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TWO WEIGHTED (L^p, L^q) ESTIMATES FOR THE FOURIER TRANSFORM

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0. Introduction and notation. In this paper we continue to study the two-weight problem for the Fourier transform. The problem is for given p and q with $1 < p \le q < \infty$, to determine necessary and sufficient conditions on w and v so that

$$(0.1) \qquad \left(\int_{\mathbf{R}^n} |\hat{f}(x)|^q w(x)\, dx \right)^{p/q} \le C \int_{\mathbf{R}^n} |f|^p v(x)\, dx,$$

where C is a positive constant independent of f.

In the case where w and $1/v$ are radial and symmetrically decreasing, this was completely solved in Theorem 2 of [**4**]. There the Fourier transform problem was reduced to the two-weight problem for the Hardy operator.

In two dimensions ($n = 2$) for $p = q = 2$, Kerman and Sawyer solved the problem when w and $1/v$ are symmetrically decreasing in each of their variables. Here they showed that the Fourier transform problem can be reduced to a two-weight problem for the two-dimensional Hardy operator, solved by Sawyer in [**6**], where he also presented these results.

Heinig and Sinnamon in [**2**] were able to generalize these results of Kerman and Sawyer to n-dimensions for conjugate exponents, where w decreases in each of its variables and v has the special form $v(x) = w(1/x)^{p/q}$ (note $1/x = (1/x_1, 1/x_2, \dots, 1/x_n)$). Furthermore, the necessary and sufficient conditions they obtain are quite easy to apply.

We obtain the following results. In Section 2, in n-dimension, for weights w and $1/v$ that are symmetrically decreasing in each of their variables, we reduce the Fourier transform problem to a two-weight problem for the n-dimensional Hardy operator. Here $p \le q$, with q a positive even integer. We should point out, though, that the Hardy problem is still open in three or higher dimensions.

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For $x \in \mathbf{R}^n$, we take $x = (y_1, y_2)$ with $y_i \in \mathbf{R}^{l_i}$, where l_1 and l_2 are nonnegative integers summing up to n. We say that a real-valued function $f(x)$ defined on $\mathbf{R}^n$ is symmetrically decreasing in the variables y_1 and y_2 if

$$(0.2) \quad \begin{cases} \text{(a)} \quad f(y_1, y_2) \text{ is radial in each of the variables } y_1, y_2, \text{ and} \\ \text{(b)} \quad f(|y_1|, |y_2|) \text{ is nonincreasing in each of its variables.} \end{cases}$$

In n-dimensions, when $1 < p \le q < \infty$, with q even, for weights w and $1/v$ that are symmetrically decreasing in the variables y_1 and y_2, we reduce the Fourier transform problem to a two-weight problem for the two-dimensional Hardy operator, and apply Sawyer's characterization (in [**6**]) of such weights. This generalization of the Kerman-Sawyer work is done in Section 3.

In n-dimensions and for all $1 < p \le q < \infty$, when $v(x)$ has the specific form

$$v(x) = (|y_1|^{l_1} |y_2|^{l_2})^{p(1-1/p-1/q)} w\left(\frac{1}{|y_1|}, \frac{1}{|y_2|}\right)^{p/q}$$

and is increasing in the y_i-variables, we solve the Fourier transform problem. This extends the Heinig-Sinnamon results, and like their results, the conditions obtained are quite easy to apply. In a similar fashion, we can solve the Fourier transform problem whenever w and $1/v$ are symmetrically decreasing in any number of variables $y_1, \dots, y_k$, with $k \le n$. This result extends Theorems 4.9 and 4.10 of [**2**] and Theorem 3 of [**4**]. This material is all presented in Section 4.

Finally, in Section 5, we obtain some further special necessary and sufficient conditions for the Fourier transform problem.

Let $f : \mathbf{R}^n \to \mathbf{C}$. If f is radial, i.e., if $f(x) = f(|x|)$, and if $s > 0$ then $\int_{|x| \le s} f(x)\, dx$ is a standard radial integral, meaning that

$$\int_{|x| \le s} f(x)\, dx = \gamma_n \int_0^s f(r) r^{n-1}\, dr,$$

where γ_n is the volume of the unit ball in $\mathbf{R}^n$.

When we suppose that $f(x)$ is radial in each of its variables, as we do in Section 2, then we write $\int_{|x| \le s} f(x)\, dx$ to denote $\int_{|x_1| \le s_1} dx_1 \cdots$

$\int_{|x_n|\le s_n} dx_n f(x)$, where $s = (s_1, s_2, \dots, s_n)$ with each $s_i > 0$. We define $\int_{|x|\ge s} f$ and $\int_{|x|\le 1/s} f$ similarly, where now $1/s = (1/s_1, \dots, 1/s_n)$.

When we deal with functions $f(x)$ radial in the variables $(y_1, y_2) \in \mathbf{R}^n$, with each $y_i \in \mathbf{R}^{l_i}$, we will write $\int_{|x|\le s} f(x)\,dx$ for $\int_{|y_1|\le s_1} dy_1 \cdot \int_{|y_2|\le s_2} dy_2 f(x)$ with these corresponding y_1 and y_2 integrals to be interpreted as the radial integrals described above. This notation will be used frequently in Sections 1, 3 and 4.

Let $f : \mathbf{R}^n \to \mathbf{C}$ and $\mu_f(s) = |\{x \in \mathbf{R}^n : |f(x)| \ge s\}|$ be finite for each $s > 0$. We let f^* denote the usual (one-dimensional) decreasing rearrangement of $|f|$. In particular, notice that

$$\mu_f(s) = |\{t \in (0,\infty) : f^*(t) \ge s\}| \qquad \text{for all } s > 0.$$

We define the radial (or symmetric) decreasing rearrangement of f by

$$\bar{f}(x) = f^*(\gamma_n |x|^n).$$

Throughout, we note that $1/p + 1/p' = 1$, and c, c_1, c_2, and so on, will denote positive constants not necessarily the same in each occurrence. We use $P \ll Q$ to abbreviate an inequality $|P| \le c|Q|$, with c independent of the specified variable quantities, and we use $P \propto Q$ to indicate $P \ll Q$ and $Q \ll P$.

1. Local L^q-estimates. Throughout this section, we let $x = (y_1, y_2)$, with the y_i's $\in \mathbf{R}^{l_i}$. By $\tilde{f}(x)$ we mean a specific function that is radial and decreasing in each of the variables y_1 and y_2, defined as follows. If $f_1 = R_1 f$ represents the radial decreasing rearrangement of $f(\cdot, y_2)$ with respect to the first variable, and if $f_{12} = R_2 R_1 f$ is the radial decreasing rearrangement of $f_1(y_1, \cdot)$ with respect to the second variable, then we denote the function f_{12} by $\tilde{f}$ (see, for instance, Section 4 of [**2**]). Similarly, we denote f_{21} as $\tilde{\tilde{f}}$.

We also define (for reasonable functions φ),

$$A\varphi(x) = \frac{1}{|y_1|^{l_1} |y_2|^{l_2}} \int_{|t|\le|x|} \varphi(t)\,dt, \tag{1.1}$$

where x is, as usual, (y_1, y_2).

We recall the well-known Riesz rearrangement theorem and its generalization to n-dimensions by Sobolev [**8**] which says: for r, f_1 and $f_2 \in L^1(\mathbf{R}^n)$, with r, f_1, and $f_2 \geq 0$ that

$$\int_{\mathbf{R}^n} r(x)(f_1 * f_2)(x)\,dx \leq c \int_{\mathbf{R}^n} \bar{r}(x)(\bar{f}_1 * \bar{f}_2)(x)\,dx \tag{1.2}$$

where the bar represents the radial, decreasing rearrangement of the function in $\mathbf{R}^n$.

It is easy to see that we obtain from (1.2),

$$\int_{\mathbf{R}^n} r(x)(f_1 * f_2 * \cdots * f_n)(x)\,dx \leq c \int_{\mathbf{R}^n} \bar{r}(x)(\bar{f}_1 * \cdots * \bar{f}_n)(x)\,dx.$$

It follows from these results that

(1.3)

$$\begin{cases} \text{(a)} \quad \displaystyle\int_{\mathbf{R}^n} r(x)(f_{1*}f_{2*}\cdots{}_*f_n)(x)\,dx \leq c \int_{\mathbf{R}^n} \tilde{r}(x)(\tilde{f}_{1*}\tilde{f}_{2*}\cdots{}_*\tilde{f}_n)(x)\,dx \\ \text{(b)} \quad \displaystyle\int_{\mathbf{R}^n} (f_{1*}f_{2*}\cdots{}_*f_n)^2\,dx \leq c \int_{\mathbf{R}^n} (\tilde{f}_{1*}\tilde{f}_{2*}\cdots{}_*\tilde{f}_n)^2\,dx. \end{cases}$$

We obtain from (6.1) and (6.5) of [**5**], that

Proposition 1.1. *For $u > 0$ and $q \geq 2$, let $f \in L^2(\mathbf{R}^n) \cap L^q(\mathbf{R}^n)$. Then*

$$\int_{|x|\leq u} \bar{\hat{f}}(x)^q\,dx \leq c \int_{|x|\leq u} \left(\int_{|t|\leq \frac{1}{|x|}} \bar{f}(t)\,dt \right)^q dx.$$

Here, the bar denotes the radial, decreasing rearrangement, and the integrals that appear are all standard radial integrals.

Throughout, we let $\chi_E(x)$ denote the characteristic function of the measurable set E.

