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MIXED CUSP FORMS AND POINCARÉ SERIES

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Introduction. Let $G \subset PSL(2, \mathbf{R})$ be a torsion-free Fuchsian group of the first kind acting on the Poincaré upper half plane $\mathcal{H}$ by linear fractional transformations, and let $X = G\backslash\mathcal{H}$. Let E be an elliptic surface over X in the sense of Kodaira [**5**]. In [**2**], Hunt and Meyer introduced mixed cusp forms of type (2,1) and showed that the space $S_{2,1}(G, \omega, \chi)$ of mixed cusp forms of type (2,1) associated to ω and χ is isomorphic to the space of holomorphic 2-forms on E, where the holomorphic map $\omega : \mathcal{H} \to \mathcal{H}$ is the period map of the elliptic fibration $E \to X$ and the homomorphism $\chi : G \to SL(2, \mathbf{R})$ is the monodromy representation of G satisfying $\omega(gz) = \chi(g)\omega(z)$ for all $g \in G$ and $z \in \mathcal{H}$.

Let E be an elliptic surface over $X = G\backslash\mathcal{H}$. If m is a positive integer, an elliptic variety E^m is obtained by resolving the singularities of the compactification of the m-fold fiber product of E over X (see, e.g., [**9**] for details). In [**6**] some of the results of Hunt and Meyer were extended to higher weight cases. In particular, cusp forms of type $(2, m)$ for $m > 1$ were introduced in that paper, and it was shown that the space $S_{2,m}(G, \omega, \chi)$ of cusp forms of type $(2, m)$ associated to ω and χ is isomorphic to the space of holomorphic $(m+1)$-forms on E^m.

It is well known that the space of modular forms for $SL(2, \mathbf{Z})$ has a basis consisting of Poincaré series (see, e.g., [**8**, Section 8.3]). In [**3**], S. Katok considered relative Poincaré series associated to hyperbolic elements γ_0 of a Fuchsian group Γ of the first kind and showed that the space of cusp forms for Γ is generated by these Poincaré series. In [**4**], Katok and Millson reformulated the main results of [**3**] in terms of the homology $H_1(\Gamma, (S^{2m}V)^*)$ of the group Γ, where $S^{2m}V$ is the mth symmetric power of the standard two-dimensional complex representation of $SL(2, \mathbf{R})$.

In this paper we give a more precise definition of mixed automorphic forms of type (k, l), where k and l are nonnegative integers with k

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even, and obtain results that extend some of the ones in [**3**] and [**4**] to mixed cusp forms. More precisely, we define the Poincaré series $\theta_{2,2k}(g_0, \omega, \chi)$ for each hyperbolic element g_0 of G and show that under certain conditions the space $S_{2,2k}(G, \omega, \chi)$ of mixed cusp forms of type $(2, 2k)$ is generated by a certain set of Poincaré series $\theta_{2,2k}(g_0, \omega, \chi)$.

1. Mixed automorphic forms. In this section we give a more precise definition of mixed cusp forms considered in [**6**] and [**7**]. Let $G \subset PSL(2, \mathbf{R})$ be a torsion-free Fuchsian group of the first kind acting on the Poincaré upper half plane $\mathcal{H}$. Let $\chi : G \to SL(2, \mathbf{R})$ be a homomorphism, and let $\omega : \mathcal{H} \to \mathcal{H}$ be a holomorphic map such that

$$\omega(gz) = \chi(g)\omega(z)$$

for all $g \in G$ and $z \in \mathcal{H}$. We assume that the inverse image of a parabolic subgroup under χ is also a parabolic subgroup. Let k and l be nonnegative integers with k even. Suppose that $f : \mathcal{H} \to \mathbf{C}$ is a holomorphic function such that

$$f(gz) = f(z) j(g, z)^k j(\chi(g), \omega(z))^l$$

for all $g \in G$ and $z \in \mathcal{H}$, where $j(h, w) = cw + d$ if

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbf{R}) \quad \text{or} \quad SL(2, \mathbf{R})$$

and $w \in \mathcal{H}$. Let $s \in \mathbf{R} \cup \{\infty\}$ be a G-cusp such that $\rho s = \infty$ and $\alpha s = s$, where α is a parabolic element of G and ρ is an element of $SL(2, \mathbf{R})$. If G_s is the subgroup of elements of G that fix s, then we have

