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A THEOREM OF MILLOUX FOR DIFFERENCE EQUATIONS

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Dedicated to Paul Waltman on the occasion of his 60th birthday

We consider the vector difference equation

$$y(t+1) = A(t)y(t) \tag{1a}$$

where $A(t)$ is a complex $n \times n$ matrix function defined on the integer interval $[t_0, \infty) \equiv \{t_0, t_0 + 1, t_0 + 2, \dots\}$. We also will consider the difference operator analog of equation (1a)

$$\Delta y(t) = B(t)y(t) \tag{1b}$$

where $B(t)$ is a complex $n \times n$ matrix function defined on the integer interval $[t_0, \infty)$. An asterisk will denote the conjugate transpose and Δ will denote the forward difference operator, that is, $\Delta y(t) = y(t+1) - y(t)$.

The main theorem will give necessary and sufficient conditions for the existence of a nontrivial solution of equation (1a) which tends to zero. The theorem is a generalization of the trivial scalar case that all solutions of $u(t+1) = a(t)u(t)$ tend to zero provided the limit of $\prod a(t)$ is zero. The related result using equation (1b) is a discrete analog of a theorem of Hartman [**1, 2**] which gave necessary and sufficient conditions for the existence of a solution for a first order system of linear differential equations which tends to zero. Hartman's result implied a result of Milloux [**4**] for a certain second order scalar differential equation which demonstrated the existence of a nontrivial zero-tending solution.

Theorem 1A. *Assume*

$$\lim_{t\to\infty} ||y(t)||$$

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exists as a finite number for every solution y of (1a). *Then equation* (1a) *has a nontrivial solution y_0 satisfying*

$$\lim_{t\to\infty} y_0(t) = 0,$$

if and only if

$$\lim_{t\to\infty} \prod_{\tau=t_0}^{t} \det[A(\tau)] = 0. \tag{2a}$$

Theorem 1B. *Assume*

$$\lim_{t\to\infty} ||y(t)||$$

exists as a finite number for every solution y of (1b). *Then equation* (1b) *has a nontrivial solution y_0 satisfying*

$$\lim_{t\to\infty} y_0(t) = 0,$$

if and only if

$$\lim_{t\to\infty} \prod_{\tau=t_0}^{t} \det[I + B(\tau)] = 0, \tag{26}$$

where I is the identity matrix.

Proof of Theorem 1A. Let $Y(t)$ be a fundamental matrix for equation (1a). We will show that

$$\lim_{t\to\infty} Y^*(t)Y(t)$$

exists. The (p,q)th element of $Y^*(t)Y(t)$ is

$$y_p^*(t)y_q(t)$$

where $y_p(t), y_q(t)$ are the p^{th} and q^{th} columns of $Y(t)$. By the polarization identity

$$4y_p^*(t)y_q(t) = \sum_{k=0}^{3} i^k ||y_q(t) + i^k y_p(t)||^2$$

where $i^2 = -1$. Since $y_q(t) + i^k y_p(t)$ is a solution of equation (1a), we have that

$$\lim_{t\to\infty} y_p^*(t)y_q(t)$$

exists and is finite. Hence

$$\lim_{t\to\infty} Y^*(t)Y(t) = H$$

exists where H is an $n \times n$ Hermitian matrix.

By Liouville's formula for (1a),

$$\det Y(t) = \left[\prod_{\tau=t_0}^{t-1} \det[A(\tau)]\right] \det Y(t_0).$$

If (2a) holds, then

$$\lim_{t\to\infty} \det Y(t) = 0.$$

Hence,

$$\det H = \lim_{t\to\infty} [\det Y(t)]^2 = 0.$$

Choose $c_0 \in \mathcal{C}^n\backslash\{0\}$ such that $Hc_0 = 0$. Set $y_0(t) = Y(t)c_0$ on $[t_0, \infty)$, then y_0 is a nontrivial solution of (1a) and

$$\begin{aligned}\lim_{t\to\infty} ||y_0(t)||^2 &= \lim_{t\to\infty} c_0^* Y^*(t)Y(t)c_0 \\ &= c_0^* H c_0 = 0.\end{aligned}$$