Proposition 1.2. *Suppose that $q \geq 2$, $\tilde{\chi}_E(x) \leq \chi(|y_2| \leq u_2)$, $\tilde{\tilde{\chi}}_E(x) \leq \chi(|y_1| \leq u_1)$, and let $u = (u_1, u_2) \in \mathbf{R}^2_+$. Then for $g \in L^2(\mathbf{R}^n) \cap L^q(\mathbf{R}^n)$,*

$$\begin{aligned} I &= \int_{\mathbf{R}^n} |\hat{g}(x)|^q \chi_E(x)\,dx \\ &\leq c \int_{|y_2|\leq u_2} dy_2 \int dy_1 \tilde{\chi}_E(x) \left[\int_{|x_1'|\leq \frac{1}{|y_1|}} dx_1' \int_{\mathbf{R}^{l_2}} dx_2' \tilde{g}(t) \right]^q \end{aligned} \tag{1.4}$$

and

$$(1.5)\quad I \le c\int_{|y_1|\le u_1} dy_1 \int_{\mathbf{R}^{l_2}} dy_2 \tilde{\tilde{\chi}}_E(x)\left[\int_{|x_2'|\le \frac{1}{|y_2|}} dx_2' \int_{\mathbf{R}^{l_1}} dx_1' \tilde{g}(t)\right]^q.$$

Proof.

$$I = \int_{\mathbf{R}^{l_2}} dy_2 \int_{\mathbf{R}^{l_1}} dy_1 \chi_E(x)\left|\int_{\mathbf{R}^{l_1}} dx_1' e^{\iota x_1'\cdot y_1} h_{y_2}(x_1')\right|^q,$$

where $h(x_1') = h_{y_2}(x_1') = \int_{\mathbf{R}^{l_2}} g(t)e^{\iota x_2'\cdot y_2}\, dx_2'$, and $t = (x_1', x_2')$. Thus we shall estimate

$$I = \int_{\mathbf{R}^{l_2}}\int_{\mathbf{R}^{l_1}} \chi_E(x)|\hat{h}(y_1)|^q\, dy_1\, dy_2.$$

For fixed y_2, $\chi(\cdot, y_2)$ is the characteristic function of some set in $\mathbf{R}^{l_1}$, and so, by Proposition 1.1, it follows that

$$\int_{\mathbf{R}^{l_1}} \chi_E(x)|\hat{h}(y_1)|^q \le c\int_{\mathbf{R}^{l_1}} dy_1 R_1\chi_E(x)\left[\int_{|x_1'|\le\frac{1}{|y_1|}} R_1 h(x_1')\, dx_1\right]^q.$$

We can safely assume that $g \ge 0$. In that case,

$$R_1 h(x_1') \le R_1\left(\int_{\mathbf{R}^{l_2}} g(x_1', x_2')\, dx_2'\right).$$

Therefore,

$$\begin{aligned} I &\le c\int_{\mathbf{R}^{l_1}} dy_1\left[\int_{|x_1'|\le\frac{1}{|y_1|}} R_1\left(\int_{\mathbf{R}^{l_2}} g\, dx_2'\right) dx_1'\right]^q \int_{\mathbf{R}^{l_2}} dy_2 R_1\chi_E(x)\\ &= c\int_{\mathbf{R}^{l_1}} dy_1\left[\int_{|x_1'|\le\frac{1}{|y_1|}} R_1\left(\int_{\mathbf{R}^{l_2}} g\, dx_2'\right) dx_1'\right]^q \int_{\mathbf{R}^{l_2}} dy_2 \tilde{\chi}_E(x), \end{aligned}$$

and now, by hypothesis, since $\tilde{\chi}_E(x) = 0$, if $|y_2| > u_2$,

(1.6)

$$I \le c\int_{|y_2|\le u_2} dy_2 \int_{\mathbf{R}^{l_1}} dy_1 \tilde{\chi}_E(x)\left[\int_{|x_1'|\le\frac{1}{|y_1|}} R_1\left(\int_{\mathbf{R}^{l_2}} g\, dx_2'\right) dx_1'\right]^q.$$

Fix a set $B = B_{y_1}$ in $\mathbf{R}^{l_1}$ with measure less than or equal to the measure of the ball of radius $1/|y_1|$. Then,

$$\begin{aligned}\int_{\mathbf{R}^{l_1}} dx_1' \chi_B(x_1') \int_{\mathbf{R}^{l_2}} g\,dx_2' &= \int_{\mathbf{R}^{l_2}} \int_{\mathbf{R}^{l_1}} \chi_B g\,dx_1'\,dx_2' \\ &\le \int_{\mathbf{R}^{l_2}} \int_{\mathbf{R}^{l_1}} R_1\chi_B(x_1') R_1 g\,dx_1'\,dx_2' \\ &= \int_{\mathbf{R}^{l_1}} R_1\chi_B(x_1')\,dx_1' \int_{\mathbf{R}^{l_2}} R_1 g\,dx_2' \\ &= \int_{\mathbf{R}^{l_1}} R_1\chi_B \int_{\mathbf{R}^{l_2}} \tilde{g}.\end{aligned}$$

In other words,

$$\int_{\mathbf{R}^{l_1}} \chi_B(x_1')\,dx_1' \int_{\mathbf{R}^{l_2}} g\,dx_2' \le \int_{|x_1'|\le \frac{1}{|y_1|}} \int_{\mathbf{R}^{l_2}} \tilde{g}\,dx_2'\,dx_1', \tag{1.7}$$

and so (1.4) follows from (1.6) and (1.7).

In order to see (1.5), we obtain instead that

$$I \le c\int_{\mathbf{R}^{l_1}} dy_1 \int_{\mathbf{R}^{l_2}} R_2\chi_E(x)\left[\int_{|x_2'|\le\frac{1}{|y_2|}} dx_2' R_2\left(\int_{\mathbf{R}^{l_1}} g\,dx_1'\right)\right]^q, \tag{1.8}$$

but since this time

$$\int_{|x_2'|\le\frac{1}{|y_2|}} dx_2' R_2\left(\int_{\mathbf{R}_{l^1}} g\,dx_1'\right) = \sup \int_{\mathbf{R}^{l_2}} \chi_B(x_2') \int_{\mathbf{R}^{l_1}} g\,dx_1'\,dx_2',$$

where the supremum is over all sets B with $R_2\chi_B \le \chi(|x_2'| \le 1/|y_2|)$. But

$$\begin{aligned}\int_{\mathbf{R}^{l_2}} \chi_B \int_{\mathbf{R}^{l_1}} g &= \int_{\mathbf{R}^{l_2}} \chi_B \int_{\mathbf{R}^{l_1}} R_1 g \\ &= \int_{\mathbf{R}^{l_1}} \int_{\mathbf{R}^{l_2}} \chi_B R_1 g \le \int_{\mathbf{R}^{l_1}} \int_{|x_2'|\le\frac{1}{|y_2|}} \tilde{g},\end{aligned}$$

and putting this estimate into (1.8), we obtain our result. □

Our next result generalizes Proposition 4.5 of [**2**] as well as Theorem 1 of [**5**]. We view it as one of the main results of this paper.

Lemma 1.3. *For $u = (u_1, u_2)$ with $u_i > 0$, and for $f \in L^1 \cap L^q(\mathbf{R}^n)$, where $q = 2, 4, 6, \dots,$ we have*

$$U = \sup_E \int_{\mathbf{R}^n} |\hat{f}|^q \chi_E(x)\, dx \le c \int_{|x|\le u} \left(\int_{|t|\le 1/|x|} \tilde{f}\, dt \right)^q dx, \tag{1.9}$$

where the supremum is over all sets E for which $\tilde{\chi}_E(x) \le \chi(|y_2| \le u_2)$, $\tilde{\tilde{\chi}}_E(x) \le \chi(|y_1| \le u_1)$, and $|E| \le c u_1^{l_1} \cdot u_2^{l_2}$.

Proof. We first show that

$$\begin{aligned} U \le c\{ u_1^{l_1} \cdot u_2^{l_2} &\left(\int_{|x|\le \frac{1}{u}} \tilde{f}(x)\, dx \right)^q \\ &+ \int_{\mathbf{R}^{l_1}} dy_1 \int_{|y_2|\le u_2} dy_2 \tilde{\chi}_E(x) \left[\int_{|x_2'|\le \frac{1}{u_2}} dx_2' \int_{|x_1'|\le \frac{1}{|y_1|}} dx_1' \tilde{f}(t) \right]^q \\ &+ \int_{\mathbf{R}^{l_2}} dy_2 \int_{|y_1|\le u_1} dy_1 \tilde{\tilde{\chi}}_E(x) \left[\int_{|x_1'|\le \frac{1}{u_1}} dx_1' \int_{|x_2'|\le \frac{1}{|y_2|}} dx_2' \tilde{f}(t) \right]^q \\ &+ \int_{|x|\ge \frac{1}{u}} \tilde{f}(x)^q (|y_1|^{l_1} |y_2|^{l_2})^{q-2}\, dx \}, \end{aligned} \tag{1.10}$$

and next we show that the right hand side of (1.10) is below the right side of (1.9).