$$\rho G_s \rho^{-1} \cdot \{\pm 1\} = \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^n | n \in \mathbf{Z} \right\}$$

for some $h \in \mathbf{R}$. Since $\chi(\alpha)$ is a parabolic element of $\Gamma = \chi(G)$, there is a Γ-cusp s' such that $\chi(\alpha)s' = s'$. Let ρ' be an element of $SL(2, \mathbf{R})$ such that $\rho' s' = \infty$, and assume that

$$\omega(\rho z) = \rho' \omega(z)$$

for all $z \in \mathcal{H}$. We set

$$\Phi_\rho(z) = f(\rho^{-1} z) j(\rho^{-1}, z)^{-k} j(\rho'^{-1}, \omega(z))^{-l}.$$

Using the relations

$$j(\rho g \rho^{-1}, z) = 1, \qquad j(\rho' \chi(g) \rho'^{-1}, \omega(z)) = 1$$

and

$$j(\alpha\beta, w) = j(\alpha, \beta w) j(\beta, w)$$

for $\alpha, \beta \in SL(2, \mathbf{R})$ and $w \in \mathcal{H}$, we obtain

$$\begin{aligned} j(\rho^{-1}, \rho g \rho^{-1} z) &= j(g\rho^{-1}, z) j(\rho g \rho^{-1}, z)^{-1} \\ &= j(g, \rho^{-1} z) j(\rho^{-1}, z) \end{aligned}$$

and

$$\begin{aligned} j(\rho'^{-1}, \omega(\rho g \rho^{-1} z)) &= j(\rho'^{-1}, \rho' \chi(g) \rho'^{-1} \omega(z)) \\ &= j(\chi(g), \rho'^{-1} \omega(z)) j(\rho'^{-1}, \omega(z)). \end{aligned}$$

Hence, we have

$$\Phi_\rho(z + h) = \Phi_\rho(\rho g \rho^{-1} z) = \Phi_\rho(z).$$

Therefore, there exists a function ϕ on the punctured unit disk $\{z \in \mathbf{C} \mid 0 < |z| < 1\}$ such that

$$\Phi_\rho(z) = \phi(e^{2\pi i z/h})$$

for all $z \in \mathcal{H}$. If the Laurent series expansion of ϕ at 0 is

$$\phi(w) = \sum_{n=N}^{\infty} a_n w^n,$$

then we obtain the expansion of Φ_ρ of the form

$$\Phi_\rho(z) = \sum_{n=N}^{\infty} a_n e^{2\pi i z/h}$$

for $z \in \mathcal{H}$ with $\operatorname{Im} z > M > 0$ for sufficiently large M. This series will be called the Fourier expansion of $f(z)$ at the cusp s. The function f is said to be holomorphic at s if $N \geq 0$, and it is said to vanish at s if $N > 0$. Now we are ready to define mixed automorphic forms.

Definition. A holomorhic function $f : \mathcal{H} \to \mathbf{C}$ is said to be a mixed automorphic form of type (k, l) if f satisfies the following conditions:

(i) $f(gz) = f(z) j(g, z)^k j(\chi(g), \omega(z))^l$ for all $g \in G$ and $z \in \mathcal{H}$.

(ii) f is holomorphic at each G-cusp.

The function f is said to be a mixed cusp form of type (k, l) if (ii) is replaced by

(ii)′ f vanishes at each G-cusp.

2. The inner product. Let G, ω, and χ be as in Section 1, and let k and l be nonnegative integers with k even. In this section we define the inner product $\langle \;\; , \;\; \rangle$ on the space $S_{k,l}(G, \omega, \chi)$ of mixed cusp forms of type (k, l) associated to G, ω, and χ, which generalizes the Petersson inner product on the usual cusp forms.

Proposition 2.1. *If $f, h \in S_{k,l}(G, \omega, \chi)$, then the integral*

$$\int_{G \backslash \mathcal{H}} f(z) \overline{h(z)} (\operatorname{Im} z)^{k-2} (\operatorname{Im} \omega(z))^l \, dx \, dy$$

is well defined.

