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# PRIME SUBMODULES OF NOETHERIAN MODULES

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**0. Introduction.** Let $R$ be a ring. A proper left ideal $L$ of $R$ is prime if, for any elements $a$ and $b$ in $R$ such that $aRb \subseteq L$, either $a \in L$ or $b \in L$. For example, any prime two-sided ideal is a prime left ideal. Prime left ideals have properties reminiscent of prime ideals in commutative rings. For example, Michler [**13**] and Koh [**7**] proved that the ring $R$ is left Noetherian if and only if every prime left ideal is finitely generated. Moreover, Smith [**14**] showed that if $R$ is left Noetherian (or even if $R$ has left Krull dimension) then a left $R$-module $M$ is injective if and only if, for every essential prime left ideal $L$ of $R$ and homomorphism $\varphi : L \to M$, there exists a homomorphism $\theta : R \to M$ such that $\theta|_L = \varphi$.

Several authors have extended the notion of prime left ideals to modules (see, for example, [**2, 3, 4, 6, 8, 9, 10, 11**]; in particular, [**3**] has a good bibliography). In this paper, we continue these investigations both in some generality and also in case $M$ is a Noetherian module.

Let $M$ be a left $R$-module. Then a proper submodule $N$ of $M$ is prime if, for any $r \in R$ and $m \in M$ such that $rRm \subseteq N$, either $rM \subseteq N$ or $m \in N$. It is easy to show that if $N$ is a prime submodule of $M$ then the annihilator $P$ of the module $M/N$ is a two-sided prime ideal of $R$. We consider which prime ideals $P$ of $R$ are the annihilators of modules $M/N$ with $N$ prime in $M$. A special class of prime submodules of $M$ are the strongly prime submodules. Let $K$ be a proper submodule of $M$, and let $Q$ denote the annihilator of $M/K$. Then $K$ is called strongly prime if (i) $Q$ is a prime ideal of $R$ and the ring $R/Q$ is a left Goldie ring, and (ii) $M/K$ is a torsion-free left $(R/Q)$-module. We investigate which prime ideals $Q$ arise in this way.

We also are interested in chain conditions on (strongly) prime submodules of $M$. It is shown that if $R$ satisfies the ascending chain condition (respectively, descending chain condition) on prime ideals then any finitely generated left $R$-module $M$ satisfies the ascending chain

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condition (descending chain condition) on strongly prime submodules.

Now suppose that the ring $R$ satisfies a polynomial identity. Let $M$ be a left $R$-module. Then every prime submodule is strongly prime. The results on chain conditions then apply to prime submodules of $M$. One interesting consequence is that if $M$ is a Noetherian module then $M$ satisfies the descending chain condition on prime submodules.

We briefly consider minimal prime submodules of a left module $M$ over an arbitrary ring $R$. It is shown that if $M$ is Noetherian then $M$ contains only a finite number of minimal prime submodules. Since every prime submodule contains a minimal prime submodule it follows that the prime radical of a Noetherian module $M$ is a finite intersection of prime submodules. Several attempts have been made to characterize the prime radical of a module $M$. Even for commutative rings, $R$, progress has been limited to a number of special cases (see, for example, [**3, 4, 9, 10, 11**]).

**1. Prime submodules.** Let $R$ be a ring and $M$ a left $R$-module. Let $N$ be a submodule of $M$. Then we define

$$(N : M) = \{r \in R : rM \subseteq N\}.$$

Note that $(N : M)$ is an ideal of $R$, in fact $(N : M)$ is the annihilator of the left $R$-module $M/N$. The submodule $N$ of $M$ is called *prime* if $N \neq M$ and, given $r \in R$ and $m \in M$ such that $rRm \subseteq N$, either $m \in N$ or $r \in (N : M)$. It is not difficult to see that $N$ is a prime submodule of $M$ if and only if $(N : K) = (N : M)$ for all submodules $K$ of $M$ properly containing $N$. Clearly, any prime (two-sided) ideal of the ring $R$ is a prime submodule of the left $R$-module $R$.

Following [**5**, p. 31], a left $R$-module $M$ will be called *fully faithful* if every nonzero submodule of $M$ is faithful.

**Proposition 1.1.** *A submodule $N$ of a left $R$-module $M$ is prime if and only if $P = (N : M)$ is a prime ideal of the ring $R$ and the left $(R/P)$-module $M/N$ is fully faithful.*

*Proof.* Elementary. □

If $R$ is a simple ring, then every nonzero left $R$-module is faithful.

Therefore, simple rings $R$ have the property that every proper submodule of every left $R$-module $M$ is prime. The converse is also true (see [**6**, Theorem 4.2, **7**, Theorem 2]). For a general ring $R$, any maximal submodule of a left $R$-module $M$ is a prime submodule of $M$. However, it is not at all difficult to give examples of modules which have no prime submodules. For example, if $\mathbf{Z}$ denotes the ring of rational integers then, for any prime $p$, as a $\mathbf{Z}$-module, the Prüfer group $\mathbf{Z}(p^\infty)$ has no prime submodules. Moreover, the zero submodule is the only prime submodule of the $\mathbf{Z}$-module $\mathbf{Q}$ of rational numbers.

Let $R$ be a ring and $\varphi : M \to M'$ a homomorphism of left $R$-modules $M, M'$. For any nonempty subset $X$ of $M'$, $\varphi^{-1}(X) = \{m \in M : \varphi(m) \in X\}$. The proof of the next result is elementary and is omitted (see [**2**, Proposition 2.2]).

**Proposition 1.2.** *Let $R$ be any ring, $M$ and $M'$ left $R$-modules, and $\varphi : M \to M'$ an $R$-homomorphism. Let $N$ be a prime submodule of $M'$ such that $\varphi(M) \nsubseteq N$. Then $\varphi^{-1}(N)$ is a prime submodule of $M$.*

Let $R$ be a ring. Then $N(R)$ will denote the intersection of all prime ideals of $R$. Proposition 1.2 has the following immediate consequence.

**Corollary 1.3.** *Let $R$ be any ring and $M$ a left $R$-module such that* $\mathrm{Hom}_R(M, R/N(R)) \neq 0$. *Then $M$ contains a prime submodule.*

Before proceeding, we give two further sources of examples of prime submodules of a module. First we recall some definitions. Let $R$ be any ring, and let $M$ be a left $R$-module. Let $N$ be a submodule of $M$. A submodule $K$ of $M$ maximal with respect to the property that $N \cap K = 0$ is called a *complement of $N$ in $M$*. A submodule $K$ of $M$ will be called a *complement in $M$* if there exists a submodule $N$ of $M$ such that $K$ is a complement of $N$ in $M$. It is not difficult to prove that $K$ is a complement in $M$ if and only if $K$ has no proper essential extension in $M$. In consequence, every submodule of $M$ is essential in a complement in $M$. Following [**12**, 6.9.3], we call a nonzero module $M$ *compressible* if every nonzero submodule contains an isomorphic copy of $M$. The next result is known (see, for example, [**2**, 1.12(1) and Proposition 2.7]), but its proof is included for completeness.