For $f \ge 0$, we define

$$\tilde{f}_1(x) = \begin{cases} \tilde{f}(x) & \text{if } |y_i| \le 1/u_i \text{ for } i = 1 \text{ and } 2, \\ 0 & \text{elsewhere} \end{cases}$$

$$\tilde{f}_2(x) = \begin{cases} \tilde{f}(|y_1|, |y_2| + 1/u_2) & \text{if } |y_1| < 1/u_1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\tilde{f}_3(x) = \begin{cases} \tilde{f}(|y_1| + \frac{1}{u_1}, |y_2|) & \text{if } |y_2| < 1/u_2, \\ 0 & \text{elsewhere} \end{cases}$$

and

$$\tilde{f}_4(x) = \tilde{f}(|y_1| + 1/u_1, |y_2| + 1/u_2)$$

in such a way that $f(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x)$. Then we have

$$\int_{\mathbf{R}^n} |\hat{f}(x)|^q \chi_E(x)\, dx \leq c \sum_{i=1}^{4} \int_{\mathbf{R}^n} |\hat{f}_i|^q \chi_E(x)\, dx = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.$$

First observe that

$$\mathrm{I} \leq c u_1^{l_1} u_2^{l_2} \left(\int_{\mathbf{R}^n} f_1(x)\, dx \right)^q = c u_1^{l_1} u_2^{l_2} \left(\int_{|x| \leq \frac{1}{u}} \tilde{f}(x)\, dx \right)^q.$$

Next, using Proposition 1.2, where we employ (1.5) for II and (1.4) for III, we obtain

$$\begin{aligned} \mathrm{II} + \mathrm{III} \leq\ & c \int_{|y_1| \leq u_1} dy_1 \int_{\mathbf{R}^{l_2}} dy_2 \tilde{\chi}_E(x) \\ & \cdot \left[\int_{|x_2'| \leq \frac{1}{|y_2|}} dx_2' \int_{|x_1'| \leq \frac{1}{u_1}} dx_1' \tilde{f}\left(|x_1'|, |x_2'| + \frac{1}{u_2} \right) \right]^q \\ & + c \int_{|y_2| \leq u_2} dy_2 \int_{\mathbf{R}^{l_1}} dy_1 \tilde{\tilde{\chi}}_E(x) \\ & \cdot \left[\int_{|x_1'| \leq \frac{1}{|y_1|}} dx_1' \int_{|x_2'| \leq \frac{1}{u_2}} dx_2' \tilde{f}\left(|x_1'| + \frac{1}{u_1}, |x_2'| \right) \right]^q. \end{aligned}$$

Next, to estimate

$$\begin{aligned} \mathrm{IV} &= \int_{\mathbf{R}^n} |\hat{f}_4(x)|^q \chi_E(x)\, dx \leq \int_{\mathbf{R}^n} |\hat{f}_4(x)|^q\, dx \\ &= \int_{\mathbf{R}^n} (f_4 * f_4 * \cdots * f_4)^2\, dx, \end{aligned}$$

where f_4 is convolved with itself n-times ($q = 2n$, here is where we need that q is even). Thus, by (1.3)(b) we get that

$$\begin{aligned} \mathrm{IV} &\leq \int_{\mathbf{R}^n} (\tilde{f}_4 * \cdots * \tilde{f}_4)^2\, dx = \int_{\mathbf{R}^n} |\hat{\tilde{f}}_4|^q\, dx \\ &\leq c \int_{\mathbf{R}^n} (\tilde{f}_4)^q (|y_1|^{l_1} |y_2|^{l_2})^{q-2}\, dx \end{aligned}$$

where we have iterated the standard inequality due to Hardy-Littlewood-Paley, namely,

$$\int_{\mathbf{R}^n} |\hat{g}(x)|^q \, dx \le c \int |g|^q |x|^{n(q-2)} \, dx.$$

Thus,

$$\mathrm{IV} \le c \int_{\mathbf{R}^n} \tilde{f}^q \left(|y_1| + \frac{1}{u_1}, |y_2| + \frac{1}{u_2} \right) (|y_1|^{l_1} |y_2|^{l_2})^{q-2} \, dx,$$

or

$$\mathrm{IV} \le c \int_{|x| \ge \frac{1}{u}} \tilde{f}^q(x) (|y_1|^{l_1} |y_2|^{l_2})^{q-2} \, dx,$$

and now putting these estimate for I, II, III and IV together, we obtain (1.10).

Next, to see that the right side of (1.10) is below the right side of (1.9), notice that

$$\int_{|x| \le u} \left(\int_{|t| \le \frac{1}{|x|}} \tilde{f}(t) \, dt \right)^q dx \ge c u_1^{l_1} u_2^{l_2} \left(\int_{|t| \le \frac{1}{u}} \tilde{f}(t) \, dt \right)^q \ge cI.$$

Also,

$$\begin{aligned}
\mathrm{III} &\le \int_{\mathbf{R}^{l_1}} dy_1 \int_{|y_2| \le u_2} \tilde{\chi}_E(x) \left[\int_{|x_2'| \le \frac{1}{u_2}} dx_2' \int_{|x_1'| \le \frac{1}{|y_1|}} dx_1' \tilde{f}(t) \right]^q \\
&= \int_{|y_2| \le u_2} dy_2 \left(\int_{|y_1| \le u_1} + \int_{|y_1| > u_1} \right) \tilde{\chi}_E(x) \left[\iint \tilde{f}(t) \right]^q \\
&\le \int_{|y_2| \le u_2} dy_2 \int_{|y_1| \le u_1} \left[\int_{|x_2'| \le \frac{1}{u_2}} dx_2' \int_{|x_1'| \le \frac{1}{|y_1|}} dx_1' \tilde{f}(t) \right]^q \\
&\quad + \int_{|y_2| \le u_2} dy_2 \int_{|y_1| \ge u_1} dy_1 \tilde{\chi}_E(x) \left[\int_{|x_2'| \le \frac{1}{u_2}} dx_2' \int_{|x_1'| \le \frac{1}{u_1}} dx_1' \tilde{f}(t) \right]^q
\end{aligned}$$

$$\begin{aligned}
&= \int_{|y_2|\le u_2} dy_2 \int_{|y_1|\le u_1} \left[\int_{|x_2'|\le \frac{1}{u_2}} dx_2' \int_{|x_1'|\le \frac{1}{|y_1|}} dx_1' \tilde{f}(t) \right]^q \\
&\quad + c u_1^{l_1} u_2^{l_2} \left(\int_{|x_1'|\le \frac{1}{|y_1|}} dx_1' \int_{|x_2'|\le \frac{1}{u_2}} dx_2' \tilde{f}(t) \right)^q \\
&\le c u_2^{l_2} \int_{|y_1|\le u_1} \left[\int_{|x_2'|\le \frac{1}{u_2}} dx_2' \int_{|x_1'|\le \frac{1}{|y_1|}} dx_1' \tilde{f}(t) \right]^q \\
&\le c \int_{|x|\le u} \left[\int_{|t|\le \frac{1}{|x|}} \tilde{f}(t)\, dt \right]^q dx.
\end{aligned}$$

A similar argument gives this same estimate for II. Furthermore,

$$\begin{aligned}
\int_{|x|\le u} \left[\int_{|t|\le \frac{1}{|x|}} \tilde{f}(t)\, dt \right]^q dx &\ge c \int_{|x|\le u} \tilde{f}^q \left(\frac{1}{|y_1|}, \frac{1}{|y_2|} \right) (|y_1|^{l_1} |y_2|^{l_2})^q \, dx \\
&\ge c\, \mathrm{IV}.
\end{aligned}$$

Putting all these estimates together, we get our result. $\square$

It follows immediately from Lemma 1.3 that

$$U = \sup_E \int_E |\hat{f}(x)|^q \, dx \le c \int_{|x|\ge \frac{1}{u}} (A\tilde{f}(x))^q (|y_1|^{l_1} |y_2|^{l_2})^{q-2} \, dx, \tag{1.12}$$

where the supremum is taken over the admissible sets E (as described in the lemma) and this holds for all $u \in \mathbf{R}_+^2$.

Let $t = (x_1', x_2')$. We define $\chi(t;x)$ to be either zero or one (i.e., for each fixed t, $\chi(t;x)$ is a characteristic function), and say it is admissible, if for each t,

(i) $\int \chi(t;x)\, dx \le c|x_1'|^{l_1} |x_2'|^{l_2}$, and

(ii) both $\tilde{\chi}(t;x) \le \chi(|y_2| \le |x_2'|)$ and $\chi(t;x) \le \chi(|y_1| \le |x_1'|)$.

Next, consider weights $w(x)$ and $w_1(x)$ such that

$$\begin{cases} \text{(a)} \quad w(x) = \displaystyle\int_{\mathbf{R}^{l_1}} dx_1' \int_{\mathbf{R}^{l_2}} dx_2' \omega(t) \chi(t;x) \\ \text{(b)} \quad w_1(x) = \displaystyle\int_{|t|\ge |x|} \omega(t)\, dt \end{cases} \tag{1.13}$$

for some $\omega(t) \geq 0$.

Notice that if, for each t, $\chi(t;x) = \chi(x_1';y_1)\chi(x_2';y_2)$, where $\int_{\mathbf{R}^{l_i}} \chi(x_i';y_i)\,dy_i \leq c|x_i|^{l_i}$, for $i = 1$ and 2, then $\chi(t;x)$ is admissible. Furthermore, if $\chi(t;x) = \chi(|y_1| \leq |x_1'|)\chi(|y_2| \leq |x_2'|)$, then it is admissible, and in this case, $w(x) = w_1(x)$.

We end this section with the following result.

