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THE DISTRIBUTION OF RELATIVELY r-PRIME INTEGERS IN RESIDUE CLASSES

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ABSTRACT. If 1 is the only r-th power which is a divisor of $m_1, m_2, \dots, m_k$, then $m_1, m_2, \dots, m_k$ are said to be relatively r-prime. If $\bar{a} = \langle a_1, a_2, \dots, a_k\rangle$ is a k-tuple of nonnegative integers, h is a positive integer and x is a positive real number, let $Q(x; \bar{a}, h, r, k)$ denote the number of k-tuples of positive integers $\langle m_1, m_2, \dots, m_k\rangle$ for which $1 \le m_i \le x$, $m_i \equiv a_i \pmod h$, $i = 1, 2, \dots, k$ and $m_1, m_2, \dots, m_k$ are relatively r-prime. An asymptotic formula with 0-estimate for $Q(x; \bar{a}, h, r, k)$ is determined. Special cases of this estimate give earlier estimates for relatively prime integers and r-free integers.

1. Introduction. For $m_1, m_2, \dots, m_k$ integers and r a positive integer we write $(m_1, m_2, \dots, m_k)_r = d^r$ if d is the largest integer for which $d^r \mid m_i (i = 1, 2, \dots, k)$. If $(m_1, m_2, \dots, m_k)_r = 1$, we say $m_1, m_2, \dots, m_k$ are relatively r-prime. Note that in the case $k = 1$, $(m)_r = 1$ means m is r-free. For $a_1, a_2, \dots, a_k$ nonnegative integers, $\bar{a}$ will denote the k-tuple $\langle a_1, a_2, \dots, a_k\rangle$. For h a positive integer and x a positive real number, $Q(x; \bar{a}, h, r, k)$ will denote the number of k-tuples of positive integers $\langle m_1, m_2, \dots, m_k\rangle$ for which $1 \le m_i \le x$, $m_i \equiv a_i \pmod h$, $i = 1, 2, \dots, k$ and $(m_1, m_2, \dots, m_k)_r = 1$.

Letting $g = (a_1, a_2, \dots, a_k)$, it is not difficult to see that if $(g, h)_r \ne 1$, then $Q(x; \bar{a}, h, r, k) = 0$ for all x. Section 3 of this paper is devoted to obtaining an asymptotic formula with 0-estimate for $Q(x; \bar{a}, h, r, k)$ in the case $(g, h)_r \ne 1$. The remaining sections are devoted to showing that special cases of this result give earlier results on the distribution of relatively prime integers and r-free integers, and examining questions of equidistribution of relatively r-prime k-tuples in the (admissible) k-tuples of residue classes $\pmod h$.

2. Preliminaries. A divisor d of n is said to be a unitary divisor if $(d, n/d) = 1$. We write $(a, n)_* = d$ if d is the largest unitary divisor of n which divides a. $\phi^*(n)$ denotes the number of positive integers $a \le n$ for which $(a, n)_* = 1$. Noting first that ϕ^* is multiplicative, it is not difficult to show

$$\phi^*(n) = \prod_{p^e \| n} (1 - p^{-e}).$$

Denote by $\phi_s(n)$ the number of positive integers $a \le n^s$ for which $(a, n^s)_s = 1$. Again noting that ϕ_s is multiplicative, we have

$$\phi_s(n) = n^s \prod_{p \mid n} (1 - p^{-s}).$$

If n is r-free, $\omega_r(n)$, called the r-complement of n, is defined by

$$\omega_r(n) = \prod_{p^e \| n} p^{r-e}. \tag{1}$$

Note that (1) makes sense for $r = 1$ (n must then be 1) if we agree that the empty product is 1.

The following result is not difficult to verify and can be found in [**2**].

Lemma 1. *If $S(x; \alpha, \beta, \gamma)$ denotes the number of solutions of the congruence $\alpha y \equiv \beta \pmod{\gamma}$ with $1 \le y \le x$, then*

$$S(x; \alpha, \beta, \gamma) = \begin{cases} \left[\frac{x}{\gamma}(\alpha, \gamma)\right] + \varepsilon & \text{if } (\alpha, \gamma) \mid \beta \\ 0 & \text{if } (\alpha, \gamma) \nmid \beta \end{cases}$$

where ε is either 0 *or* 1.

The following result plays a central role in the proof of the asymptotic formula with 0-estimate for $Q(x; \bar{a}, h, r, k)$.

Lemma 2. *If $(g, h)_r = 1$ and $rk > 1$, then*

$$\sum_{\substack{d=1 \\ (d^r, h) \mid g}}^{\infty} \frac{\mu(d)}{d^{rk}} (d^r, h)^k = \frac{1}{\zeta(rk)} \frac{h^{rk}}{\phi_{rk}(h)} \frac{\phi^*((\omega_r((g, h)_*))^k)}{(\omega_r((g, h)_*))^k}$$

where ζ denotes the Riemann zeta function.