**Proposition 1.4.** *Let $R$ be any ring. Let $M$ be a left $R$-module and $N$ a proper submodule of $M$.*

(i) *If $M/N$ is a compressible module, then $N$ is a prime submodule of $M$.*

(ii) *If $N$ is a prime submodule of $M$ and $K$ is a submodule containing $N$ such that $K/N$ is a complement in $M/N$, then $K$ is a prime submodule of $M$.*

*Proof.* (i) Let $L$ be a submodule of $M$ properly containing $N$ such that $rL \subseteq N$ for some $r \in R$. There exists a monomorphism $\varphi : M/N \to L/N$. Now $r\varphi(M/N) = 0$ implies $\varphi(r(M/N)) = 0$ and hence $r(M/N) = 0$, i.e., $rM \subseteq N$. It follows that $N$ is a prime submodule of $M$.

(ii) Let $L$ be a submodule of $M$ properly containing $K$ such that $rL \subseteq K$ for some $r \in R$. Because $K/N$ is a complement in $M/N$, we know that $K/N$ is not essential in $L/N$. Thus there exists a submodule $L'$ of $L$ such that $N \subset L'$ and $K \cap L' = N$. Now $rL' \subseteq rL \cap L' \subseteq K \cap L' \subseteq N$. It follows that $rM \subseteq N \subseteq K$, because $N$ is prime. Hence, $K$ is a prime submodule of $M$. □

Note that Proposition 1.4 generalizes [**6**, Lemma 3.5]. A consequence of Proposition 1.1 is that if $M$ is a left $R$-module and $P$ is a maximal ideal of $R$ such that $M \neq PM$ then every proper submodule $K$ of $M$ containing $PM$ is prime and satisfies $(K : M) = P$ (see [**8**, Proposition 3]). We now address the question: Given a prime ideal $P$ of $R$ and a left $R$-module $M$, does there exist a prime submodule $N$ of $M$ with $P = (N : M)$? Note that if such a submodule $N$ exists then $M \neq PM$. In fact, we can say more:

**Lemma 1.5.** *Let $A$ be an ideal of a ring $R$ and let $M$ be a left $R$-module. Then there exists a proper submodule $N$ of $M$ such that $A = (N : M)$ if and only if $AM \neq M$ and $A = (AM : M)$.*

*Proof.* The sufficiency is clear. Conversely, suppose that $A = (N : M)$ for some proper submodule $N$ of $M$. Then $AM \subseteq N$, and, hence, $AM \neq M$. Moreover, clearly, $A \subseteq (AM : M)$. On the other hand, $(AM : M)M \subseteq AM \subseteq N$, so that $(AM : M) \subseteq A$. Thus $A = (AM : M)$. □

A left $R$-module $M$ will be called *weakly Noetherian* if, for every element $a$ in $R$ and element $m$ in $M$, the submodule $RaRm$ is finitely generated. For any ring $R$, every Noetherian module is weakly Noetherian. If $R$ is a commutative ring, then any $R$-module is weakly Noetherian. On the other hand, if the (not necessarily commutative) ring $R$ has the property that every ideal is finitely generated as a left ideal, in particular if $R$ is left Noetherian, then every left $R$-module is weakly Noetherian. Let $R$ be any ring. It is easy to check that if a left $R$-module $M$ is weakly Noetherian, then so too is any submodule of $M$ and any homomorphic image of $M$.

Let $M$ be a left $R$-module, and let $P$ be a prime ideal of $R$. Then we shall denote by $M(P)$ the following subset of $M$:

$$M(P) = \{m \in M : Am \subseteq PM \text{ for some ideal } A \nsubseteq P\}.$$

It is clear that $M(P)$ is a submodule of $M$ and $PM \subseteq M(P)$. Note the following fact about $M(P)$.

**Lemma 1.6.** *Let $P$ be a prime ideal of a ring $R$. Let $M$ be a left $R$-module such that there exists a prime submodule $K$ of $M$ with $(K : M) = P$. Then $M(P) \subseteq K$.*

*Proof.* Let $m \in M(P)$. There exists an ideal $A$ of $R$ such that $A \nsubseteq P$ and $Am \subseteq PM$. However, $PM \subseteq K$, and hence $Am \subseteq K$. Because $A \nsubseteq P$, $m \in K$. It follows that $M(P) \subseteq K$. □

Now we show that in many situations $M(P)$ itself is a prime submodule of $M$.

**Proposition 1.7.** *Let $P$ be a prime ideal of a ring $R$. Let $M$ be a left $R$-module such that the left $(R/P)$-module $M/PM$ is weakly Noetherian. Let $N = M(P)$. Then $N = M$ or $N$ is a prime submodule of $M$ such that $P = (N : M)$.*

*Proof.* Suppose that $N \neq M$. Let $r \in R$, $m \in M$ satisfy

$rRm \subseteq N$. If $r \in P$, then $rM \subseteq N$. Suppose that $r \notin P$. Let $A = RrR$. Then $M/PM$ weakly Noetherian implies that $Am + PM = Rm_1 + \cdots + Rm_k + PM$, for some positive integer $k$ and elements $m_i \in Am$, $1 \leq i \leq k$. For each $1 \leq i \leq k$, $m_i \in Am \subseteq N$, and, hence, there exists an ideal $B_i \nsubseteq P$ such that $Bm_i \subseteq PM$. Let $B = B_1 \cap \cdots \cap B_k$. Note that $B$ is an ideal of $R$ and $B \nsubseteq P$ because $P$ is prime. Moreover, $BAm \subseteq Bm_1 + \cdots + Bm_k + PM \subseteq PM$. However, $P$ prime implies $BA \nsubseteq P$. Thus, $m \in N$. It follows that $N$ is a prime submodule of $M$.

Let $C = (N : M)$. Clearly $P \subseteq C$. Suppose that $P \neq C$. Let $c \in C$, $c \notin P$. Let $x \in M$. Then $RcRx \subseteq N$. By the above argument, $x \in N$. It follows that $M = N$, a contradiction. Thus, $P = C$. □

Proposition 1.7 raises the question: When does $M = M(P)$? We know that $M \neq M(P)$ if $M$ contains a prime submodule $K$ such that $P = (K : M)$ (Lemma 1.6).

