(u), 0 < u < 1\} \equiv M. \tag{6}$$

Since
$$\max \frac{c}{(1+\varepsilon c)^2} = \frac{1}{4\varepsilon}$$
(at $c = \varepsilon^{-1}$), a necessary condition for (6) is $\varepsilon < 1/(4M)$.

Theorem 1. *Assume $f(0) = f(1) = 0$, $D(0) = 0$, and for $0 < u < 1$, $f(u) < 0$, $D(u) > 0$, $D'(u) > 0$. Let $d(\varepsilon) \in (\varepsilon^{-1}, 0]$ satisfy*
$$\frac{d}{(1+\varepsilon d)^2} \le m.$$
Assume $0 < \varepsilon \le 1/(4M)$. Then there is a nonnegative constant $c(\varepsilon)$ satisfying
$$\frac{c}{(1+\varepsilon c)^2} \ge M,$$
and there is a solution $u(w,\varepsilon)$ of (2) and (3) so that
$$f(u)(1+\varepsilon c) \le du/dw \le f(u)(1+\varepsilon d) \tag{7}$$
for $0 \le u \le 1$.

Proof. Suppose first that the strict inequalities (5) and (6) are true. Then we have constructed a trapping region T for solutions of (4). Let $u_0 \in (0,1)$ and define $S = T \cap \{(u,v) : u = u_0\}$. Each trajectory that intersects S must remain in T and go to the origin as $w \to \infty$. (Note that this is true even if (4) is singular at the origin.)

In reverse time, these solutions must either exit through the boundary of T or go to the equilibrium point at $u = 1$, $v = 0$ as $w \to -\infty$. By the Wazewski retract method (see [**2**]), some solution must remain in T and approach $(1,0)$ as $w \to -\infty$. The proof is completed by a standard limiting argument. □

Note that inequality (7) can be integrated to obtain uniformly valid upper and lower bounds on the travelling wave solution. Also, we can compute optimal values for c and d by solving the quadratic equations corresponding to inequalities (5) and (6):
$$\begin{aligned} d(\varepsilon) &= \frac{1 - 2m\varepsilon - \sqrt{1-4m\varepsilon}}{2\varepsilon^2 m} \\ &= m - 2\varepsilon m^2 + \mathcal{O}(\varepsilon^2), \qquad \varepsilon \to 0, \end{aligned} \tag{8}$$

$$c(\varepsilon) = \frac{1 - 2M\varepsilon - \sqrt{1 - 4M\varepsilon}}{2\varepsilon^2 M} \tag{9}$$
$$= M - 2\varepsilon M^2 + \mathcal{O}(\varepsilon^2), \qquad \varepsilon \to 0.$$

Example. Consider the equation

$$u_t = (u^p)_{xx} + u(1-u).$$

Here $D(u) = u^p$ $(p \geq 1)$ and $f(u) = u(u-1)$. Thus the travelling wave solution $u(w, \varepsilon)$ satisfies

$$\varepsilon \frac{d^2}{dw^2} u^p + \frac{du}{dw} + u(1-u) = 0,$$
$$u(-\infty) = 1, \quad u(0) = .5, \quad u(\infty) = 0.$$

Now

$$-(D'f)'(u) = pu^{p-1}[p - (p+1)u]$$

and

$$m = -p, \quad M = \begin{cases} 1 & \text{if } p = 1 \\ p(\frac{p-1}{p+1})^{p-1} & \text{if } p > 1. \end{cases}$$

By Theorem 1, for $0 < \varepsilon \leq 1/(4M)$

$$u(u-1)(1 + \varepsilon c) \leq du/dw \leq u(u-1)(1 + \varepsilon d)$$

on the interval $0 \leq u \leq 1$, where d and c are given by (8) and (9), respectively. Integrating the inequality, we have

$$\frac{1}{1 + e^{(1+\varepsilon c)w}} \leq u(w, \varepsilon) \leq \frac{1}{1 + e^{(1+\varepsilon d)w}}$$

if $w \geq 0$ and

$$\frac{1}{1 + e^{(1+\varepsilon d)w}} \leq u(w, \varepsilon) \leq \frac{1}{1 + e^{(1+\varepsilon c)w}}$$

if $w \leq 0$.