Now assume (1a) has a nontrivial solution y_0 such that

$$\lim_{t\to\infty} y_0(t) = 0.$$

Then $y_0(t) = Y(t)c_0$ for some $c_0 \in \mathcal{C}^n\backslash\{0\}$. Hence

$$0 = \lim_{t\to\infty} ||y_0(t)||^2 = c_0^* H c_0.$$

It follows that $\det H = 0$. This implies that

$$\lim_{t\to\infty} \det Y(t) = 0.$$

Using Liouville's formula we get that (2a) holds. □

The next theorem gives criteria in terms of the coefficient matrix $A(t)$ for equation (1a) to have the property that every solution has magnitude with finite limit. The proof of this result is very elementary. Here, inequalities between matrices will be in the negative semi-definite sense.

Theorem 2A. *If*

$$A^*(t)A(t) \leq I \tag{3a}$$

on $[t_0, \infty)$, *then*

$$\lim_{t\to\infty} ||y(t)||$$

exists as a finite number for every solution y *of* (1a).

Theorem 2B. *If*

$$B(t) + B^*(t) + B^*(t)B(t) \leq 0 \tag{3b}$$

on $[t_0, \infty)$, *then*

$$\lim_{t\to\infty} ||y(t)||$$

exists as a finite number for every solution y *of* (1b).

Proof of Theorem 2A. Let y be a solution of (1a) and set

$$w(t) = ||y(t)||^2 = y^*(t)y(t).$$

Then

$$\Delta w(t) = y^*(t+1)\Delta y(t) + \Delta y^*(t)y(t).$$

Hence,

$$\begin{aligned}\Delta w(t) &= y^*(t)A^*(t)[A(t) - I]y(t) + y^*(t)[A^*(t) - I]y(t)\\ &= y^*(t)[A^*(t)A(t) - I]y(t)\\ &\leq 0\end{aligned}$$

on $[t_0, \infty)$. Hence $\lim_{t\to\infty} ||y(t)||$ exists and is finite. □

The following corollary follows immediately from Theorem 1A and Theorem 2A.

Corollary A. *If*

$$A^*(t)A(t) \leq I$$

on $[t_0, \infty)$ *and*

$$\lim_{t\to\infty} \prod_{\tau=t_0}^{t} \det[A(\tau)] = 0,$$

then equation (1a) *has a nontrivial solution* y_0 *with*

$$\lim_{t\to\infty} y_0(t) = 0.$$

Corollary B. *If*

$$B(t) + B^*(t) + B^*(t)B(t) \leq 0$$

on $[t_0, \infty)$ *and*

$$\lim_{t\to\infty} \prod_{\tau=t_0}^{t} \det[I + B(\tau)] = 0,$$

then equation (1b) *has a nontrivial solution* y_0 *with*

$$\lim_{t\to\infty} y_0(t) = 0.$$

We now consider two examples. First, the second order scalar euation

$$u(t+2) + a(t)u(t+1) + b(t)u(t) = 0 \tag{4}$$

where a and b are complex valued functions with $b(t) \neq 0$ on $[t_0, \infty)$. Write equation (4) as a second order system by letting

$$v(t) = u(t+1).$$

Then the matrix A for equation (1a) where y is the vector function $[u,v]^T$ is given by

$$A(t) = \begin{bmatrix} 0 & 1 \\ -b(t) & -a(t) \end{bmatrix}.$$

Therefore,

$$A^*(t)A(t) - I = \begin{bmatrix} |b(t)|^2 - 1 & a(t)b^*(t) \\ a^*(t)b(t) & |a(t)|^2 \end{bmatrix}. \tag{5}$$

Equation (5) is negative semi-definite if and only if $a(t) \equiv 0$ and $|b(t)| \le 1$ on $[t_0, \infty)$. Note that the determinant of $A(t)$ is $b(t)$. Hence, equation (4) has a nontrivial zero-tending solution, if $a(t) \equiv 0$, $|b(t)| \le 1$ and $\prod_{t_0}^{\infty} b(t) = 0$. This result is trivial for difference equations, but note it is not true that under these conditions every solution of equation (5) tends to zero. To see this, consider the solution u of the initial value problem of equation (4) with $b(t) = 1$ when t is an even integer and $b(t) = 1/t$ when t is odd, and the initial conditions $u(0) = 1$, $u(1) = 1$.