**Proposition 1.8.** *Let $P$ be a prime ideal of a ring $R$. Let $M$ be a left $R$-module such that the left $(R/P)$-module $M/PM$ is finitely generated and weakly Noetherian. Then the following statements are equivalent.*

(i) $M(P) \neq M$.

(ii) *$M(P)$ is a prime submodule of $M$.*

(iii) *There exists a prime submodule $K$ of $M$ such that $P = (K : M)$.*

(iv) $P = (PM : M)$.

*Proof.* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) by Proposition 1.7.

(iii) $\Rightarrow$ (iv) by Lemma 1.5.

(iv) $\Rightarrow$ (i). Let $N = M(P)$. Suppose that $N = M$. There exist a positive integer $k$ and elements $m_i$, $1 \leq i \leq k$, in $M$ such that $M = Rm_1 + \cdots + Rm_k + PM$. For each $1 \leq i \leq k$, there exists an ideal $A_i \nsubseteq P$ such that $A_i m_i \subseteq PM$. Let $A = A_1 \cap \cdots \cap A_k$. Note that $A$ is an ideal of $R$, $A \nsubseteq P$ and $AM \subseteq PM$. It follows that $A \subseteq (PM : M)$, and, hence, $P \neq (PM : M)$. Thus, $N \neq M$. □

Proposition 1.8 has the following immediate consequence.

**Corollary 1.9.** *Let $P$ be a prime ideal of a ring $R$, and let $M$ be a left $R$-module such that the module $M/PM$ is Noetherian. Then $M$ contains a prime submodule $K$ such that $P = (K : M)$ if and only if $P = (PM : M)$. In this case, $K = M(P)$ is one such prime submodule.*

Consider statement (iii) in Proposition 1.8 for a moment. Let $R$ be a commutative ring and $M$ a finitely generated $R$-module. Suppose that $P \neq (PM : M)$. Let $c \in (PM : M)$, $c \notin P$. Then $cM \subseteq PM$ gives, by the usual determinant argument, $(c^k + p)M = 0$ for some positive integer $k$ and element $p$ in $P$. Thus, $M_P = 0$, where $M_P$ is the localization of the module $M$ at the prime ideal $P$. Thus, $P \neq (PM : M)$ implies that $M_P = 0$. Conversely, if $M_Q = 0$, for some prime ideal $Q$ of $R$, then it is easy to check that $Q \neq (QM : M)$. The *support* of the module $M$ is defined to be the set of prime ideals $P$ of $R$ such that $M_P \neq 0$. Thus, the support of $M$ consists precisely of all prime ideals $P$ of $R$ such that $M$ contains a prime submodule $N$ with $P = (N : M)$. In particular, if, in addition, $M$ is faithful, then every prime ideal of $R$ belongs to the support of $M$. Thus, Proposition 1.8 generalizes [**8**, Theorem 2].

Let $\text{spec}_P(M)$ denote the collection of all prime submodules $K$ of $M$ such that $P = (K : M)$, together with the module $M$. If $N \in \text{spec}_P(M)$ and $N \neq M$, then we shall call $N$ a *proper* member of $\text{spec}_P(M)$. Compare the next result with [**9**, Lemma 1].

**Proposition 1.10.** *Let $P$ be a prime ideal of a ring $R$. Let $M$ be a left $R$-module.*

(i) *Let $K_i$, $i \in I$, be any collection of submodules of* $\text{spec}_P(M)$. *Then $\cap_I K_i$ also belongs to* $\text{spec}_P(M)$.

(ii) *Now suppose that $M/PM$ is weakly Noetherian. If $\{L_i : i \in I\}$ is any chain in* $\text{spec}_P(M)$, *then $\cup_I L_i$ also belongs to* $\text{spec}_P(M)$. *If, in addition, $M$ is finitely generated, then* $\text{spec}_P(M)$ *contains maximal proper members.*

*Proof.* (i) Let $K = \cap_I K_i$. Then $K$ is a submodule of $M$ and $PM \subseteq K$. Let $A$ be an ideal of $R$ and $m$ an element in $M$ such that $Am \subseteq K$. If $A \subseteq P$, then $AM \subseteq K$. Suppose that $A \nsubseteq P$. For each $i \in I$, $Am \subseteq K_i$, and, hence, $m \in K_i$. It follows that $m \in \cap_I K_i = K$.

Hence, $K \in \mathrm{spec}_P(M)$.

(ii) Let $L = \cup_I L_i$. Then $L$ is a submodule of $M$ and $PM \subseteq L$. Suppose that $a \in R$, $m \in M$ and $aRm \subseteq L$. There exist a positive integer $k$ and elements $m_i \in RaRm$, $1 \leq i \leq k$, such that $RaRm \subseteq Rm_1 + \cdots + Rm_k + PM$. There exists $j \in I$ such that $m_i \in L_j$ for all $1 \leq i \leq k$. Thus, $aRm \subseteq L_j$, and, hence, $aM \subseteq L_j \subseteq L$ or $m \in L_j \subseteq L$. It follows that $L$ is a prime submodule of $M$. Now suppose that $L_i$ is proper for all $i \in I$. Let $b \in (L : M)$. If $b \notin P$, then the above argument gives $M \subseteq L$, and, hence, $M = L$. Thus, $(L : M) = P$ or $L = M$. Thus, $L \in \mathrm{spec}_P(M)$. If $M$ is finitely generated, then $L \neq M$ so that $L$ is proper. By Zorn's Lemma, $\mathrm{spec}_P(M)$ has maximal proper members. □

Let $R$ be any ring and $M$ a left $R$-module. Suppose that $N$ is a prime submodule of $M$. Then we shall call $N$ *irreducible* if $N \neq K \cap L$, where $K$ and $L$ are prime submodules of $M$ properly containing $N$.

**Proposition 1.11.** *Let $R$ be any ring and $M$ a Noetherian left $R$-module. Then every prime submodule of $M$ is a finite intersection of irreducible prime submodules. Moreover, if $N$ is an irreducible prime submodule of $M$, then $M/N$ is a uniform module.*

*Proof.* Suppose that not every prime submodule of $M$ is a finite intersection of irreducible prime submodules. Let $P$ be a prime submodule maximal with respect to the property that $P$ is not a finite intersection of irreducible prime submodules. In particular, $P$ is not irreducible, so that $P = K \cap L$, for some prime submodules $K$ and $L$, both properly containing $P$. By the choice of $P$, both $K$ and $L$ are finite intersections of irreducible prime submodules, and, hence, so too is $P$, a contradiction.