Proof. It can be easily shown that both

$$f(z) \overline{h(z)} (\operatorname{Im} z)^k (\operatorname{Im} \omega(z))^l$$

and

$$(\operatorname{Im} z)^{-2} \, dx \, dy$$

are G-invariant. Therefore, it suffices to show that the integral converges. To prove the convergence, it is sufficient to show that $f(z) \overline{h(z)} (\operatorname{Im} z)^k (\operatorname{Im} \omega(z))^l$ is continuous as a function on the quotient space $G \backslash \mathcal{H}^*$ at the points corresponding to the cusps of G. Thus, let s be a cusp of G, and let ρ be an element of $SL(2, \mathbf{R})$ such that $\rho(s) = \infty$ as in Section 1. If $G_s = \{g \in G | gs = s\}$, then

$$\rho G_s \rho^{-1} \cdot \{\pm 1\} = \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^n \mid n \in \mathbf{Z} \right\}$$

for some $h \in \mathbf{R}$. If $q = e^{2\pi i z/h}$, there are functions $\phi(q)$ and $\psi(q)$ holomorphic at $q = 0$ such that

$$f(\rho^{-1}z)j(\rho^{-1}, z)^{-k}j((\rho')^{-1}, \omega(z))^{-l} = \phi(q)$$

and

$$h(\rho^{-1}z)j(\rho^{-1}, z)^{-k}j((\rho')^{-1}, \omega(z))^{-l} = \psi(q),$$

where ρ' is an element of $SL(2, \mathbf{R})$ associated to ρ as in Section 1. Hence, we have

$$\begin{aligned} f(\zeta)\overline{h(\zeta)}(\operatorname{Im}\zeta)^k(\operatorname{Im}\omega(\zeta))^l \\ &= f(\rho^{-1}z)\overline{h(\rho^{-1}z)}(\operatorname{Im}\rho^{-1}z)^k(\operatorname{Im}\omega(\rho^{-1}z))^l \\ &\quad \cdot \phi(e^{2\pi i z/h})\overline{\psi(e^{2\pi i z/h})}(\operatorname{Im} z)^k(\operatorname{Im}\omega(z))^l. \end{aligned}$$

Since $\phi(0) = \psi(0) = 0$, we see that this function is continuous at the point in $G\backslash\mathcal{H}^*$ corresponding to the cusp s. □

Now we define the inner product $\langle f, h\rangle$ of $f, h \in S_{k,l}(G, \omega, \chi)$ by the integral in Proposition 2.1, which is a generalization of the Petersson inner product for classical cusp forms.

3. Poincaré series. Let G, ω, and χ be as in Section 1. We denote by $S_{k,l}(G, \omega, \chi)$ the space of mixed cusp forms of type (k, l) associated to G, ω, χ. If χ is a monodromy representation of an elliptic surface E, then it is known [**6**, Theorem 3.2] that the space $S_{2,2k}(G, \omega, \chi)$ is canonically isomorphic to the space of holomorphic $(2k+1)$-forms on an elliptic variety E^{2k}. In this section we discuss Poincaré series associated to a mixed cusp form in $S_{k,l}(G, \omega, \chi)$. If $f : \mathcal{H} \to \mathbf{C}$ is a holomorphic function and $g \in G$, we set

$$(f|_{(k,l)}g)(z) = j(g, z)^{-k}j(\chi(g), \omega(z))^{-l}f(gz)$$

Proposition 3.1. *Let G_0 be a subgroup of G, and let $f : \mathcal{H} \to \mathbf{C}$ be a holomorphic function such that*

$$f|_{(2,2k)}g = f$$

for all $g \in G_0$ *and*

$$\int_{G_0 \backslash \mathcal{H}} |f(z)| (\operatorname{Im} z)(\operatorname{Im} \omega(z))^k \, dV < \infty.$$

Then

(i) *The series*

$$F(z) = \sum_{g \in G_0 \backslash G} (f|_{(2,2k)} g)(z)$$

converges absolutely on $\mathcal{H}$ *and uniformly on compact sets.*

(ii) $F(z)$ *is a mixed cusp form of type* $(2, 2k)$ *in* $S_{2,2k}(G, \omega, \chi)$.