Theorem 1.4. *If q is a positive even integer, and if $w(x)$, $w_1(x)$ satisfy* (1.13), *then*

$$\int_{\mathbf{R}^n} |\hat{f}(x)|^q w(x)\,dx \leq c \int_{\mathbf{R}^n_+} (A\tilde{f}(x))^q (|y_1|^{l_1}|y_2|^{l_2})^{q-2} w_1\left(\frac{1}{|w|}\right) dx.$$

Proof. From (1.13)(a), we have

$$\begin{aligned}\int_{\mathbf{R}^n} |\hat{f}(x)|^q w(x)\,dx &= \int_{\mathbf{R}^n} |\hat{f}(x)|^q \int_{\mathbf{R}^n} \omega(t)\chi(t;x)\,dt\,dx \\ &= \int_{\mathbf{R}^n} \omega(t) \int_{\mathbf{R}^n} |\hat{f}(x)|^q \chi(t;x)\,dx\,dt.\end{aligned}$$

This χ, of course, is admissible. Thus, by Lemma 1.3 with $u = t$

$$\begin{aligned}\int_{\mathbf{R}^n} |\hat{f}(x)|^q w(x)\,dx &\leq c \int_{\mathbf{R}^n} \omega(t)\,dt \int_{|x|\leq|t|} \left(\int_{|v|\leq\frac{1}{|x|}} \tilde{f}(v)\,dv\right)^q dx \\ &= c \int_{\mathbf{R}^n} dx \left(\int_{|v|\leq\frac{1}{|x|}} \tilde{f}(v)\,dv\right)^q \int_{|t|\geq|x|} \omega(t)\,dt \\ &= c \int_{\mathbf{R}^n_+} dx \left(\int_{|t|\leq\frac{1}{|x|}} \tilde{f}(t)\,dt\right)^q w_1(x)\end{aligned}$$

and now we obtain our result. □

2. Necessary and sufficient conditions with weights. In this setion we generalize the results of Proposition 1 of [**6**] to n-dimensions and q even. We will reduce the Fourier transform problem for weights

$w(x)$ and $1/v(x)$ decreasing in each of their variables to a two-weight problem for the n-dimensional Hardy operator.

Throughout this section, the weights $w(x)$ and $v(x)$ will be radial in each of the variables x_i, and $\tilde{f}(x)$ will denote a function that is radial and decreasing in each of its variables,

$$\tilde{f}(x) = R_n R_{n-1} \cdots R_1 f(x),$$

where $R_j f$ represents the radial decreasing rearrangement of f in the x_j-variable.

In this section, we define

$$A\varphi(x) = \frac{1}{|x_1| \cdots |x_n|} \int_{|t_1| \le |x_1|} dt_1 \cdots \int_{|t_n| \le |x_n|} dt_n \varphi(t) \tag{2.1}$$

and set

$$w(x) = \int_{|t| \ge |x|} \omega(t)\, dt \tag{2.2}$$

for some $\omega \ge 0$. This $w(x)$ corresponds to (the appropriate) $w_1(x)$ of Section 1.

We begin with a generalization of Theorem 2 of [**5**]:

Theorem 2.1. *If $1 < q < \infty$, then for $0 \le f \in L^1(\mathbf{R}^n)$ and $s = (s_1, \ldots, s_n)$ with each $s_i > 0$,*

$$\int_{|x| \le s} |\hat{f}(x)|^q \, dx \ge c \int_0^s dy \bigg(\int_{|t| \le \frac{1}{y}} f(t)\, dt \bigg)^q,$$

where c is independent of f and s.

Proof. For $f \ge 0$, as in the proof of Theorem 2 of [**5**], it follows that

$$A|\hat{f}|(y) \ge c \int_{|t| \le \frac{1}{y}} f(t)\, dt,$$

where $A\varphi$ is defined by (2.1).

Hence,

$$\int_0^s (A|\hat{f}|(y))^q\,dy \geq c\int_0^s dy\bigg(\int_{|t|\leq\frac{1}{y}} f(t)\,dt\bigg)^q.$$

Now by iterating Hardy's inequality, we get that

$$\int_0^s (A|\hat{f}|(y))^q\,dy \leq c\int_{|x|\leq s} |\hat{f}|^q\,dx$$

and this last inequality implies our result. □

By Theorem 2.1, we get (just as we did in Theorem 1.4) that for $0 \leq f \in L^1 \cap L^q$,

$$\int_{\mathbf{R}^n} |\hat{f}(x)|^q w(x)\,dx \geq c\int_{\mathbf{R}^n} (Af(x))^q u(x)\,dx, \tag{2.3}$$

where $w(x)$ satisfies (2.2) and

$$u(x) = |x_1\cdots x_n|^{q-2} w(1/|x|). \tag{2.4}$$

In our next result, we obtain a version of Lemma 1.3,

Lemma 2.2. *If $q \geq 2$ and is even, then for $u \in \mathbf{R}^n_+$,*

$$\int_{|x|\leq u} |\hat{f}(x)|^q\,dx \leq c\int_{|x|\geq\frac{1}{u}} [A\tilde{f}(x)]^q |x_1\cdots x_n|^{q-2}\,dx.$$

Proof. Take $\psi_u(t) = (1/u_1)\chi_1(t)\cdots(1/u_n)\chi_n(t)$, where each χ_i is the characteristic function of the set $\{|t_i| \leq u_i\}$. If $g \geq 0$ and is decreasing in each of its variables, then $g * \psi_u(x)$ is also decreasing in the variables x_i, and it follows that

$$g * \psi_u(x) \leq cAg(x+u). \tag{2.5}$$

Next, since $|\hat{\psi}_u(x)| \geq c$ for $|x_i| \leq 1/u_i$, $1 \leq i \leq n$, we have, for even q,

$$\begin{aligned}\int_{|x|\leq \frac{1}{u}} |\hat{f}(x)|^q\,dx &\leq c\int_{\mathbf{R}^n} |\hat{f}(x)|^q |\hat{\psi}_u(x)|^q\,dx \\ &= c\int_{\mathbf{R}^n} |f*\psi_u*\cdots*f*\psi_u|^2\,dx \\ &\leq c\int_{\mathbf{R}^n} [\tilde{f}*\psi_u*\cdots*\tilde{f}*\psi_u]^s\,dx, \qquad \text{by (1.3)(b)} \\ &= c\int_{\mathbf{R}^n} |(\tilde{f}*\psi_u)^\wedge|^q\,dx \\ &\leq c\int_{\mathbf{R}^n_+} (\tilde{f}*\psi_u)^q |x_1\cdots x_n|^{q-2}\,dx,\end{aligned}$$

by the Hardy-Littlewood-Paley inequality in iterated form.

Now, from (2.5), it follows that

$$\begin{aligned}\int_{|x|\leq \frac{1}{u}} |\hat{f}(x)|^q\,dx &\leq c\int_{\mathbf{R}^n_+} [A\tilde{f}(x+u)]^q |x_1\cdots x_n|^{q-2}\,dx \\ &\leq c\int_{|x|\geq u} [A\tilde{f}(x)]^q |x_1\cdots x_n|^{q-2}\,dx,\end{aligned}$$

and we get our result simply by replacing $1/u$ with u. □

The next result follows immediately from Lemma 2.2.

Proposition 2.3. *If $q = 2, 4, 6, \ldots$, and if $w(x)$ satisfies (2.2), then*

$$\int_{\mathbf{R}^n} |\hat{f}(x)|^q w(x)\,dx \leq c\int_{\mathbf{R}^n_+} [A\tilde{f}(x)]^q u(x)\,dx,$$

where $u(x)$ is defined in (2.4).

Now we are in a position to obtain the main result of this section.

Theorem 2.4. *Suppose $q = 2, 4, 6, \ldots$, $u(x)$ is defined in (2.4), $w(x)$ in (2.2), and $1/v(x)$ is decreasing in each of its variables. Then*

$$\left(\int_{\mathbf{R}^n} |\hat{f}(x)|^q w(x)\,dx\right)^{p/q} \leq c\int_{\mathbf{R}^n} |f(x)|^p v(x)\,dx \tag{2.6}$$

holds if and only if

$$\left(\int_{\mathbf{R}^n} (Af(x))^q u(x)\,dx\right)^{p/q} \le c\int_{\mathbf{R}^n} |f|^p v(x)\,dx. \tag{2.7}$$

Proof. That (2.7) implies (2.6) follows immediately from Proposition 2.3. To argue the converse, we note that, by (2.3),

$$\int_{\mathbf{R}^n} |\hat{f}(x)|^q w(x)\,dx \ge c\int_{\mathbf{R}^n} [Af(x)]^q u(x)\,dx,$$

and now apply (2.6), which gets our result. Notice that (2.3), and so this direction, holds for all $1 < q < \infty$. □

The estimates given in Lemma 2.2 can be improved, compare this with Lemma 1.3, but this will be left to another project. It also should be noted that, in Theorem 2.4, there is no restriction on p.

3. Estimating the Hardy operator. We continue with the generalization of Proposition 1 of [**6**]. In Theorem 2.4, we showed that the Fourier transform problem can be reduced to a two-weighted problem for the Hardy operator. In this section we return to the case when $x = (y_1, y_2) \in \mathbf{R}^n$ and both $w(x)$ and $1/v(x)$ are decreasing in the variables y_1 and y_2. We will appeal to Theorem 1.4 (note that the Hardy operator is defined by (1.1)). Next we apply Sawyer's result for the Hardy operator in [**6**] to complete the result.