Now suppose that $N$ is an irreducible prime submodule of $M$. Note first that the module $M/N$ is nonzero. Suppose further that there exist submodules $X$ and $Y$ of $M$ such that $N = X \cap Y$. Let $P = (N : M)$. We define submodules $X'$ and $Y'$ of $M$ as follows:

$$X' = \{m \in M : Am \subseteq X \text{ for some ideal } A \text{ of } R \text{ with } A \nsubseteq P\},$$

and

$$Y' = \{m \in M : Am \subseteq Y \text{ for some ideal } A \text{ of } R \text{ with } A \not\subseteq P\}.$$

Clearly, $X \subseteq X'$ and $Y \subseteq Y'$. If $m \in X' \cap Y'$, then there exist ideals $B$ and $C$ of $R$, neither contained in $P$, such that $Bm \subseteq X$ and $Cm \subseteq Y$. Thus, $(B \cap C)m \subseteq X \cap Y = N$, and, hence, $m \in N$. It follows that $N = X' \cap Y'$. But, by Proposition 1.7, $X'$ and $Y'$ are prime submodules of $M$ or are equal to $M$. In any case, $N = X'$ or $N = Y'$. It follows that $N = X$ or $N = Y$. Hence, $M/N$ is uniform. □

**2. Strongly prime submodules.** Let $R$ be a ring. An element $c$ in $R$ is called *regular* if $cr \neq 0$ and $rc \neq 0$ for every nonzero element $r$ in $R$. If $A$ is a proper ideal of $R$, then $C(A)$ will denote the set of elements $c$ in $R$ such that $c + A$ is a regular element in the ring $R/A$. Clearly, $c \in C(A)$ if and only if, for any $r \in R$, $cr \in A$ or $rc \in A$ implies $r \in A$.

Let $R$ be a prime left Goldie ring. Let $M$ be a left $R$-module. Then the singular submodule of $M$ is given by

$$Z(M) = \{m \in M : cm = 0 \text{ for some } c \in C(0)\}.$$

Recall that $M$ is called a *torsion* module if $M = Z(M)$, and $M$ is called *torsion-free* if $Z(M) = 0$.

Let $R$ be any ring. Let $M$ be a left $R$-module. A proper submodule $N$ of $M$ will be called *strongly prime* if there exists a prime ideal $P$ of $R$ such that

(i) the ring $R/P$ is left Goldie,

(ii) $PM \subseteq N$, and

(iii) for any $c \in C(P)$ and $m \in M$, $cm \in N$ implies $m \in N$.

Note that, in this case, $P = (N : M)$. For, let $A = (N : M)$. Clearly, $P \subseteq A$. If $P \neq A$, then there exists $c \in A \cap C(P)$, by [**5**, Proposition 5.9 or **12**, 2.3.4 and 2.3.5]. In this case, $cM \subseteq N$ implies that $M = N$, a contradiction.

**Proposition 2.1.** *Let $N$ be a strongly prime submodule of a left $R$-module $M$. Then $N$ is prime.*

*Proof.* There exists a prime ideal $P$ of $R$ satisfying (i), (ii) and (iii). Let $A$ be an ideal of $R$ and $m$ an element of $M$ such that $Am \subseteq N$. Suppose that $A \nsubseteq P$. Then there exists $c \in A \cap C(P)$, as above. Now $cm \in N$ gives that $m \in N$. It follows that $N$ is prime. ◻

The converse of Proposition 2.1 is true for commutative rings $R$. In fact, the converse is true for a wider class of rings and we shall consider such rings at the end of this section.

Let $P$ be a prime ideal of a ring $R$ such that $R/P$ is a left Goldie ring. Let $M$ be a left $R$-module. For any submodule $N$ of $M$ such that $PM \subseteq N$, let $\mathrm{cl}_P(N)$ denote the submodule $T$ of $M$ containing $N$ such that $T/N$ is the singular submodule of the left $(R/P)$-module $M/N$, i.e.,

$$\mathrm{cl}_P(N) = \{m \in M : cm \in N \text{ for some } c \in C(P)\}.$$

Before we proceed to the next result, note that, for any submodule $K$ containing $PM$, the submodule $K/PM$ is a complement in the left $(R/P)$-module $M/PM$ if $M/K$ is a torsion-free left $(R/P)$-module (see [**5**, Proposition 3.27]). Compare the next result with Proposition 1.4 (ii).

**Proposition 2.2.** *Let $P$ be a prime ideal of a ring $R$ such that the ring $R/P$ is* (*prime*) *left Goldie. Let $M$ be a left $R$-module. Let $T = \mathrm{cl}_P(PM)$. Then the following statements are equivalent for a submodule $N$ of $M$.*

(i) *$N$ is a strongly prime submodule of $M$ such that $P = (N : M)$.*

(ii) *The module $M/N$ is a nonzero torsion-free left $(R/P)$-module.*

(iii) *$T \subseteq N$ and $N/T$ is a proper complement in $M/T$.*

*Proof.* Elementary. ◻

**Corollary 2.3.** *Let $P$ be a prime ideal of a ring $R$ such that the ring $R/P$ is a* (*prime*) *left Goldie ring. Let $M$ be a left $R$-module. Then the following statements are equivalent.*

(i) *$\mathrm{cl}_P(PM)$ is a strongly prime submodule of $M$.*

(ii) *There exists a strongly prime submodule $K$ of $M$ such that*

$P = (K : M)$.

(iii) *$M/PM$ is not a torsion left $(R/P)$-module.*

*Proof.* (i) $\Rightarrow$ (ii). Let $T = \mathrm{cl}_P(PM)$. Let $A = (T : M)$. Clearly, $P \subseteq A$. If $P \neq A$, then there exists $c \in A \cap C(P)$. If $m \in M$, then $cm \in T$, and, hence, $m \in T$. It follows that $T = M$, which contradicts the fact that $T$ is strongly prime. Thus, $P = (T : M)$.

(ii) $\Rightarrow$ (iii). We know that $PM \subseteq K \subset M$, and $M/K$ is torsion-free as a left $(R/P)$-module. (iii) follows.