Proof. (i) This follows from a more general statement in [**1**, Theorem 9.1].

(ii) If $h \in G$, then we have

$$\begin{aligned}
&F(hz) j(h,z)^{-2} j(\chi(h), \omega(z))^{-2k} \\
&= \sum_{g \in G_0 \backslash G} f(g(hz)) j(g, hz)^{-2} j(\chi(g), \omega(hz))^{-2k} j(h,z)^{-2} j(\chi(h), \omega(z))^{-2k} \\
&= \sum_{g \in G_0 \backslash G} f(ghz) j(gh, z)^{-2} j(\chi(gh), \omega(z))^{-2k} \\
&= \sum_{g \in G_0 \backslash G} (f|_{(2,2k)} gh)(z) = F(z).
\end{aligned}$$

The cusp condition follows from the finiteness of the integral. Hence, it follows that $F(z) \in S_{2,2k}(G, \omega, \chi)$. □

The series in Proposition 3.1(i) will be called a relative Poincaré series determined by the holomorphic function f.

Proposition 3.2. *Let* $\phi \in S_{2,2k}(G, \omega, \chi)$, *and let*

$$F(z) = \sum_{g \in G_0 \backslash G} (f|_{(2,2k)} g)(z)$$

for some holomorphic function $f : \mathcal{H} \to \mathbf{C}$ *that satisfies* $f|_{(2,2k)} g = f$ *for all* $g \in G_0$. *If* $\langle \quad , \quad \rangle$ *is the inner product defined in Section* 2, *then we have*

$$\langle \phi, F \rangle = \int_{G_0 \backslash \mathcal{H}} \phi(z) \overline{f(z)} (\operatorname{Im} z)^2 (\operatorname{Im} \omega(z))^{2k} \, dV$$

where $dV = dx\, dy / (\operatorname{Im} z)^2$.

Proof. Using the definition of the inner product $\langle \quad , \quad \rangle$, we have

$$\begin{aligned}
\langle \phi, F \rangle &= \int_{G \backslash \mathcal{H}} \phi(z) \Big(\overline{\sum_{g \in G_0 \backslash G} (f|_{(2,2k)} g)(z)} \Big) (\operatorname{Im} z)^2 (\operatorname{Im} \omega(z))^{2k} \, dV \\
&= \sum_{g \in G_0 \backslash G} \int_{G \backslash \mathcal{H}} \phi(z) \overline{f(z) j(g,z)}^{-2} \overline{j(\chi(g), \omega(z))}^{-2k} (\operatorname{Im} z)^2 \\
&\qquad\qquad \cdot (\operatorname{Im} \omega(z))^{2k} \, dV \\
&= \sum_{g \in G_0 \backslash G} \int_{G \backslash \mathcal{H}} \phi(gz) j(G,z)^{-2} j(\chi(g), \omega(z))^{-2k} \overline{f(z) j(g,z)}^{-2} \\
&\qquad\qquad \cdot \overline{j(\chi(g), \omega(z))}^{-2k} \\
&\qquad\qquad \cdot (\operatorname{Im} gz)^2 |j(g,z)|^4 (\operatorname{Im} \omega(gz))^{2k} |j(\chi(g), \omega(z))|^{4k} \, dV \\
&= \sum_{g \in G_0 \backslash G} \int_{G \backslash \mathcal{H}} \phi(gz) \overline{f(gz)} (\operatorname{Im} gz)^2 (\operatorname{Im} \omega(gz))^{2k} \, dV \\
&= \int_{G_0 \backslash \mathcal{H}} \phi(z) \overline{f(z)} (\operatorname{Im} z)^2 (\operatorname{Im} \omega(z))^{2k} \, dV. \qquad \square
\end{aligned}$$

4. Hyperbolic elements. Let g_0 be a hyperbolic element of G, and let $\chi : G \to SL_2(\mathbf{R})$ be as in Section 1. We assume that the fixed points of g_0 and the fixed points of $\chi(g_0)$ correspond in the sense described in the next paragraph.