Theorem 3.1. *Let q be a positive even integer, and let*

$$u(x) = (|y_1|^{l_1}|y_2|^{l_2})^{q-2} w(1/|y_1|, 1/|y_2|), \tag{3.1}$$

$w(x) = w_1(x)$ satisfies (1.13), and $1/v$ is decreasing in the y_i variables. Then

$$\left(\int_{\mathbf{R}^n} |\hat{f}(x)|^q w(x)\,dx\right)^{p/q} \le c\int_{\mathbf{R}^n} |f|^p v(x)\,dx \tag{3.2}$$

if and only if

$$\left(\int_{\mathbf{R}^n}(A\tilde f(x))^q u(x)\,dx\right)^{p/q}\le c\int_{\mathbf{R}^n}\tilde f^p v(x)\,dx, \tag{3.3}$$

where Af is defined in (1.1).

Proof. By Theorem 1.4, we get that

$$\int_{\mathbf{R}^n}|\hat f(x)|^q w(x)\,dx\le c\int_{\mathbf{R}^n}[A\tilde f(x)]^q u(x)\,dx,$$

and now by (3.3) this implies that

$$\left(\int_{\mathbf{R}^n}|\hat f(x)|^q w(x)\,dx\right)^{p/q}\le c\int_{\mathbf{R}^n}\tilde f^p(x)v(x)\,dx,$$

but since $\tilde f$ is decreasing in the variables y_1 and y_2 and $v(x)$ is increasing y_1, y_2, we obtain (3.2).

Next to argue that (3.2) implies (3.3), we note that, since $w(x) = w_1(x)$ and satisfies (1.13), it follows from Theorem 2.1 (as it applies here) that

$$\int_{\mathbf{R}^n}|\hat f(x)|^q w(x)\,dx\ge c\int_0^\infty[Af(x)]^q u(x)\,dx \tag{3.4}$$

and now in (3.4) replace f by $\tilde f$. Next apply (3.2) with $\tilde f$ in place of f to obtain our result. □

Just as in Theorem 2.4, the $\tilde f$ in (3.3) can be replaced by f, to give us the necessary and sufficient condition for (3.2). Also notice that (3.3) reduces to

$$\begin{aligned}\int_0^\infty dr_1\int_0^\infty dr_2 r_1^{-l_1-1}r_2^{-l_2-1}w\left(\frac{1}{r_1},\frac{1}{r_2}\right)\\ \cdot\left[\int_0^{r_2}ds_2\int_0^{r_1}ds_1 s_2^{l_2-1}s_1^{l_1-1}\tilde f(s_1,s_2)\right]q\\ \le c\left[\int_0^\infty dr_2\int_0^\infty dr_1(r_1^{l_1-1}r_2^{l_2-1})^p v_1(r_1,r_2)\tilde f(r_1,r_2)^p\right]^{q/p}\end{aligned} \tag{3.5}$$

where $v(r_1, r_2) = (r_1^{l_1-1} r_2^{l_2-1})^{p-1} v_1(r_1, r_2)$, with v the weight in (3.3). The characterization of all weights satisfying (3.5), when $p \le q$ was settled in [**6**]. So we are finished once we identify these weights with our original weights w and v. We shall write out Sawyer's result.

Set

$$If(r_1, r_2) = \int_0^{r_1} \int_0^{r_2} f(t_1, t_2)\, dt_2\, dt_1$$

and

$$I^* f(r_1, r_2) = \int_{r_1}^{\infty} \int_{r_2}^{\infty} f(t_1, t_2)\, dt_2\, dt_1.$$

Given weights α and β, we set $\sigma = \beta^{1-p'}$ and define these three conditions:

(A) $$\sup_{a,b>0} (I^* \alpha(a,b))^{1/q} (I\sigma(a,b))^{1/p'} < \infty$$

(B) $I[\alpha(I\sigma)^q](a,b) \le c^q [I\sigma(a,b)]^{q/p}$, for all a and $b > 0$, and

(C) $I^*[\sigma(I^*\alpha)^{p'}(a,b) \le c^{p'} [I^*\alpha(a,b)]^{p'/q'}$, for all a and $b > 0$.

The elegant central result of [**6**] is

Theorem 3.2 (Sawyer). *Suppose* $1 < p \le q < \infty$ *and* α *and* β *are nonnegative weights on* $\mathbf{R}_+ \times \mathbf{R}_+$, *with* $\sigma = \beta^{1-p'}$. *Then*

$$\left(\int_0^{\infty} \int_0^{\infty} |If|^q \alpha(x) \right)^{p/q} \le c \int_0^{\infty} \int_0^{\infty} |f|^p \beta(x)$$

if and only if (A), (B) *and* (C) *hold.*

Theorem 3.3. *Let* $1 < p \le q < \infty$ *and* $q = 2, 4, 6, \ldots$. *Let* $u(x)$ *satisfy* (3.1), $w = w_1$ *satisfy (1.13) and* $1/v(x)$ *be decreasing in each of the variables* y_1 *and* y_2. *Put*

$$\alpha(x) = r_1^{l_1-1-l_1 q} r_2^{l_2-1-l_2 q} u(r_1, r_2)$$
$$\beta(x) = r_1^{l_1-1} r_2^{l_2-1} v(r_1, r_2),$$

and

$$\sigma(x) = (\beta(x))^{1-p'}.$$

Then (3.2) *holds if and only if* (A), (B) *and* (C) *hold.*

Proof. Simply apply Sawyer's Theorem 3.2 to Theorem 3.1. □

We have now generalized the Kerman-Sawyer Proposition 1 of [**6**] to the more general setting of $1 < p \le q < \infty$, and q even.

We close off this section with interpolation results that will prove useful in the next section.

Proposition 3.4. *Suppose* $q_0 > 1$ *and*

$$\int_{\mathbf{R}^n} |\hat{f}(x)|^{q_0} w(x)\,dx \le c \int_{\mathbf{R}^n} |f|^{q_0} v(x)\,dx.$$

If $1 \le p \le q_0$ *and if* $q = (q_0 - 1)p'$, *then*

$$\left(\int_{\mathbf{R}^n} |\hat{f}|^q w\right)^{p/q} \le c \int_{\mathbf{R}^n} |f|^p v^p/q. \tag{3.6}$$

When we apply this proposition, $v(x)$ will be $(|y_1|^{l_1}|y_2|^{l_2})^{q_0-2}$

$w(1/|y_1|, 1/|y_2|)$, $w = w_1$ satisfies (1.13). If

$$u(x) = (|y_1|^{l_1}|y_2|^{l_2})^{p(1-1/p-1/q)}(w(1/|y_1|, 1/|y_2|))^{p/q},$$

then (3.6) simplifies to

$$\left(\int_{\mathbf{R}^n} |\hat{f}|^q w\right)^{p/q} \le c \int_{\mathbf{R}^n} |f|^p u. \tag{3.7}$$

Proof of the Proposition. Take $T_z f = w^{z/q_0}(f v^{-z/q_0})^\wedge(x)$, for $z = \theta + \imath y$, where $0 \le \theta \le 1$ and $-\infty < y < \infty$. This generates an analytic family (taking $v > 0$ at first), and by analytic interpolation, (3.6) holds with $1/q = \theta/q_0$ and $1/p = \theta/q_0 + 1 - \theta$. Hence $\theta = q_0'/p'$ and $q = (q_0 - 1)p'$. □

Proposition 3.5. *Let $q_0 > 2$. If*

(i) $$\int_{\mathbf{R}^n} |\hat{f}(x)|^2 w_1(x)\,dx \le c \int_{\mathbf{R}^n} |f(x)|^2 v_1(x)\,dx,$$

and if

(ii) $$\int_{\mathbf{R}^n} |\hat{f}(x)|^{q_0} w_2(x)\,dx \le c \int_{\mathbf{R}^n} |f(x)|^{q_0} v_2(x)\,dx,$$

then we get that

$$\int_{\mathbf{R}^n} |\hat{f}|^q (w_1^{q_0-q} w_2^{q-2})^{(q_0-2)^{-1}} \le c \int_{\mathbf{R}^n} |f|^q (v_1^{q_0-q} v_2^{q-2})^{(q_0-2)^{-1}}, \tag{3.8}$$

where $1/q = \theta/2 + (1-\theta)/q_0$ and $0 \le \theta \le 1$.