(iii) $\Rightarrow$ (i). $T/PM$ is the singular submodule of $M/PM$, so that, by (iii), $T \neq M$. The left $(R/P)$-module $M/T$ is a torsion-free left $(R/P)$-module, so that $T$ is strongly prime by Proposition 2.2. $\square$

Let $P$ be a prime ideal of a ring $R$ such that $R/P$ is a left Goldie ring. Let $M$ be a left $R$-module. Let $\mathrm{Spec}_P(M)$ denote the set of all strongly prime submodules $K$ of $M$ such that $P = (K : M)$, together with the module $M$. By Proposition 2.1, $\mathrm{Spec}_P(M) \subseteq \mathrm{spec}_P(M)$. Any member of $\mathrm{Spec}_P(M)$ other than $M$ will be called *proper*. The following analogue of Proposition 1.10 can be proved by adapting its proof.

**Proposition 2.4.** *Let $P$ be a prime ideal of a ring $R$ such that $R/P$ is a (prime) left Goldie ring. Let $M$ be a left $R$-module.*

(i) *Let $K_i$, $i \in I$, be any collection of submodules in $\mathrm{Spec}_P(M)$. Then $\cap_I K_i$ also belongs to $\mathrm{Spec}_P(M)$.*

(ii) *Let $\{L_i : i \in I\}$ be any chain in $\mathrm{Spec}_P(M)$. Then $\cup_I L_i$ also belongs to $\mathrm{Spec}_P(M)$. If, in addition, $M/PM$ is finitely generated, then $\mathrm{Spec}_P(M)$ contains maximal proper members.*

Let $R$ be a prime left Goldie ring. Let $M$ be a torsion-free left $R$-module. Let $\Omega$ denote the collection of submodules $N$ of $M$ such that the module $M/N$ is torsion-free. Then $\Omega$ is a lattice, where we define

$$K \wedge L = K \cap L, \quad \text{and} \quad K \vee L = \mathrm{cl}_0(K + L),$$

for all $K$ and $L$ in $\Omega$. The ring $R$ has a simple Artinian classical left quotient ring $Q$ (see, for example, [**5**, Theorem 5.12, or **12**, 2.3.6]).

Consider the left $Q$-module $M' = Q \otimes_R M$. Any $Q$-submodule of $M'$ has the form $N' = Q \otimes_R N$, where $N \in \Omega$. The mapping $N \to N'$ from the lattice $\Omega$ to the lattice $\Lambda$ of $Q$-submodules of $M'$ is an isomorphism with inverse $\varphi : \Lambda \to \Omega$ defined by

$$\varphi(K') = \{m \in M : 1 \otimes m \in K'\}.$$

**Theorem 2.5.** *Let $P$ be a prime ideal of a ring $R$ such that the ring $R/P$ is a (prime) left Goldie ring. Let $Q$ denote the classical left quotient ring of $R/P$. Let $M$ be a left $R$-module. Let $M'$ denote the left $Q$-module $Q \otimes_{R/P} (M/PM)$. Then* $\mathrm{Spec}_P(M)$ *is a lattice isomorphic to the lattice of $Q$-submodules of the left $Q$-module $M'$. Moreover,* $\mathrm{Spec}_P(M)$ *is a complete complemented modular lattice.*

*Proof.* Let $T = \mathrm{cl}_P(PM)$. Because $T/PM$ is the singular submodule of the left $(R/P)$-module $M/PM$, the module $M/T$ is a torsion-free left $(R/P)$-module and $\mathrm{Spec}_P(M)$ consists of all submodules $N$ of $M$ such that $T \subseteq N$ and $M/N$ is a torsion-free left $(R/P)$-module (Proposition 2.2). Thus, by the remarks immediately preceding the theorem, $\mathrm{Spec}_P(M)$ is a lattice isomorphic to the lattice of submodules of the left $Q$-module $M'$. The $Q$-module $M'$ is semisimple, so that its lattice of submodules is complete, complemented and modular. The result follows. □

Let $R$ be a ring. The prime ring $R$ will be called *left bounded* if, for each regular element $c$ in $R$, there exists an ideal $A$ of $R$ and a regular element $d$ such that $Rd \subseteq A \subseteq Rc$. A general ring $R$ will be called *left fully bounded* if every prime homomorphic image of $R$ is left bounded. The relevance of this notion to our present discussion can be seen from the next result.

**Lemma 2.6.** *Let $R$ be a ring and $P$ a prime ideal of $R$ such that the ring $R/P$ is left bounded left Goldie. Let $M$ be a left $R$-module. Then the following statements are equivalent for a submodule $N$ of $M$.*

(i) *$N$ is a prime submodule of $M$ such that $P = (N : M)$.*

(ii) *$N$ is a strongly prime submodule of $M$ such that $P = (N : M)$.*

*Proof.* (ii) $\Rightarrow$ (i) by Proposition 2.1.

(i) $\Rightarrow$ (ii). Suppose that $N$ is a prime submodule of $M$. Suppose that $c \in C(P)$ and $m \in M$ satisfy $cm \in N$. Because $R/P$ is left bounded, there exists an ideal $A$ of $R$ such that $A \nsubseteq P$ and $A \subseteq Rc + P$. Then $Am \subseteq N$ and, hence, $m \in N$. It follows that $N$ is strongly prime. $\square$

Note further that, with the notation of Lemma 2.6,

$$M(P) = \mathrm{cl}_P(PM).$$

This fact is easy to prove because, for any ideal $A$ of $R$, $A \nsubseteq P$ if and only if the set $A \cap C(P)$ is nonempty (see, for example, [**5**, Proposition 5.9 or **12**, 2.3.4 and 2.3.5]). A ring $R$ is called a *left* FBN-*ring* if $R$ is left fully bounded and left Noetherian. Lemma 2.6 has the following immediate consequence.

**Corollary 2.7.** *Let $R$ be a left* FBN-*ring. Let $M$ be a left $R$-module. Then a submodule $N$ of $M$ is prime if and only if $N$ is strongly prime.*

Another class of rings for which prime submodules of modules are strongly prime is the class of rings with polynomial identity (PI-rings). It is well known that if $R$ is a PI-ring and $P$ is a prime ideal of $R$, then the ring $R/P$ is (left and right) bounded and (left and right) Goldie [**12**, 13.6.6]. Thus, Lemma 2.6 has the following consequence.

**Corollary 2.8.** *Let $R$ be a* PI-*ring. Let $M$ be a left $R$-module. Then a submodule $N$ of $M$ is prime if and only if $N$ is strongly prime.*

We make one final observation in this section.

**Proposition 2.9.** *Let $R$ be a* PI-*ring. Let $M$ be a finitely generated left $R$-module. Then $N$ is an irreducible prime submodule of $M$ if and only if $M/N$ is a compressible uniform module.*

*Proof.* Suppose first that $M/N$ is a compressible uniform module. Then $N$ is a prime submodule of $M$ by Proposition 1.4, and clearly $N$ is irreducible.