Let $z_1, z_2 \in \mathbf{R} \cup \{i\infty\}$ be the two fixed points of g_0, and let $h \in PSL(2, \mathbf{R})$ be an element that transforms the geodesic between z_1 and z_2 to the ray $[0, i\infty]$ with $hz_1 = 0$ and $hz_2 = i\infty$. Similarly, let

$w_1, w_2 \in \mathbf{R} \cup \{i\infty\}$ be the fixed points of $\chi(g_0)$, and let $h' \in SL(2, \mathbf{R})$ be an element that transforms the geodesic between w_1 and w_2 to the ray $[0, i\infty]$ with $h'w_1 = 0$ and $h'w_2 = i\infty$. Then we assume that h and h' satisfy the relation

$$\omega(hz) = h'\omega(z)$$

for all $z \in \mathcal{H}$.

If the hyperbolic element g_0 of $G \subset PSL(2, \mathbf{R})$ can be represented by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$$

with $\operatorname{tr} g_0 > 0$, then we set

$$Q_{g_0}(z) = cz^2 + (d - a)z - b.$$

Similarly, we set

$$Q_{\chi(g_0)}(\omega(z)) = c_\chi \omega(z)^2 + (d_\chi - a_\chi)\omega(z) - b_\chi$$

if

$$\chi(g_0) = \begin{pmatrix} a_\chi & b_\chi \\ c_\chi & d_\chi \end{pmatrix} \in SL(2, \mathbf{R}).$$

Lemma 4.1. *If $\alpha \in PSL(2, \mathbf{R})$ and $\beta \in SL(2, \mathbf{R})$, then we have*

$$Q_{g_0}(\alpha z) j(\alpha, z)^2 = Q_{\alpha^{-1}\cdot g_0 \cdot \alpha}(z)$$

and

$$Q_{\chi(g_0)}(\beta\omega(z)) j(\beta, \omega(z))^2 = Q_{\beta^{-1}\cdot \chi(g_0)\cdot \beta}(\omega(z))$$

for all $z \in \mathcal{H}$.

Proof. Let g_0 and α be represented by the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix},$$

respectively, in $SL(2, \mathbf{R})$ with $a + b > 0$ and $a_1 + b_1 > 0$. Then the matrix $\alpha^{-1} \cdot g_0 \cdot \alpha$ is represented by

$$\begin{pmatrix} aa_1d_1 - a_1b_1c + bc_1d_1 - b_1c_1d & ab_1d_1 - b_1^2c + bd_1^2 - b_1dd_1 \\ -aa_1c_1 + a_1^2c - bc_1^2 + a_1c_1d & -ab_1c_1 + a_1b_1c - bc_1d_1 + a_1dd_1 \end{pmatrix}$$

and

$$\begin{aligned} Q_{g_0}(\alpha z) &= (c_1+d_1)^{-2}[c(a_1z+b_1)^2+(d-a)(a_1z+b_1)(c_1z+d) \\ &\qquad\qquad - b(c_1z+d_1)^2] \\ &= j(\alpha,z)^{-2}Q_{\alpha^{-1}\cdot g_0\cdot\alpha}(z). \end{aligned}$$

Thus, the first relation of the lemma follows. The second relation follows easily from the first. □

5. Periods of mixed cusp forms. In this section we define Poincaré series associated to a hyperbolic element of G and express the inner product of this Poincaré series and a mixed cusp form in terms of the period of the mixed cusp form. Let g_0 be a hyperbolic element of G, and, as in Section 4, let $h \in PSL(2,\mathbf{R})$ (respectively, $h' \in SL(2,\mathbf{R})$) be an element that transforms the geodesic between the fixed points of g_0 (respectively, $\chi(g_0)$) to the ray $[0, i\infty]$ with $\omega(hz) = h'(z)$, $z \in \mathcal{H}$. Then we have

$$h^{-1}\cdot g_0\cdot h = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}$$

where $\delta = \operatorname{tr} g_0 + \sqrt{(\operatorname{tr} g_0)^2-4}$. We also have

$$h'^{-1}\cdot \chi(g_0)\cdot h' = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$$

where

$$\varepsilon = \begin{cases} \operatorname{tr}(\chi(g_0)) + \sqrt{(\operatorname{tr}\chi(g_0))^2-4} & \text{if } \operatorname{tr}\chi(g_0) > 0 \\ \operatorname{tr}(\chi(g_0)) - \sqrt{(\operatorname{tr}\chi(g_0))^2-4} & \text{if } \operatorname{tr}\chi(g_0) < 0. \end{cases}$$