Furthermore, if $w_1 = w_2 = w$ and $v_1 = w(1/|x|)$, $v_2 = (|y_1|^{l_1} \cdot |y_2|^{l_2})^{q_0-2} w(1/|x|)$, then for $2 \le q \le q_0$, we get that

$$\int_{\mathbf{R}^n} |\hat{f}(x)|^q w(x)\,dx \le c \int_{\mathbf{R}^n} |f|^q (|y_1|^{l_1} |y_2|^{l_2})^{q-2} w(1/|x|)\,dx. \tag{3.9}$$

Proof. Take $T_z f = w_1^{z/2} w_2^{(1-z)/q_0} (f v_1^{-z/2} v_2^{(z-1)/q_0})$, where $z = \theta + \iota y$, $-\infty < y < \infty$ and $0 \le \theta \le 1$. This is an analytic family (assuming again first that $v_i > 0$). Hence, it follows that

$$\int_{\mathbf{R}^n} |(f v_1^{-\theta/2} v_2^{(\theta-1)/q_0})^\wedge|^q (w_1^{\theta/2} w_2^{(1-\theta)/q_0})^q\,dx \le c \int_{\mathbf{R}^n} |f|^q, \tag{3.10}$$

where $1/q = \theta/2 + (1-\theta)/q_0$, or $(q_0 - q)/(q_0 q) = 1/q - 1/q_0 = (\theta(q_0-2))/(2q_0)$. Hence (3.8) follows from (3.10). In order to see (3.9) simply notice that

$$(v_1^{\theta/2} v_2^{(1-\theta)/q_0})^q = w(1/|x|)(|y_1|^{l_1} |y_2|^{l_2})^{(q-2)}. \qquad \square$$

4. A necessary and sufficient condition that applies to a class of regular weights. In this section we suppose that $w = w_1$, w satisfies (1.13), and that

$$v(x) = (|y_1|^{l_1} |y_2|^{l_2})^{p(1-p^{-1}-q^{-1})} (w(1/|y_1|, 1/|y_2|))^{p/q}$$

is nondecreasing. Furthermore, we shall restrict our attention to w's satisfying the regularity condition

$$(4.1)\qquad \begin{cases} \text{(a)} & \displaystyle\int_{|y_1|\le s} w(y_1,y_2)\,dy_1 \le cw(s,y_2)s^{l_1} \quad \text{for a.a. } y_2, \\ \text{(b)} & \displaystyle\int_{|y_2|\le s} w(y_1,y_2)\,dy_2 \le cw(y_1,s)s^{l_2} \quad \text{for a.a. } y_1. \end{cases}$$

We will show that (4.1)(a) and (b) is a necessary and sufficient condition for (0.1) with w and v as above. For some p's and q's, obviously if w is decreasing, then v is increasing. The requirement that v is increasing becomes relevant only when $1/p + 1/q > 1$.

This result will generalize Theorem 3 of [**4**] as well as Theorems 4.9 and 4.10 of [**2**]. To see this last generalization, take w decreasing in each of its variables, with w satisfying (1.13) as it applies here, and replace (4.1) by the corresponding n conditions in each of the variables,

$$\int_{|x_i|\le s} w\,dx_i \le csw(\cdots, s, \cdots).$$

We begin with a sufficiency result.

Lemma 4.1. *Let w satisfy* (1.13) *and $w = w_1$. Let $q = 2, 4, 6, \dots$, and let $u(x)$ satisfy* (3.1). *Then* (4.1)(a) *and* (b) *imply*

$$(4.2)\qquad \int_{\mathbf{R}^n} |\hat{f}(x)|^q w(x)\,dx \le c\int_{\mathbf{R}^n} |f|^q u(x)\,dx.$$

Proof. We first suppose

$$(4.3a)\quad \left[\int_{|y_1|\le \frac{1}{s}} w\left(y_1, \frac{1}{|y_2|}\right) dy_1\right]^{q'-1} \cdot \left[\int_{|y_1|\le s} \frac{dy_1}{(|y_1|^{l_1(q-2)} w(1/|x|))}^{1/(q-1)}\right] \le c$$

and

$$\text{(4.3b)}\quad \left[\int_{|y_2|\le \frac{1}{s}} w(1/|y_1|, y_2)\,dy_2\right]^{q'-1} \cdot \left[\int_{|y_2|\le s} \frac{dy_2}{(|y_2|^{l_2(q-2)}w(1/|x|))^{1/(q-1)}}\right] \le c$$

for almost all y_2 and y_1, respectively.

Next consider

$$I = \int_{\mathbf{R}_+^{l_2}} dx_2' \int_{\mathbf{R}_+^{l_1}} dx_1' (|x_1'|^{l_1}|x_2'|^{l_2})^{q-2} w(1/|x_1'|, 1/|x_2'|)(A\tilde{f})^q$$

with $A\tilde{f}$ as defined by (1.1). Now by (4.3a) we get (this is just an iterated Hardy's inequality)

$$I \le \int_{\mathbf{R}_+^{l_1}} dy_1 \int_{\mathbf{R}_+^{l_2}} dx_2' (|y_1|^{l_1}|x_2'|^{l_2})^{q-2} w(1/|y_1|, 1/|x_2'|) \cdot \left(1/|x_2'|^{l_2} \int_{|y_2|\le|x_2'|} \tilde{f}\right)^q$$

and by (4.3b) and another Hardy, we get that

$$I \le c \int_{\mathbf{R}_+^{l_2}} dy_2 \int_{\mathbf{R}_+^{l_1}} dy_1 (|y_1|^{l_1}|y_2|^{l_2})^{q-2} w(1/|y_1|, 1/|y_2|)(\tilde{f})^q.$$

According to Theorem 3.1, then, (4.3a) and (4.3b) imply (4.2).

So we must derive (4.3) from (4.1). By Theorem 3 of [**4**], we get that (4.1a) implies

$$\text{(4.4a)}\quad \int_{\mathbf{R}^{l_1}} |\hat{f}(y_1)|^q w(y_1, 1/|y_2|)\,dy_1 \le c \int_{\mathbf{R}^{l_1}} |f(y_1)|^q |y_1|^{l_1(q-2)} w(1/|y_1|, 1/|y_2|)\,dy_1,$$

but (4.3a) follows from (4.4a) by Theorem 2 of [**4**].

Again by Theorem 3 of [**4**] we get that (4.1b) implies

$$\text{(4.4b)}\qquad \int_{\mathbf{R}^{l_2}} |\hat{f}(y_2)|^q w(1/|y_1|, y_2)\, dy_2 \le c \int^{\mathbf{R}^{l_2}} |f(y_2)|^q |y_2|^{l_2(q-2)} w(1/|y_1|, 1/|y_2|)\, dy_2,$$

and in the same manner, (4.4b) implies (4.3b). Hence, we obtain our result. □

Theorem 4.2. *Let* $q \ge 2$ *and suppose that* $w = w_1$ *satisfies* (1.13). *Then* (4.1) *implies* (4.2).

Proof. By (4.1) we get, by the lemma, that

$$\text{(4.5)}\qquad \int_{\mathbf{R}^n} |\hat{f}(x)|^{q_0} w(x)\, dx \le c \int_{\mathbf{R}^n} |f(x)|^{q_0} u(x)\, dx,$$

where $u(x) = (|y_1|^{l_1}|y_2|^{l_2})^{q_0-2} w(1/|x|)$ and q_0 is an even integer. For other q's, we apply (3.9) of Proposition 3.5. □

And now to the main point of this section.

Theorem 4.3. *Let* $1 < p \le q < \infty$ *and suppose that* $w = w_1$ *satisfies* (1.13). *Let*

$$v_{p,q}(x) = (|y_1|^{l_1}|y_2|^{l_2})^{p(1-1/p-1/q)} (w(1/|y_1|, 1/|y_2|))^{p/q}$$

and suppose that $v_{p,q}$ *is increasing in the* y_i *variables. Then* (4.1) *holds if and only if* (0.1) *holds with* $v(x) = v_{p,q}(x)$.

Proof. Set $q_0 = 1 + q/p'$. Then $q_0 \ge 1 + p/p' = p$, and so we can apply the interpolation Proposition 3.4. By Theorem 4.2, if (4.1) holds, we have

$$\int_{\mathbf{R}^n} |\hat{f}|^{q_0} w \le c \int_{\mathbf{R}^n} |f|^{q_0} v_{q_0,q_0}(x)\, dx$$

(notice that $v_{q_0,q_0}^{p/q} = v_{p,q}$ is increasing), and so by that Proposition, (0.1) holds.

In order to see the necessity, we begin wtih (0.1) for w and v as in the hypotheses. Take $f(x) = h(y_1)g(y_2)$, so that $\hat{f}(x) = \hat{h}(y_1)\hat{g}(y_2)$, and (0.1) becomes

$$(4.6)\quad \left(\int_{\mathbf{R}^{l_1}}\int_{\mathbf{R}^{l_2}} |\hat{h}(y_1)|^q|\hat{g}(y_2)|^q w(x)\,dx\right)^{p/q} \leq c\int_{\mathbf{R}^{l_1}} dy_1 \int_{\mathbf{R}^{l_2}} dy_2|h(y_1)|^p|g(y_2)|^p v(x).$$

In this, take $h(y_1) = 1$ for $|y_1| \leq s$ and zero elsewhere, and hence

$$(4.7)\quad \hat{h}(y_1) = \int_{|x_1'|\leq s} e^{\iota y_1\cdot x_1'}\,dx_1'.$$

From (4.6) and (4.7), it follows that

$$\left(\int_{|y_1|\leq \frac{1}{s}} dy_1 s^{l_1 q}\int_{\mathbf{R}^{l_2}} dy_2|\hat{g}(y_2)|^q w(x)\right)^{p/q} \leq c\int_{|y_1|\leq s} dy_1|y_1|^{l_1p(1-p^{-1}-q^{-1})} \cdot \int_{\mathbf{R}^{l_2}} dy_2|g(y_2)|^p|y_2|^{l_2p(1-p^{-1}-q^{-1})} \cdot (w(1/|y_1|, 1/|y_2|))^{p/q}$$

but, since w is decreasing, this yields

$$\left(\int_{\mathbf{R}^{l_2}} dy_2|\hat{g}(y_2)|^q w(1/s, |y_2|)\right)^{p/q} s^{l_1(q-1)p/q} \leq c\int_{\mathbf{R}^{l_2}} dy_2|g(y_2)|^p|y_2|^{l_2p(1-p^{-1}-q^{-1})} \cdot (w(1/|s|, 1/|y_2|))^{p/q}\cdot s^{l_1p(1-p^{-1}-q^{-1})+l_1},$$

or

$$\left(\int_{\mathbf{R}^{l_2}} dy_2|\hat{g}(y_2)|^q w(1/s, |y_2|)\right)^{p/q} \leq c\int_{\mathbf{R}^{l_2}} dy_2|g(y_2)|^p|y_2|^{l_2p(1-p^{-1}-q^{-1})} \cdot (w(1/|s|, 1/|y_2|))^{p/q}.$$

Appealing once again to Theorem 3 of [**4**], we see that (4.1b) holds, and a similar argument yields (4.1a). □

We remark here that when w satisfies (4.1) we obtain necessary and sufficient conditions for a certain version of the two-dimensional Hardy operator. This version is much easier to apply than the general case. Note also that in the case when w is decreasing in k variables, $y_1, \dots, y_k$, with $k \leq n$, we obtain simplifications to a k-dimensional Hardy inequality.