Conversely, suppose that $N$ is an irreducible prime submodule of $M$. Without loss of generality, we can suppose that $N = 0$. Let $P = (0 : M)$. Without loss, we can suppose that $P = 0$, and thus $R$ is a prime Goldie ring and $M$ a nonzero finitely generated torsion-free left $R$-module. By Proposition 1.11, $M$ is uniform. Now [**5**, Corollary 6.20] gives $M$ isomorphic to a left ideal $A$ of $R$. Let $K$ be a nonzero submodule of $M$. Note that $A \neq 0$ implies $AK \neq 0$. There exists $k \in K$ such that $Ak \neq 0$. Define $\theta : A \to K$ by $\theta(a) = ak$, $a \in A$. Clearly, $\theta$ is a homomorphism. Suppose that $\theta(a) = 0$ for some $0 \neq a \in A$. Let $b \in A$. Because $A$ is uniform, there exists $c \in C(0)$ such that $cb \in Ra$ (see, for example, [**5**, Proposition 5.9]). Then $ak = 0$ gives $cbk = 0$ and, hence, $bk = 0$. It follows that $Ak = 0$, a contradiction. Thus, $\theta$ is a monomorphism. It follows that $M$ is compressible. □

**3. Chain conditions.** Let $R$ be a ring and $M$ a left $R$-module. A proper submodule $N$ of $M$ will be called *virtually maximal* if $M/N$ is a direct sum of isomorphic simple modules. By [**1**, Proposition 9.4], any proper submodule $K$ containing $N$ is also virtually maximal. Moreover, for the submodule $N$, if $P$ is the annihilator of the simple direct summands of $M/N$, then it is clear that $P = (N : M) = (N : L)$ for any submodule $N \subseteq L \subseteq M$ with $L \neq N$. It follows that $N$ is a prime submodule of $M$. We record this fact as follows.

**Lemma 3.1.** *Let $M$ be a left $R$-module. Then any virtually maximal submodule of $M$ is prime.*

Now we prove a partial converse of Lemma 3.1 in the case of finitely generated Artinian modules $M$.

**Proposition 3.2.** *Let $R$ be a* PI*-ring, and let $M$ be a finitely generated Artinian left $R$-module. Then every prime submodule of $M$ is virtually maximal.*

*Proof.* Let $N$ be a prime submodule of $M$. Let $P = (N : M)$. Note that $M/N$ is a faithful left $(R/P)$-module. By [**12**, 13.6.6], the ring $R/P$ is a left bounded left Goldie ring. Now [**5**, Proposition 8.7] gives that $R/P$ embeds as a left $R$-module in a finite direct sum of copies of

$M/N$. It follows that the ring $R/P$ is left Artinian, and, hence, $R/P$ is simple Artinian. Thus, the left $(R/P)$-module $M/N$ is a direct sum of isomorphic simple modules. Therefore, $N$ is virtually maximal. $\square$

Proposition 3.2 is not true in general. In fact, we have the following result. Recall that a ring $R$ is called a *left $V$-ring* if every simple left $R$-module is injective.

**Proposition 3.3.** *Let $R$ be a simple ring such that, for every essential left ideal $L$ of $R$, the left $R$-module $R/L$ is Artinian and has the property that every prime submodule is virtually maximal. Then $R$ is a left $V$-ring.*

*Proof.* Let $U$ be a simple left $R$-module. Let $A$ be an essential left ideal of $R$ and $\varphi : A \to U$ a nonzero homomorphism. Let $B = \operatorname{Ker} \varphi$. Note that $A/B$ is simple. If $B$ is not an essential submodule of $A$, then $A$ contains a minimal submodule. Thus, $R$ has a minimal left ideal and, hence, $R$ is semiprime Artinian whence a left $V$-ring. Suppose now that $B$ is essential in $A$. Then $B$ is an essential left ideal of $R$ [**1**, Proposition 5.16]. Moreover, $B \neq A$ so that 0 is a prime submodule of $R/B$. By hypothesis, $R/B$ is semisimple. There exists a maximal left ideal $P$ of $R$, containing $B$, such that $R/B = (A/B) \oplus (P/B)$. It follows that the mapping $\varphi$ can be lifted to $R$. Hence, $U$ is injective. Thus, $R$ is a left $V$-ring. $\square$

The first Weyl algebra $R = A_1(\mathbf{C})$ is a simple Noetherian domain such that $R/L$ is Artinian for each essential left ideal $L$ of $R$ (see [**12**, 1.3.5 and 6.6.15]). However, $R$ is not a left $V$-ring.

A partial converse of Proposition 3.2 is proved next.

**Proposition 3.4.** *Let $R$ be a left Noetherian* PI*-ring. Let $M$ be a finitely generated left $R$-module such that $P = (PM : M)$ for every prime ideal $P$ of $R$. Suppose further that every prime submodule of $M$ is virtually maximal. Then $M$ is Artinian.*

*Proof.* Let $P$ be any prime ideal of $R$. By Proposition 1.8, there exists a prime submodule $K$ of $M$ such that $P = (K : M)$. Now $M/K$ is a direct sum of isomorphic simple modules, so that $P$ must be the annihilator of these simples. Hence, $P$ is left primitive, so that $R/P$ is a simple Artinian ring by [**12**, 13.3.8]. Thus, $R/P$ is Artinian for every prime ideal $P$ of $R$. It follows that $R$ is left Artinian (see, for example, the proof of [**5**, Proposition 3.20]), and, hence, $M$ is Artinian. □

The condition that $R$ be left Noetherian can be dropped in case $R$ is commutative, and we have the following result.

**Theorem 3.5.** *Let $R$ be a commutative ring. A finitely generated $R$-module $M$ is Artinian if and only if $M$ is Noetherian and every prime submodule of $M$ is virtually maximal.*

*Proof.* Without loss of generality, we can suppose that $M$ is a faithful $R$-module. Suppose first that $M$ is Artinian. Then $R$ embeds in $M^n$ for some positive integer $n$. It follows that $R$ is an Artinian ring, and, hence, $R$ is a Noetherian ring. It follows that $M$ is Noetherian. By Proposition 3.2, every prime submodule is virtually maximal.

Conversely, suppose that $M$ is Noetherian and every prime submodule is virtually maximal. If $P$ is any prime ideal of $R$, then $P = (PM : M)$ (see the remarks after Corollary 1.9). Thus, we can apply Proposition 3.4 to obtain that $M$ is Artinian. □

Now we consider more general chain conditions on strongly prime submodules.

