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THE MODULI OF WEIERSTRASS FIBRATIONS OVER $\mathbf{P}^1$: RATIONALITY

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1. The moduli of Weierstrass fibrations over $\mathbf{P}^1$: Rationality. Let K be an algebraically closed field of characteristic $\neq 2, 3$. A Weierstrass fibration is a pair $(p : Y \to S, s)$ where p is a morphism of algebraic varieties over K whose fibers are elliptic curves or rational curves with a node or a cusp, and where s is a section of S not passing through the nodes or cusps of the fibers. As is well known, a Weierstrass fibration can be defined by an equation of the form $y^2 = x^3 + ax + b$ due to Weierstrass (see [**1**]).

In [**3**], Miranda defines the moduli W_N, $N \geq 1$ of Weierstrass fibrations over $S = \mathbf{P}^1_K$ whose section has self-intersection $-N$ in the associated elliptic surface. Seiler [**4**] generalizes this work and defines moduli for Weierstrass fibrations over an arbitrary complete nonsingular connected curve. In spite of their obvious interest very little is known about the structure of these moduli. Here we prove:

Theorem 1.1. *When $K = \mathbf{C}$, W_N is a rational variety for all $N \geq 1$.*

Define V_n to be the set of binary homogeneous forms of degree n in t, s. Let $f = f(t, s)$ be an element of V_n and $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be an element of $\mathbf{GL}_2$. Then

$$f \cdot g = f(\alpha t + \beta s, \gamma t + \delta s)$$

defines an action of $\mathbf{GL}_2$ on V_n. There are induced actions of $\mathbf{GL}_2$ on products $V_m \times V_n$. Define $(V_{4N} \times V_{6N})^s$ and V_{6N}^s to be the open sets of SL_2-stable (= finite stabilizer and closed orbit) points of $V_{4N} \times V_{6N}$ and V_{6N}, respectively. By means of the Weierstrass equation, W_N is defined as the quotient $(V_{4N} \times V_{6N})^s / \mathbf{GL}_2$. On the other side the quotient

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$V_{6N}^s/\mathbf{GL}_2$ is birationally equivalent to the moduli of hyperelliptic curves of genus $g = 3N - 1$. This last variety is rational over $\mathbf{C}$ by [**1**].

Now the group μ_n of n-th roots of unity is embedded in $\mathbf{GL}_2$ as scalar matrices. Define $G = \mathbf{GL}_2/\mu_{2N}$ and μ as the image of μ_{3N} inside G. With the natural projection $(V_{4N} \times V_{6N}^s)/\mu$ is a vector bundle over the base V_{6N}^s/μ because μ acts freely outside the zero section of the trivial vector bundle $V_{4N} \times V_{6N}^s$ over V_{6N}^s. We have that G/μ acts freely on V_{6N}^s/μ. Taking into account that $(V_{6N}^s/\mu)/(G/\mu) = V_{6N}^s/G = V_{6N}^s/\mathbf{GL}_2$, it follows from [**2**, Corollary 1], that

$$((V_{4N} \times V_{6N}^s)/\mu)/(G/\mu) = (V_{4N} \times V_{6N}^s)/G = (V_{4N} \times V_{6N}^s)/\mathbf{GL}_2$$

is rational. It is enough to remark that $V_{4N} \times V_{6N}^s$ is open in $(V_{4N} \times V_{6N})^s$ to conclude the proof.

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