For $g \in G$, by Lemma 4.1 we have

$$\begin{aligned} &Q^{-k}_{\chi(g_0)}(\omega(z))Q^{-1}_{g_0}(z)|_{(2,2k)} \\ &\qquad = j(g,z)^{-2}j(\chi(g),\omega(z))^{-2k}Q^{-k}_{\chi(g_0)}(\omega(gz))Q^{-1}_{g_0}(gz) \\ &\qquad = Q^{-k}_{\chi(g^{-1}g_0g)}(\omega(z))Q^{-1}_{g^{-1}g_0g}(z). \end{aligned}$$

Thus, if $g \in G_0$, then we have

$$Q^{-k}_{\chi(g_0)}(\omega(z))Q^{-1}_{g_0}(z)|_{(2,2k)} = Q^{-k}_{\chi(g_0)}(\omega(z))Q^{-1}_{g_0}(z).$$

Now we denote by $\theta_{2,2k}(g_0,\omega,\chi)$ the Poincaré series determined by $Q^{-k}_{\chi(g_0)}(\omega(z))Q^{-1}_{g_0}(z)$, i.e.,

$$\theta_{2,2k}(g_0,\omega,\chi)=\sum_{g\in G_0\backslash G} Q^{-k}_{\chi(g_0)}(\omega(z))Q^{-1}_{g_0}(z)|_{(2,2k)}.$$

Theorem 5.1. *Let $\langle\ ,\ \rangle$ be the generalized Petersson inner product defined in Section 2, and let $\phi(z)$ be a mixed cusp form in $S_{2,2k}(G,\omega,\chi)$. Let $z=re^{i\theta}$ and $\omega(z)=Re^{i\theta}$ be the polar representations of z and $\omega(z)$. If we assume that φ is a function of θ only, then we have*

$$\langle\phi(z),\theta_{2,2k}(g_0,\omega,\chi)\rangle = C\int_{z_0}^{\gamma z_0}\phi(z)Q^{k}_{\chi(g_0)}(\omega(z))\,dz,$$

where

$$C=D_0^{1/2}(D_0')^k\left(\int_0^{2\pi}(\sin 2\varphi)^{2k}\,d\theta\right)^{-1},$$

z_0 is an arbitrary point in $\mathcal{H}$, and the integral is taken along a piecewise continuous path joining z_0 and gz_0.

Proof. By Proposition 3.2, we have

$$\begin{aligned}
&\langle\phi(z),\theta_{2,2k}(g_0,\omega,\chi)\rangle\\
&=\int\!\!\int_{\mathcal{F}}\phi(z)\overline{Q^{-k}_{\chi(g_0)}(\omega(z))Q^{-1}_{g_0}(z)}(\operatorname{Im} z)^2(\operatorname{Im}\omega(z))^{2k}(\operatorname{Im} z)^{-2}\,dx\,dy\\
&=\int\!\!\int_{\mathcal{F}_1}\phi(hz)\overline{Q^{-k}_{\chi(g_0)}(h'\omega(z))Q^{-1}_{g_0}(hz)}(\operatorname{Im} hz)^2(\operatorname{Im} h'\omega(z))^{2k}\\
&\qquad\qquad\cdot(\operatorname{Im} z)^{-2}\,dx\,dy\\
&=\int\!\!\int_{\mathcal{F}_1}\phi(hz)\overline{((-1)^{\operatorname{tr}\chi(g_0)}\sqrt{D_0'}\omega(z))}^{-k}\,\overline{j(h',\omega(z))}^{\,2k}(\overline{\sqrt{D_0}z})^{-1}\overline{j(h,z)}^{2}\\
&\qquad\cdot|j(h,z)|^{-4}(\operatorname{Im} z)^2|j(h',\omega(z))|^{-4k}(\operatorname{Im}\omega(z))^{2k}(\operatorname{Im} z)^{-2}\,dx\,dy,
\end{aligned}$$