**Theorem 3.6.** *Let $R$ be any ring which satisfies* ACC (*respectively,* DCC) *on prime ideals. Let $M$ be any finitely generated left $R$-module. Then $M$ satisfies* ACC (*respectively,* DCC) *on strongly prime submodules.*

*Proof.* We prove the result in the DCC case; the ACC case is similar. Suppose that $R$ satisfies DCC on prime ideals. Suppose that $M$ is a finitely generated left $R$-module which does not satisfy DCC on strongly prime submodules. Let $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ be any infinite properly descending chain of strongly prime submodules of $M$. Then $(N_1 : M) \supseteq (N_2 : M) \supseteq \cdots$ is a descending chain of prime ideals of $R$. By hypothesis, there exists a positive integer $k$ such that $(N_k : M) = (N_{k+1} : M) = (N_{k+2} : M) = \cdots$.

Without loss of generality, we can suppose that $P = (N_i : M)$ for all $i \geq 1$. This means that $N_i \in \mathrm{Spec}_P(M)$ for all $i \geq 1$. Let $Q$ denote the classical left quotient ring of the prime left Goldie ring $R/P$. Let $M' = Q \otimes_{R/P} (M/PM)$. Because $M$ is finitely generated, it follows that $M'$ is a finitely generated left $Q$-module and, hence, $M'$ is Artinian. By Theorem 2.5, it follows that $\mathrm{Spec}_P(M)$ is Artinian, a contradiction. Thus, $M$ satisfies DCC on prime submodules. □

Theorem 3.6 has a particularly pleasing consequence in case $R$ is a PI-ring. It is well known that any commutative Noetherian ring satisfies DCC on prime ideals. This is also true for any left Noetherian PI-ring (see, for example, [**12**, 13.7.15]). We now extend this fact to modules.

**Theorem 3.7.** *Let $R$ be a* PI-*ring, and let $M$ be a Noetherian left $R$-module. Then $M$ satisfies* DCC *on prime submodules.*

*Proof.* Suppose that the result is false. Let $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ be any infinite properly descending chain of prime submodules of $M$. Let $N = \cap N_i$. It is clear (or see [**9**, Lemma 1]) that $N$ is a prime submodule of $M$. Without loss of generality, we can suppose that $N = 0$. Let $P = (0 : M)$. Then $P$ is a prime ideal of $R$ and $R/P$ is a prime bounded Goldie ring. By [**5**, Proposition 8.7], $R/P$ embeds in $M^n$ for some positive integer $n$ and, hence, $R/P$ is left Noetherian. By [**12**, 13.7.15], $R/P$ satisfies DCC on prime ideals. Now apply Theorem 3.6 and Lemma 2.6 to obtain a contradiction. □

For commutative rings, we can go further. Let $R$ be a ring and $M$ a left $R$-module. Let $N$ be a prime submodule of $M$. Then we define the *height* of $N$ to be the maximal positive integer $k$, if such exists, such that there exists a chain of prime submodules of $M$ as follows:

$$N = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_k.$$

We shall denote the height of $N$ in $M$ by $\mathrm{ht}_M(N)$. In particular, if $P$ is a prime ideal of the ring $R$, then $\mathrm{ht}_R(P)$ is the height of $P$ in $R$. Let $M$ be a finitely generated module. Then $g(M)$ will denote the minimal number of elements required to generate $M$.

**Theorem 3.8.** *Let $R$ be a commutative ring, and let $M$ be a finitely generated $R$-module. Let $P$ be any prime ideal of $R$ such that $A \subseteq P$, where $A$ is the annihilator of $M$. Let $R'$ denote the ring $R/A$. Then*

$$\mathrm{ht}_{R'}(P/A) \leq \sup\{ht_M(N) : N \in \mathrm{spec}_P(M)\} \leq g(M)[\mathrm{ht}_R(P)+1].$$

*Proof.* Suppose that $\mathrm{ht}_R(P) = n < \infty$. Let $N \in \mathrm{spec}_P(M)$. Let

$$N = N_0 \supset N_1 \supset N_2 \supset \cdots \tag{1}$$

be any descending chain of prime submodules of $M$. Then

$$(N : M) = (N_0 : M) \supseteq (N_1 : M) \supseteq (N_2 : M) \supseteq (N_3 : M) \supseteq \cdots$$

is a descending chain of prime ideals of $R$. Because $\mathrm{ht}_R(P) = n$, the collection $\{(N_i : M) : i \geq 1\}$ contains at most $n+1$ distinct prime ideals. If $Q = (N_i : M)$ for some $i \geq 1$, then $\mathrm{spec}_Q(M)$ is lattice isomorphic to the lattice of submodules of the vector space $V = F \otimes (M/QM)$ over $F$, where $F$ is the field of fractions of the integral domain $R/Q$. Because $M$ can be generated by $g(M)$ elements, it follows that $V$ can be spanned by $g(M)$ vectors, so that $V$ has dimension at most $g(M)$ over $F$. Thus, any descending chain in $\mathrm{spec}_Q(M)$ contains at most $g(M)$ terms. It follows that the number of "steps" in (1) is at most $(n+1)g(M)$. Thus, $\mathrm{ht}_M(N) \leq g(M)[\mathrm{ht}_R(P)+1]$.

Now suppose that $\sup\{\mathrm{ht}_M(N) : N \in \mathrm{spec}_P(M)\} = k < \infty$. Let

$$P = P_0 \supset P_1 \supset \cdots \supset P_{k+1} \supseteq A,$$

where $P_i$, $1 \leq i \leq k+1$ is a prime ideal. There exists a prime submodule $L_{k+1}$ such that $(L_{k+1} : M) = P_{k+1}$, by Proposition 1.8. Now consider the finitely generated faithful $(R/P_{k+1})$-module $M/L_{k+1}$. By another application of Proposition 1.8, there exists a prime submodule $L_k/L_{k+1}$ of $M/L_{k+1}$ such that $P_k = (L_k/L_{k+1} : M/L_{k+1})$ for some submodule

$L_k$ of $M$ containing $L_{k+1}$. It is easy to check that $L_k$ is a prime submodule of $M$ and $P_k = (L_k : M)$. In this way we can produce a chain $L_{k+1} \subseteq L_k \subseteq \cdots \subseteq L_0$ of prime submodules of $M$ such that $P_i = (L_i : M)$ for all $0 \leq i \leq k+1$. However, $P = P_0$ means that $L_0 \in \text{spec}_P(M)$ and $\text{ht}_M(L_0) \geq k+1$, a contradiction. It follows that $\text{ht}_{R'}(P/A) \leq k$. The result follows. □

**4. Minimal prime submodules.** Let $R$ be any ring. Let $M$ be a left $R$-module. A prime submodule $N$ of $M$ is called *minimal* if, for any prime submodule $K$ of $M$ such that $K \subseteq N$, $K = N$. Let $L$ be a prime submodule of $M$. Let

$$\Lambda = \{K : K \text{ is a prime submodule of } M \text{ and } K \subseteq L\}.$$

If $\{K_i : i \in I\}$ is any chain in $\Lambda$, then it can easily be checked that $\cap_I K_i$ also belongs to $\Lambda$ (see [**9**, Lemma 1]). By Zorn's Lemma, $\Lambda$ contains a minimal member which is clearly a minimal prime submodule of $M$. Thus, every prime submodule of $M$ contains a minimal prime submodule of $M$. If the module $M$ is finitely generated, then $M$ has maximal submodules which are prime and, hence, $M$ contains minimal prime submodules. Note the following elementary fact, whose proof is left to the reader.