We now consider a more general second order system

$$\begin{aligned} u(t+1) &= a(t)u(t) + b(t)v(t) \\ v(t+1) &= c(t)u(t) + d(t)v(t) \end{aligned} \tag{6}$$

on $[t_0, \infty)$, where here we assume a, b, c, d are complex valued functions defined on $[t_0, \infty)$. Then

$$A^*A - I = \begin{bmatrix} |a|^2 + |c|^2 - 1 & a^*b + c^*d \\ ab^* + cd^* & |b|^2 + |d|^2 - 1 \end{bmatrix}. \tag{7}$$

Since a Hermitian matrix is negative semi-definite if and only if its principal minor determinants satisfy certain sign conditions, (7) satisfies (3a) when

$$|a|^2 + |c|^2 \le 1 \tag{8}$$

$$|b|^2 + |d|^2 \le 1 \tag{9}$$

and

$$[|a|^2 + |c|^2 - 1][|b|^2 + |d|^2 - 1] \ge |a^*b + c^*d|^2. \tag{10}$$

Also, condition (2a) is satisfied when

$$\lim_{t\to\infty}\prod_{s=t_0}^{t}[a(s)d(s)-b(s)c(s)]=0. \tag{11}$$

Hence, the system (6) has a nontrivial zero-tending solution provided conditions (8–11) are satisfied.

We now give an example which satisfies the conditions (8–11). If $-0.1 < b(t) \le 0.98$ then the system

$$y(t+1)=\begin{bmatrix}0.1 & b(t)\\ 0.1 & 0.1\end{bmatrix}y(t)$$

has a nontrivial solution $y_0(t)$ satisfying

$$\lim_{t\to\infty} y_0(t)=0.$$

Using methods similar to the proofs of the earlier results in this paper, one can easily prove the following results.

Theorem 3A. *Assume*

$$\lim_{t\to\infty}||y(t)||\le\infty$$

exists for all solutions y *of equation* (1a) (*in particular, this is true if* $A^*(t)A(t)\ge I$ *in a neighborhood of infinity*). *If*

$$\lim_{t\to\infty}\prod_{\tau=t_0}^{t}|\det A(\tau)|=\infty \tag{12a}$$

then there is a solution y_0 *of* (1a) *such that*

$$\lim_{t\to\infty}||y_0(t)||=\infty. \tag{12b}$$

Theorem 3B. *Assume*

$$\lim_{t\to\infty}||y(t)||\le\infty$$

exists for all solutions y *of equation* (1b) (*in particular, this is true if* $B(t) + B^*(t) + B^*(t)B(t) \geq 0$ *in a neighborhood of infinity*). *If*

$$\lim_{t\to\infty} \prod_{\tau=t_0}^{t} |\det[I + B(\tau)]| = \infty \tag{12b}$$

then there is a solution y_0 *of* (1b) *such that*

$$\lim_{t\to\infty} ||y_0(t)|| = \infty. \tag{13b}$$

Unlike Theorem 1A and Theorem 1B where conditions (2a) and (2b) were necessary and sufficient, the conditions (12a) and (12b) are sufficient but not necessary as illustrated in the following example. Consider the system

$$y(t+1) = \begin{bmatrix} 1/t & 0 \\ 0 & t \end{bmatrix} y(t) \tag{14}$$

on the larger interval $[1, \infty)$. Note that $y_0(t) = [0 \quad (t-1)!]^T$ is a solution of (14) satisfying (13a) and that

$$\lim_{t\to\infty} \prod_{\tau=1}^{t} |\det A(\tau)| = 1,$$

that is, the limit in (12a) is finite.

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