where $\mathcal{F}$ is a fundamental domain of G in $\mathcal{H}$, $\mathcal{F}_1=h^{-1}\mathcal{F}$, $D_0=(\operatorname{tr} g_0)^2$ and $D_0'=(\operatorname{tr}\chi(g_0))^2-4$. Thus, if

$$C_1=(-1)^{k\operatorname{tr}\chi(g_0)}(D_0')^{-k/2}(D_0)^{-1/2},$$

then the last integral becomes

$$
\begin{aligned}
&C_1 \int\int_{\mathcal{F}_1} \phi(hz)\overline{\omega(z)}^{\,-k}\bar{z}^{-1} j(h',\omega(z))^{-4k} j(h,z)^{-2} (\operatorname{Im}\omega(z))^{2k}\, dx\, dy \\
&= C_1 \int\int_{\mathcal{F}_1} \phi(hre^{i\theta})\rho^{-k}(e^{-i\varphi})^{-k}(re^{-i\theta})^{-1} j(h',\omega(z))^{-2k} \\
&\qquad\qquad \cdot j(h,z)^{-2}\rho^{2k}(\sin\varphi)^{2k} r\, dr\, d\theta \\
&= C_1 \int\int_{\mathcal{F}_1} \phi(hre^{i\theta})\rho^{k} e^{-ik\varphi} j(h',\omega(z))^{-2k} j(h,z)^{-2} (\sin\varphi)^{2k} e^{i\theta}\, dr\, d\theta \\
&= C_1 \int\int_{\mathcal{F}_1} \phi(hz)\omega(z)^{k} j(h',\omega(z))^{-2k} j(h,z)^{-2} (\sin\varphi)^{2k}\, dz\, d\theta \\
&= C_1 C_2 \int_0^{\infty} \phi(hz)\omega(z)^{k} j(h',\omega(z))^{-2k} j(h,z)^{-2}\, dz,
\end{aligned}
$$

where

$$C_2 = \int_0^{2\pi} (\sin 2\varphi)^{2k}\, d\theta.$$

On the other hand, if z_0 is a point in $\mathcal{H}$, we have

$$
\begin{aligned}
&\int_{z_0}^{\gamma z_0} \phi(z) Q^k_{\chi(g_0)}(\omega(z))\, dz \\
&= \int_0^{i\infty} \phi(hz) Q^k_{\chi(g_0)}(\omega(hz)) j(h,z)^{-2}\, dz \\
&= \int_0^{i\infty} \phi(hz)((-1)^{\operatorname{tr}\chi(g_0)}\sqrt{D_0'}\,\omega(z))^k (j(h',\omega(z))^{-2})^{-k} j(h,z)^{-2}\, dz \\
&= C_3 \int_0^{i\infty} \phi(hz)(\omega(z)^k) j(h',\omega(z))^{-2k} j(h,z)^{-2}\, dz,
\end{aligned}
$$

where

$$C_3 = (-1)^{k \operatorname{tr}\chi(g_0)} (D_0')^{k/2}.$$

Now the theorem follows from the above computations. □

6. The Kronecker pairing. Let V be the space of the standard two-dimensional complex representation of $SL(2,\mathbf{R})$, and let V^* be its dual; let $S^{2k}V$ be the $2k^{\text{th}}$ symmetric power of V, and let $(S^{2k}V)^*$ be its dual. If $\{u_1, u_2\}$ is the standard basis for V^*, then $(S^{2k}V)^*$ can be identified with the space of all homogeneous polynomials $P_{2k}(u_1, u_2)$ of degree $2k$ in u_1 and u_2. Let $\chi : G \to SL(2, \mathbf{R})$ and $\omega : \mathcal{H} \to \mathcal{H}$ be as in Section 2. Then χ induces an action of $G \subset PSL(2, \mathbf{R})$ on V and on $S^{2k}V$. G also acts on $(S^{2k}V)^*$ by

$$g \cdot P_{2k}(u_1, u_2) = P_{2k}(d_\chi u_1 - b_\chi u_2, -c_\chi u_1 + a_\chi u_2)$$

