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A NOTE ON ARTINIAN GORENSTEIN ALGEBRAS DEFINED BY MONOMIALS

MARK B. BEINTEMA

In this note we prove a result which has been discovered independently by H. Charalambous [**1**] and is possibly known to many commutative algebraists. However, we know of no statement of this result in the literature. Our proof uses a mix of algebra and combinatorics and is quite elementary. We assume familiarity with the usual notions of commutative algebra and recommend [**2**] as a general reference.

Throughout, we let $A = k[X_1, \dots, X_n]$ be a polynomial ring over a field k. If I is an ideal of A, we denote the image of X_i in A/I by x_i. If $\mathbf{a} = a_1, \dots, a_n$ is a sequence of nonnegative integers, then $\mathbf{X}^{\mathbf{a}}$ denotes the monomial $X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}$. Similarly, $\mathbf{x}^{\mathbf{a}}$ denotes $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. We observe that if I is generated by monomials, then $R = A/I$ is a noetherian graded k-algebra. That is, $R = \oplus_{n \geq 0} R_n$, where $R_0 = k$, and $R_i R_j \subset R_{i+j}$. Moreover, the nonzero monomials $\mathbf{x}^{\mathbf{a}}$ form a k-basis for R. This is a direct consequence of the following lemma, which is implicit in the work of Macaulay (see [**3**], Theorems 2.1 and 2.2]).

Lemma. *Let $I \subset A = k[X_1, \dots, X_r]$ be an ideal which is generated by monomials. If x_i denotes the image of X_i in A/I, then the nonzero monomials in the x_i are linearly independent over k.*

Proof. Let $I = (m_1, m_2, \dots, m_t)$, with $m_j = \mathbf{X}^{\mathbf{a}_j}$, $j = 1, \dots, t$. Let $\mathbf{x}^{\mathbf{b}_1}, \dots, \mathbf{x}^{\mathbf{b}_m}$ be nonzero monomials, and suppose that $\sum_{i=1}^m \beta_i \mathbf{x}^{\mathbf{b}_i} = 0$ for some $\beta_1, \dots, \beta_m \in k$. Hence there exist $f_j \in A$ such that

$$\sum_{i=1}^{m} \beta_i \mathbf{X}^{\mathbf{b}_i} = \sum_{j=1}^{t} f_j m_j.$$

Regarding each f_j as a linear combination of monomials, we may

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rewrite this expression as

$$\sum_{i=1}^{m} \beta_i \mathbf{X}^{\mathbf{b}_i} = \sum_{i=1}^{r} \gamma_i \mathbf{X}^{\mathbf{c}_i}$$

where each $\gamma_i \in k$, and the $\mathbf{X}^{\mathbf{c}_i}$ are distinct monomials, each of which is divisible by some m_j. Using the fact that monomials in A form a k-basis for A, and comparing coefficients, we conclude that if some $\beta_i \neq 0$, then $\mathbf{X}^{\mathbf{b}_i}$ is divisible by one of the m_j's. This contradicts the assumption that $\mathbf{x}^{\mathbf{b}_i} \neq 0$ in A/I. Hence, we have $\beta_i = 0$ for each $i = 1, \ldots, m$. □

Let $I \subset A$ be an ideal of height n which is generated by monomials, and set $R = A/I$. Evidently, R is a finite dimensional k-vector space. We let R_+ denote the image of $(X_1, \ldots, X_n)$ in R. The *socle* of R is defined by $\operatorname{socle} R = \operatorname{ann} R_+$. Recall that R is Gorenstein provided that $\dim_k \operatorname{socle} R = 1$. I is said to be a *complete intersection* if I can be generated by precisely n elements. Equivalently, I is a complete intersection if it can be generated by an A-sequence. It is well known that if I is a complete intersection, then R is Gorenstein.

J. Watanabe [**5**] observed that the monomials $\mathbf{x}^{\mathbf{a}}$ in R form a partially ordered set, ordered by divisibility. That is, $\mathbf{x}^{\mathbf{a}} \leq \mathbf{x}^{\mathbf{b}}$ if and only if $a_i \leq b_i$ for each $i = 1, \ldots, n$.

Proposition. *Let $A = k[X_1, \ldots, X_n]$, and let $I \subset A$ be an ideal of height n which is generated by monomials. Then A/I is Gorenstein if and only if I is a complete intersection.*

Proof. Suppose R is Gorenstein, and let P be the partially ordered set consisting of all nonzero monomials $\mathbf{x}^{\mathbf{b}}$, ordered by divisibility. It is clear that if $u \in P$ is maximal, then $u \in \operatorname{socle} R$. Since R is Gorenstein, this implies that there is a unique maximal element in P; that is, P is the lattice of divisors of some monomial $\mathbf{x}^{\mathbf{b}}$. Thus, we need only prove:

Claim. *If P is the divisor lattice of some monomial $\mathbf{x}^{\mathbf{b}}$, then I is a complete intersection.*

Proof of Claim. Suppose that P is the divisor lattice of $x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$.

Since I has height n and is generated by monomials, we have $\operatorname{rad} I = (X_1, \dots, X_n)$. For each $i = 1, \dots, n$, let e_i be the least integer such that $X_i^{e_i} \in I$. Clearly, $e_i \geq b_i + 1$, else $0 = x_i^{b_i}$ in R, contradicting the fact that $\mathbf{x}^{\mathbf{b}} \in P$. If some $e_i > b_i + 1$, then $X_i^{b_i+1} \notin I$ and, consequently, $x_i^{b_i+1} \in P$, also a contradiction. Thus,

$$(X_1^{b_1+1}, \dots, X_n^{b_n+1}) \subset I.$$

Given $y \in I$, we write $y = \sum \alpha_i \mathbf{X}^{\mathbf{a}_i} + z$, where $\alpha_i \in k$, $\mathbf{x}^{\mathbf{a}_i} \in P$, and $z \in (X_1^{b_1+1}, \dots, x_n^{b_n+1})$. Reducing modulo I, we have $\sum \alpha_i \mathbf{x}^{\mathbf{a}_i} = 0$. It follows from the lemma that $\alpha_i = 0$ for all i. That is, $y = z$, and hence $I = (X_1^{b_1+1}, \dots, X_n^{b_n+1})$. □

Remark . A non-Artinian graded algebra A/I may be Gorenstein without being a complete intersection, even if I is generated by monomials (see Example 1.2 and Corollary 5.2 of [**4**, Chapter II]). There are also many examples of graded Artinian Gorenstein algebras which are neither complete intersections nor defined by monomials. See [**3**, Example 4.3].)

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Department of Mathematics, Southern Illinois University, Carbondale, IL 62901