If, in (4.6), we select $h(y_i) = 1$, $g(y_2) = 1$ when $|y_i| \leq s_i$ and 0 elsewhere, it follows that

(4.8)

$$\left(\int_{|y_1|\leq \frac{1}{s_1}} dy_1 s_1^{l_1 q} \int_{|y_2|\leq \frac{1}{s_2}} s_2^{l_2 q} w(x)\right)^{p/q} \leq c \int_{|y_1|\leq s_1} dy_1 \int_{|y_2|\leq s_2} dy_2 v(x).$$

So, if (0.1) holds, then we get (4.7) for general w and v. If we assume furthermore that w and $1/v$ are decreasing in each of the variables y_1 and y_2, it then follows that

$$v(s_1, s_2) \gg (s_1^{l_1} s_2^{l_2})^{p(1-p^{-1}-q^{-1})}(w(1/s_1, 1/s_2))^{p/q}$$

or

$$v(x) \gg v_{p,q}(x).$$

Thus for a given w, radial and decreasing in each of the variables y_1 and y_2, the smallest v radial and increasing in each of these y_i variables is the weight $v_{p,q}(x)$ (within constants, of course). We wish to thank Peter Knopf for pointing this out to us.

It can be shown that $w(x) = (|y_1|^\alpha + |y_2|^\beta)^{-1}$ satisfies (4.1) if both α and β are below $n/2$. For α or β greater than $n/2$, Theorem 2.7 of [**1**] furnishes a sufficient condition.

5. More necessary and sufficient conditions. Thanks to Theorem 1 of Sawyer's celebrated paper [**7**], we can conclude that for $0 < \alpha < n$ and $1 < p \leq 2$,

$$\left(\int_{\mathbf{R}^n} |\hat{f}(x)|^2 |x|^{-\alpha}\, dx\right)^{p/2} \leq c \int_{\mathbf{R}^n} |f|^p v(x)\, dx \tag{5.1}$$

if and only if both

$$\int_{\mathbf{R}^n}\left|\int_{\mathbf{R}^n}\chi_Q(x-t)|t|^{\alpha/2-n}v(x-t)^{1-p'}\,dt\right|^2 dx \leq c\left(\int_Q v(t)^{1-p'}\,dt\right)^{2/p} < \infty \tag{5.2}$$

$$\int_{\mathbf{R}^n}\left|\int_{\mathbf{R}^n}\chi_Q(x-t)|t|^{\alpha/2-n}\,dt\right|^{p'} v(x)^{1-p'}\,dx \leq c\left(\int_Q dt\right)^{p'/2} < \infty \tag{5.3}$$

for every dyadic cube Q in $\mathbf{R}^n$.

Even though the weight on the left in (5.1) is very special, namely $|x|^{-\alpha}$, this allows for general weights v on the right. We begin by simplifying the two conditions (5.2) and (5.3). In this section we will use the notation

$$\int_0^h f(t)\,dt = \int_0^h dt_1 \cdots \int_0^h dt_n f(t),$$

and

$$\int_{|t|\leq u} f(t)\,dt = \int_{t_1^2+\cdots+t_n^2\leq u^2} f(t)\,dt_1\cdots dt_n.$$

Proposition 5.1. *Let $0<\alpha<n$ and $1<p\leq 2$. Then (5.1) holds if and only if both*

$$\int_0^h v(t+a)^{1-p'}\,dt\int_0^h v(s+a)^{1-p'}|s-t|^{\alpha-n}\,ds \ll \left(\int_0^h v(t+a)^{1-p'}\,dt\right)^{2/p} < \infty \tag{5.4}$$

and

$$\int_{|x|\leq 2h} v(t+a)^{1-p'} \ll h^{p'(n-\alpha)/2} \tag{5.5}$$

for all $h > 0$ and $a \in \mathbf{R}^n$.

Proof. We begin with the integrand in (5.3). Notice that

$$\int_a^{a+h} |x-t|^{\alpha/2-n}\,dt = \int_a^{a+h} |u-(t-a)|^{\alpha/2-n}\,dt = \int_0^h |u-v|^{\alpha/2-n}\,dv$$

with the variable changes $u = x - a$ and $v = t - a$. Thus (5.3) becomes

$$\int_{\mathbf{R}^n} \left| \int_0^h |x-t|^{\alpha/2-n}\,dt \right|^{p'} v(x+a)^{1-p'}\,dx \ll h^{np'/2}. \tag{5.6}$$

For $|x| \le 2h$, $\int_0^h |x-t|^{\alpha/2-n}\,dt \propto h^{\alpha/2}$, while for $|x| \ge 2h$, it behaves like $|x|^{\alpha/2-n}h^n$. So the integrand on the left of (5.6) is completely under control, and (5.6) is equivalent to

$$\int_{|x|\le 2h} v(x+a)^{1-p'}\,dx + \int_{|x|\ge 2h} v(x+a)^{1-p'}(h/|x|)^{(n-\alpha/2)p'}\,dx \ll h^{(n-\alpha)p'/2}. \tag{5.7}$$

(5.7) certainly implies (5.5) and is in fact equivalent to (5.5), for if (5.5) holds, then

$$\begin{aligned}
&\int_{|x|\ge 2h} v(x+a)^{1-p'}(h/|x|)^{(n-\alpha/2)p'}\,dx \\
&\qquad \propto \sum_k \int_{|x|\propto 2^k h} v(x+a)^{1-p'}(h/|x|)^{(n-\alpha/2)p'}\,dx \\
&\qquad \ll \sum_k 2^{k(\alpha/2-n)p'} \int_{|x|\le 2^{k+1}h} v(x+a)^{1-p'}\,dx \\
&\qquad \ll h^{(n-\alpha)p'/2} \sum_k 2^{-knp'/2} \ll h^{(n-\alpha)p'/2}.
\end{aligned}$$

Hence, (5.6), and so (5.3) simplifies to (5.5).

In the same manner, (5.2) reduces to

$$\int_{\mathbf{R}^n} \left| \int_0^h |x-t|^{\alpha/2-n} v(t+a)^{1-p'}\,dt \right|^2 dx \ll \left(\int_0^h v(t+a)^{1-p'}\,dt \right)^{2/p}.$$

But

$$\int_{|x|\gg h} \left| \int_0^h |x-t|^{\alpha/2-n} v(t+a)^{1-p'}\, dt \right|^2 dx$$
$$\ll \int_{|x|\gg h} |x|^{\alpha-2n}\, dx \left| \int_0^h v(t+a)^{1-p'}\, dt \right|^2$$
$$\ll h^{\alpha-n} \left| \int_0^h v(t+a)^{1-p'}\, dt \right|^2$$
$$\ll \left(\int_0^h v(t+a)^{1-p'}\, dt \right)^{2/p},$$

if (5.5) holds. Thus (5.2) is implied by both (5.5) and

$$\int_{|x|\ll h} \left(\int_0^h v(t+a)^{1-p'} |x-t|^{\alpha/2-n}\, dt \right)^2 dx \ll \left(\int_0^h v(t+a)^{1-p'}\, dt \right)^{2/p}, \tag{5.8}$$

and obviously (5.8) follows from (5.2). So we are done if we can simplify (5.8) to (5.4).

We start by rewriting the left side of (5.8) as

$$\int_0^h v(t+a)^{1-p'}\, dt \int_0^h v(s+a)^{1-p'}\, ds \int_{|x|\ll h} |x-s|^{\alpha/2-n} |x-t|^{\alpha/2-n}\, dx.$$

Now since each $t_i \in [0,h]$, $\{x : |x| \ll h\} \subset \{x : |x-t| \ll h\}$, given appropriate choice of constants in these $\ll$ inequalities. The inclusion goes the other way for different choices of constants, and so the left side of (5.8) can be replaced by

$$\int_0^h v(t+a)^{1-p'}\, dt \int_0^h v(s+a)^{1-p'}\, ds \int_{|x-t|\ll h} |x-s|^{\alpha/2-n} |x-t|^{a/2-n}\, dx.$$

Now

$$\int_{|x-t|\le|s-t|/2} |x-s|^{\alpha/2-n} |x-t|^{\alpha/2-n}\, dx \propto |s-t|^{\alpha-n},$$

since $|x-s| = |(x-t)+(t-s)| \propto |s-t|$. Furthermore,

$$\int_{\frac{|s-t|}{2}\le|x-t|\le 2|s-t|} |x-s|^{\alpha/2-n}|x-t|^{\alpha/2-n}\,dx$$
$$\propto |s-t|^{\alpha/2-n}\int_{|s-t|/2\le|x-t|\le 2|s-t|}|x-s|^{\alpha/2-n}\,dx$$
$$\propto |s-t|^{\alpha/2-n}\int_{|u|\ll|s-t|}|u|^{\alpha/2-n}\,du \propto |s-t|^{\alpha-n}$$

also.