where

$$\chi(g) = \begin{pmatrix} a_\chi & b_\chi \\ c_\chi & d_\chi \end{pmatrix}.$$

Thus, we can consider the cohomology $H^1(G, S^{2k}V)$ and the homology $H_1(G, (S^{2k}V)^*)$ with respect to the actions of G on $S^{2k}V$ and on $(S^{2k}V)^*$ described above. Then there is a canonical injective homomorphism

$$\Psi : S_{2,2k}(G, \omega, \chi) \to H^1(G, S^{2k}V).$$

We shall denote by $\langle\langle \quad , \quad \rangle\rangle$ the canonical pairing

$$H^1(G, S^{2k}V) \otimes H_1(G, (S^{2k}V)^*) \to \mathbf{C},$$

called the Kronecker pairing (see [**4**, p. 738 and p. 745]). We set

$$\mathcal{Q}_g^k = (c_\chi u_1^2 + (d_\chi - a_\chi) u_1 u_2 - b_\chi u_2^2)^k \in (S^{2k}V)^*.$$

Then $g \otimes \mathcal{Q}_G^k$ is a cycle in $H_1(G, (S^{2k}V)^*)$ (see [**7**, Lemma 3.3]), and we have the following theorem:

Theorem 6.1. *If $g \in G$ and $f \in S_{2,2k}(G, \omega, \chi)$, then we have*

$$\langle\langle \Psi(f), g \otimes \mathcal{Q}_g^k \rangle\rangle = \int_{z_0}^{g z_0} f(z)(c_\chi \omega(z)^2 + (d_\chi - a_\chi)\omega(z) - b_\chi)^k \, dz,$$

where z_0 is an arbitrary point in the Poincaré upper half plane $\mathcal{H}$ and the integral is taken along any piecewise continuous path joining z_0 and gz_0.

Proof. See [**7**, Theorem 3.4]. □

Corollary 6.2. *If ρ is a parabolic element of G and $f \in S_{2,2k}(G, \omega, \chi)$, then*

$$\langle\langle \Psi(f), \rho \otimes \mathcal{Q}_\rho^k \rangle\rangle = 0.$$

Proof. This follows easily by considering the limit of the integral in Theorem 6.1 as z_0 approaches the fixed point of ρ. □

Theorem 6.3. *Let Γ be a subset of G, and let $\mathcal{S}$ be the set of cycles in*

$$H_1(G, (S^{2k}V)^*)$$

of the form $\gamma \otimes \mathcal{Q}_\gamma^k$ for $\gamma \in \Gamma$. Suppose that ω satisfies the condition described in Theorem 5.1 and that $\mathcal{S}$ spans $H_1(G, (S^{2k}V)^*)$. Then the set of Poincaré series $\Theta = \{\theta_{2,2k}(g_0, \omega, \chi) \mid g_0 \in \mathcal{S}\}$ spans $S_{2,2k}(G, \omega, \chi)$.

Proof. Let W be the subspace of $S_{2,m}(G, \omega, \chi)$ spanned by the set Θ, and suppose that $f \in S_{2,m}(G, \omega, \chi)$ is orthogonal to W relative to the generalized Petersson inner product $\langle \ , \ \rangle$ defined in Section 2. It suffices to show that $f = 0$. Using Theorems 5.1 and 6.1, if g_0 is a hyperbolic element of G, then we have

$$\begin{aligned} 0 = \langle \phi(z), \theta_{2,2k}(g_0, \omega, \chi) \rangle &= C \int_{z_0}^{\gamma z_0} \phi(z) Q_{\chi(g_0)}^k(\omega(z))\, dz \\ &= \langle\langle \Psi(f), g_0 \otimes \mathcal{Q}_{g_0}^k \rangle\rangle. \end{aligned}$$

Thus, $\Psi(f)$ is orthogonal to all hyperbolic cycles in $\mathcal{S}$. By Corollary 6.2, $\Psi(f)$ is also orthogonal to parabolic cycles in $\mathcal{S}$. Since the Kronecker pairing $\langle\langle \ , \ \rangle\rangle$ is nondegenerate, we have $\Psi(f) = 0$ (note that G and $\chi(G)$ do not contain elliptic elements by our assumption). Since Ψ is injective, it follows that $f = 0$. □

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