**Lemma 4.1.** *Let $R$ be a ring, and let $M$ be a left $R$-module. Let $K \subseteq N$ be submodules of $M$. Then $N$ is a prime submodule of $M$ if and only if $N/K$ is a prime submodule of $M/K$. Moreover, if $N$ is a minimal prime submodule of $M$, then $N/K$ is a minimal prime submodule of $M/K$.*

**Theorem 4.2.** *Let $R$ be a ring, and let $M$ be a Noetherian left $R$-module. Then $M$ contains only a finite number of minimal prime submodules.*

*Proof.* Suppose that the result is false. Let $\Lambda$ denote the collection of proper submodules $N$ of $M$ such that the module $M/N$ has an infinite number of minimal prime submodules. The collection $\Lambda$ is nonempty, because $0 \in \Lambda$ and, hence, has a maximal member $K$. Clearly, $K$ is not a prime submodule of $M$. Thus, there exists a submodule $L$ of

$M$ properly containing $K$ and an ideal $A$ in $R$ such that $AL \subseteq K$ but $AM \nsubseteq K$. Hence, $K \subset K + AM$. Let $V$ be a submodule of $M$ containing $K$ such that $V/K$ is a minimal prime submodule of $M/K$. Then $AL \subseteq K \subseteq V$. By Lemma 4.1, $AM \subseteq V$ or $L \subseteq V$. Again, by Lemma 4.1, $V/(K + AM)$ is a minimal prime submodule of $M/(K + AM)$ or $V/L$ is a minimal prime submodule of $M/L$. But by the choice of $K$, both the modules $M/(K + AM)$ and $M/L$ have only finitely many minimal prime submodules. Thus, there are only a finite number of possibilities for the module $V$ and, hence, also for $V/K$, a contradiction. Thus, $M$ has only a finite number of minimal prime submodules. □

Let $R$ be a ring and $M$ a left $R$-module. Then the *prime radical* $\operatorname{rad} M$ of $M$ is defined to be the intersection of $M$ and all prime submodules of $M$. Because every maximal submodule of $M$ is a prime submodule, it is clear that $\operatorname{rad} M$ is contained in the Jacobson radical $\operatorname{Rad} M$ of $M$. If every prime submodule of $M$ is virtually maximal it is not hard to see that $\operatorname{rad} M = \operatorname{Rad} M$.

**Corollary 4.3.** *Let $R$ be a ring and $M$ a Noetherian left $R$-module. Then $\operatorname{rad} M = M$ or there exist a positive integer $k$ and prime ideals $P_i$, $1 \leq i \leq k$, such that $P_i = (P_i M : M)$, $1 \leq i \leq k$, and $\operatorname{rad} M = M(P_1) \cap \cdots \cap M(P_k)$.*

*Proof.* By Lemma 1.6, Proposition 1.8 and Theorem 4.2. □

It has proved a difficult task to determine which elements of $M$ belong to $\operatorname{rad} M$ in general. Corollary 4.3 gives less help than might be supposed at first, because of the difficulty of knowing which prime ideals $P_i$ feature there. In general, the prime ideals $P_i$, $1 \leq i \leq k$, need not all be minimal prime ideals of $R$ or of $R/A$, where $A$ is the annihilator of $M$. Perhaps a simple example would be helpful here.

Let $M$ be a finitely generated $\mathbf{Z}$-module. In the most general case,

$$M = M_1 \oplus \cdots \oplus M_k \oplus M',$$

for some positive integer $k$, where, for each $1 \leq i \leq k$, $M_i$ is a cyclic module of prime power order $p_i^{k(i)}$, for some prime $p_i$ and positive integer $k(i)$, and $M'$ is a free module of finite rank. The prime ideals of $\mathbf{Z}$ are, of course, 0, $\mathbf{Z}p$ ($p$ prime). Clearly,

$$M(0) = M_1 \oplus \cdots \oplus M_k,$$

$$M(\mathbf{Z}_p) = M_1 \oplus \cdots \oplus M_k \oplus pM', \qquad \text{if } p \neq p_i,\ 1 \leq i \leq k,$$

and

$$M(\mathbf{Z}_{p_i}) = M_1 \oplus \cdots \oplus M_{i-1} \oplus p_i M_i \oplus M_{i+1} \oplus \cdots \oplus M_k \oplus p_i M', \qquad \text{if } 1 \leq i \leq k.$$

Thus, the minimal prime submodules of $M$ are $M(0)$, $M(\mathbf{Z}p_i)$, $1 \leq i \leq k$, and

$$\operatorname{rad} M = p_1 M_1 \oplus \cdots \oplus p_k M_k.$$

In fact, this calculation can readily be extended to a finitely generated module over any Dedekind domain. More generally, if $R$ is a one-dimensional commutative domain (i.e., all nonzero prime ideals are maximal), then

$$\operatorname{rad} M = T \cap \operatorname{Rad} M,$$

for any $R$-module $M$, where $T$ denotes the torsion submodule of $M$. For, in this case, $M(0)$ is again the torsion submodule of $M$ and, for any nonzero prime ideal $P$ of $R$, $M(P) = PM$, because $P$ is a maximal ideal. Moreover, for any nonzero prime ideal $P$ of $R$, $M/PM$ is a vector space over the field $R/P$ so that $\operatorname{Rad} M \subseteq PM$. Thus,

$$\begin{aligned} T \cap \operatorname{Rad} M &\subseteq M(0) \cap \{\cap_P PM\} = M(0) \cap \{\cap_P M(P)\} = \operatorname{rad} M \\ &\subseteq T \cap \operatorname{Rad} M. \end{aligned}$$

Hence, $\operatorname{rad} M = T \cap \operatorname{Rad} M$.

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