Finally,

$$\int_{2|s-t|\le|x-t|\le 2h}|x-s|^{\alpha/2-n}|x-t|^{\alpha/2-n}\,dx$$
$$\ll \int_{2|s-t|\le u}|u|^{\alpha-2n}\,du \ll |s-t|^{\alpha-n},$$

and this lets us replace (5.8) with (5.4). □

We can use this Proposition to obtain necessary and sufficient conditions for

$$\int_{\mathbf{R}^n}|\hat{f}(x)|^2|y_1|^{-\alpha}\,dx \le c\int_{\mathbf{R}^n}|f(x)|^2 v(y_1)\,dx, \tag{5.9}$$

where $x=(y_1,y_2)$, $y_i\in\mathbf{R}^{l_i}$, and $0<\alpha<l_1$.

Proposition 5.2. *Let $0<\alpha<l_1$ and a,s and $t\in\mathbf{R}^{l_1}$. Then (5.9) holds if and only if both*

$$\int_0^h\frac{dt}{v(t+a)}\int_0^h\frac{ds}{v(s+a)}|s-t|^{\alpha-l_1}\le c\int_0^h\frac{dt}{v(t+a)}<\infty, \tag{5.10}$$

and

$$\int_{|t|\le 2h}\frac{dt}{v(t+a)}\le ch^{l_1-\alpha} \tag{5.11}$$

hold for all $h > 0$ *and* $a \in \mathbf{R}^{l_1}$.

Proof. To see the sufficiency,

$$\begin{aligned}\int_{\mathbf{R}^n} |\hat{f}(x)|^2 |y_1|^{-\alpha}\, dx &= \int_{\mathbf{R}^{l_2}} dy_2 \int_{\mathbf{R}^{l_1}} |\hat{f}(x)|^2 |y_1|^{-\alpha}\, dy_1 \\ &\le c \int_{\mathbf{R}^{l_2}} dy_2 \int_{\mathbf{R}^{l_1}} |\hat{f}_{y_1}(y_2)|^2 v(y_1)\, dy_1 \\ &\qquad \text{by Proposition 5.1} \\ &= c \int_{\mathbf{R}^{l_1}} v(y_1)\, dy_1 \int_{\mathbf{R}^{l_2}} |f|^2\, dy_2.\end{aligned}$$

To see the necessity, take $f(x) = h(y_1) g(y_2)$. Hence,

$$\begin{aligned}&\int_{\mathbf{R}^{l_2}} dy_2 \int_{\mathbf{R}^{l_1}} dy_1 |\hat{h}(y_1)|^2 |\hat{g}(y_2)|^2 |y_1|^{-\alpha} \\ &\qquad \le c \int_{\mathbf{R}^{l_2}} dy_2 \int_{\mathbf{R}^{l_1}} dy_1 |h(y_1)|^2 |g(y_2)|^2 v(y_1).\end{aligned}$$

But this implies that

$$\int_{\mathbf{R}^{l_1}} dy_1 |\hat{h}(y_1)|^2 |y_1|^{-\alpha} \le c \int_{\mathbf{R}^{l_1}} dy_1 |h(y_1)|^2 v(y_1)$$

and now (5.10) and (5.11) follow from Proposition 5.1. □

Now we have some principles in place to generate necessary and sufficient conditions for the Fourier transform problem. When the weights factor, the corresponding conditions on the lower dimensional spaces combine as expected, that is:

Proposition 5.3. *Let* $1 < q < \infty$, $x = (y_1, y_2)$ *with* $y_i \in \mathbf{R}^{l_i}$, *and* $l_1 + l_2 = n$, *and suppose the weights* $w_i(y_i)$ *are not identically zero. If we have found conditions* (A_i) *where*

$$\int_{\mathbf{R}^{l_i}} |\hat{f}(y_i)|^q w_i(y_i)\, dy_i \le c \int_{\mathbf{R}^{l_i}} |f(y_i)|^q v_i(y_i)\, dy_i$$

holds if and only if (A_i) *holds, then*

$$\int_{\mathbf{R}^n} |\hat{f}(x)|^q w_1(y_1) w_2(y_2)\, dx \le c \int_{\mathbf{R}^n} |f|^q v_1(y_1) v_2(y_2)\, dx$$

holds if and only if both conditions (A_1) *and* (A_2) *are satisfied.*

Proof. If both conditions are satisfied, then by iterating the Fourier transform, we get

$$\begin{aligned}\int_{\mathbf{R}^{l_1}} dy_1 w_1(y_1) \int_{\mathbf{R}^{l_2}} dy_2 |\hat{f}(x)|^q w_2(y_2) \\ &\le c \int_{\mathbf{R}^{l_1}} dy_1 w_1(y_1) \int_{\mathbf{R}^{l_2}} dy_2 |\hat{f}_{y_2}(y_1)|^q v_2(y_2) \\ &\le c \int_{\mathbf{R}^{l_2}} dy_2 v_2(y_2) \int_{\mathbf{R}^{l_1}} dy_1 |f(x)|^q v_1(y_1)\end{aligned}$$

by (A_2) and then by (A_1).

For necessity, take $h(x) = f(y_1) g(y_2)$, and if we know that

$$\int_{\mathbf{R}^n} |\hat{h}(x)|^q w_1(y_1) w_2(y_2)\, dx \le c \int_{\mathbf{R}^n} |h|^q v_1(y_2) v_2(y_2)\, dx,$$

we get

$$\int_{\mathbf{R}^{l_2}} |\hat{g}|^q w_2 \cdot \int_{\mathbf{R}^{l_1}} |\hat{f}|^q w_1 \le c \int_{\mathbf{R}^{l_2}} |g|^q v_2 \cdot \int_{\mathbf{R}^{l_1}} |f|^q v_1.$$

Now take any appropriately nice g with $\hat{g}$ nonzero somewhere on the support of w_2, to obtain

$$\int_{\mathbf{R}^{l_1}} |\hat{f}|^q w_1 \le c \int_{\mathbf{R}^{l_1}} |f|^q v_1$$

with this c depending on g but independent of f. This implies that (A_1) holds, and an identical argument establishes (A_2). □

We can, for example, apply this to Proposition 5.2 when $q = 2$ and $w_i = |y_i|^{-\alpha_i}$ to obtain necessary and sufficient conditions for weights of the form

$$w(x) = |y_1|^{-\alpha_1} |y_2|^{-\alpha_2}.$$

Proposition 5.4. *Suppose that for $i = 1$ and 2, we have found conditions (A_i) where*

$$\int_{\mathbf{R}^n} |\hat{f}()|^2 w_i(y_i)\,dx \le c \int_{\mathbf{R}^n} |f(x)|^2 v_i(y_i)\,dx$$

holds if and only if (A_i) holds, then if $v_i(a_i) = 0$ and if v_i is continuous at a_i, for some $a_i \in \mathbf{R}^{l_i}$, then

$$\int_{\mathbf{R}^n} |\hat{f}(x)|^2 [w_1(y_1) + w_2(y_2)]\,dx \le c \int_{\mathbf{R}^n} |f|^2 [v_1(y_1) + v_2(y_2)]\,dx$$

holds if and only if both conditions (A_1) and (A_2) are satisfied.

Proof. The sufficiency is trivial. For the necessity, take $f = g_\varepsilon(y_1)h(y_2)$, to get

$$\begin{aligned}(5.12)\qquad \int_{\mathbf{R}^{l_1}} |\hat{g}_\varepsilon(y_1)|^2 w_1(y_1) \int_{\mathbf{R}^{l_2}} |h(y_2)|^2 \\ + \int_{\mathbf{R}^{l_1}} |g_\varepsilon(y_1) + |^2 \int_{\mathbf{R}^{l_2}} \hat{h}(y_2)|^2 w_2(y_2) \\ \le c \int_{\mathbf{R}^{l_1}} |g_\varepsilon(y_1)|^2 v_1(y_1) \int_{\mathbf{R}^{l_2}} |h(y_2)|^2 \\ + \int_{\mathbf{R}^{l_1}} |g_\varepsilon(y_1)|^2 \int_{\mathbf{R}^{l_2}} |h(y_2)|^2 v_2(y_2).\end{aligned}$$

Now since $v_i(a_i)$ is zero at a_i, and continuous there, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|y_1 - a_1| < \delta$, then $v_1(y_1) < \varepsilon$. Take $g_\varepsilon = \varepsilon^{-1/2}\delta^{-l_1/2}$ for $|y_1 - a_1| < \delta$ and zero elsewhere. Therefore,

$$\int |g_\varepsilon(y_1)|^2 v_1(y_1)\,dy_1 \le 1$$

while

$$\int |g_\varepsilon(y_1)|^2\,dy_1 = 1/\varepsilon.$$

Hence, if we multiply (5.12) by ε and let $\varepsilon \to 0$, we get

$$\int_{\mathbf{R}^{l_2}} |\hat{h}(y_2)|^2 w(y_2) \le c \int_{\mathbf{R}^{l_2}} |h(y_2)|^2 v(y_2)\,dy_2,$$

which implies that (A_1) holds.

This time, using Propositions 5.2 and 5.4, we get necessary and sufficient conditions for weights of the form

$$w(x) = |y_1|^{-\alpha_1} + |y_2|^{-\alpha_2}.$$

Notice that when $\alpha_1 = \alpha_2$, we can dispense with the extra continuity condition on the v_i. □

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