Our assumption $\varepsilon \leq 1/(4M)$ is not necessarily the best possible condition for existence of the travelling wave. See [**6**] for a discussion of existence in case $D(u) = u^p$ and $f(u) = u^n(u-1)$.

3. Higher order approximations. We will now obtain a more accurate approximation of the solution of (2) and (3) when ε is sufficiently small. Inequality (7) suggests seeking a solution of the form

$$u' = f(u)(1 + \varepsilon C(u, \varepsilon)),$$

where $C(u, \varepsilon)$ is to be determined. If we substitute this form into (2), we find the following equation for C:

$$C + D'f' + D''f + \varepsilon(2D'f'C + D'fC' + 2D''fC) \\ + \varepsilon^2(D'f'C^2 + D''fC^2 + D'fC'C) = 0.$$

This suggests that

$$C(u, \varepsilon) = -(D'f)'(u) + \mathcal{O}(\varepsilon), \qquad \varepsilon \to 0$$

(compare with (5) and (6)). Thus we define

$$g(u) = -(D'f)'(u) \tag{10}$$

and ask whether

$$R \equiv \{(u, v) : f(u)(1+\varepsilon g(u)+a\varepsilon^2) < v < f(u)(1+\varepsilon g(u)+b\varepsilon^2), 0 < u < 1\}$$

is a trapping region for bounded functions $a(\varepsilon) \geq b(\varepsilon)$ and small $\varepsilon > 0$. At the upper boundary, we need the inequality

$$0 \geq f'[1 + \varepsilon g + \varepsilon^2 b]^2 + \varepsilon f g'[1 + \varepsilon g + \varepsilon^2 b] \\ + \frac{D''}{D'} f[1 + \varepsilon g + \varepsilon^2 b]^2 + \frac{\varepsilon b + g}{D'},$$

which simplifies (using (10)) to

$$0 \geq -g[2g + (2b + g^2)\varepsilon + 2gb\varepsilon^2 + b^2\varepsilon^3] + fg'D'(1 + \varepsilon g + \varepsilon^2 b) + b.$$

Consequently, for ε sufficiently small it suffices to require

$$b < \min\{(2g^2 - fg'D')(u) : 0 < u < 1\}. \tag{11}$$

Similarly, we require

$$a > \max\{(2g^2 - fg'D')(u) : 0 < u < 1\}. \tag{12}$$

Now the proof of the following theorem is similar to that of Theorem 1.

Theorem 2. *If ε is sufficiently small, f and D satisfy the hypotheses of Theorem* 1, *and b and a satisfy* (11) *and* (12), *respectively, then the solution $u(w,\varepsilon)$ of* (2) *and* (3) *satisfies*

$$f(u)(1+\varepsilon g(u)+a\varepsilon^2) < du/dw < f(u)(1+\varepsilon g(u)+b\varepsilon^2)$$

for $0 < u < 1$, where $g(u)$ is given by (10).

Theorem 2 provides an estimate for the solution trajectory in the phase plane. If an approximation of the solution as a function of w is desired, we define

$$F(u) = \int_{.5}^{u} \frac{ds}{f(s)(1+\varepsilon g(s))}$$

for $0 < u < 1$ and note that F is invertible for small ε. Then

$$F^{-1}\bigg(w + \frac{|a|w\varepsilon^2}{1+M\varepsilon}\bigg) < u(w,\varepsilon) < F^{-1}\bigg(w + \frac{|b|w\varepsilon^2}{1+m\varepsilon}\bigg)$$

for $w \geq 0$, while the inequalities are reverse for $w \leq 0$.

REFERENCES

1. D.G. Aronson, *Density-dependent interaction-diffusion systems*, Proc. Adv. Seminar on Dynamics and Modeling of Reactive Systems, Academic Press, New York, 1980.

2. P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964.

3. W. Kelley, *Travelling wave solutions of reaction-diffusion equations with convection*, Recent Trends in Ordinary Differential Equations, World Scientific Series in Applicable Analysis, Vol. 1, 1991.

4. J.D. Murray, *Mathematical biology*, Biomathematics Texts **19**, Springer, New York, 1989.

5. A. Okubo, *Diffusion and ecological problems*: *Mathematical models*, Springer, New York, 1980.

6. A. DePablo and J.L. Vazquez, *Travelling waves and finite propagation in a reaction-diffusion equation*, J. Differential Equations **93** (1991), 19–61.

Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73019