Proof. Let

$$f(d) = \begin{cases} \frac{\mu(d)}{d^{rk}} (d^r, h)^k & \text{if } (d^r, h) \mid g \\ 0 & \text{if } (d^r, h) \nmid g \end{cases}.$$

It is easy to show that f is a multiplicative function. Since $|f(d)| \le g^k/d^{rk}$ and $\sum g^k/d^{rk}$ converges, $\sum f(d)$ converges absolutely. Therefore the series in the statement of the lemma, $\sum f(d)$, can be expressed as an Euler product; i.e.

$$\sum_{d=1}^{\infty} f(d) = \prod_p (1 + f(p) + f(p^2) + \dots) = \prod_p (1 + f(p)). \tag{2}$$

The last equality above holds because $f(p^a) = 0$ for $a \ge 2$.

Now assume $p^s \| h$ and $p^t \| g$. Then, if $s > t$, $f(p) = 0$ while if $s \le t$, $f(p) = -p^{-k(r-s)}$. Therefore we have

$$\prod_p (1+f(p)) = \prod_{p \nmid h} (1-p^{-kr}) \prod_{\substack{p \| h \\ p \mid g}} (1-p^{-k(r-1)}) \prod_{\substack{p^2 \| h \\ p^2 \mid g}} (1-p^{-k(r-2)}) \cdots \prod_{\substack{p^{r-1} \| h \\ p^{r-1} \mid g}} (1-p^{-k}). \tag{3}$$

Noting that $p^e \| h$ and $p^e \mid g$ is equivalent to $p^e \| (g,h)_*$ and using the fact that $\zeta(rk) = \prod_p (1-p^{-kr})^{-1}$, the right hand side of (3) can be rewritten as

$$(\zeta(rk))^{-1} \prod_{p \mid h} (1-p^{-kr})^{-1} \prod_{p^e \| (g,h)_*} (1-p^{-k(r-e)}). \tag{4}$$

Recalling the product expansion for $\phi_{rk}(h)$ and using the fact that $p^e \| n$ is equivalent to $p^{r-e} \| \omega_r(n)$ (for n r-free), (4) can be expressed as

$$(\zeta(rk))^{-1} h^{rk} (\phi_{rk}(h)))^{-1} \prod_{p^f \| \omega_r((g,h)_*)} (1-p^{-kf}). \tag{5}$$

Using the product expansion for ϕ^*, we have

$$\prod_{p^f \| \omega_r((g,h)_*)} (1-p^{-kf}) = \frac{\phi^*((\omega_r((g,h)_*))^k)}{(\omega_r((g,h)_*))^k}. \tag{6}$$

Combining (2), (3), (4), (5) and (6) proves the lemma. □

3. Asymptotic formula with 0-estimate for $Q(x;\bar{a},h,r,k)$. We will now state and prove the main result.

Theorem 1. *If $(g,h)_r = 1$ and $rk > 1$, then*

$$Q(x;\bar{a},h,r,k) = A\frac{x^k}{\zeta(rk)} + \begin{cases} O(x^{1/r}) & \text{if } k = 1 \\ O(x\log x) & \text{if } r(k-1) = 1 \\ O(x^{k-1}) & \text{if } r(k-1) \geq 2 \end{cases}$$

where

$$A = \frac{h^{k(r-1)}}{\phi_{rk}(h)}\,\frac{\phi^*((\omega_r((g,h)_*))^k)}{(\omega_r((g,h)_*))^k}.$$

Proof. Since

$$\sum_{d^r \mid n^r} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

we have

$$Q(x;\bar{a},h,r,k) = \sum\nolimits^* \sum_{d^r \mid (m_1,m_2,\dots,m_k)_r} \mu(d)$$

where $\sum^*$ denotes the sum over all k-tuples $\langle m_1, m_2, \dots, m_k\rangle$ of integers with $1 \leq m_i \leq x$ and $m_i \equiv a_i \pmod h$, $i = 1,2,\dots,k$. We now count the number of times $\mu(d)$ occurs in the above sum noting that d cannot exceed $x^{1/r}$. If $d^r \mid (m_1,m_2,\dots,m_k)_r$, then $m_i = d^r y_i \equiv a_i \pmod h$ for some integer y_i with $1 \leq y_i \leq x/d^r$, $i = 1,2,\dots,k$. Now, using the notation of Lemma 1, this system of congruences has $\prod_{i=1}^k S(x/d^r; d^r, a_i, h)$ solutions. By Lemma 1, this product is zero unless $(d^r,h) \mid a_i$ for all $i = 1,2,\dots,k$; i.e. unless $(d^r,h) \mid g$. Furthermore, by Lemma 1, if $(d^r,h) \mid g$, the above product is $(x/(d^r h)(d^r,h) + 0(1))^k$. Hence we have

$$\begin{aligned} Q(x;\bar{a},h,r,k) &= \sum_{\substack{1\leq d\leq x^{1/r} \\ (d^r,h) \mid g}} \Big(\frac{x}{d^r h}(d^r,h) + 0(1)\Big)^k \mu(d) \\ &= \frac{x^k}{h^k}\sum_{\substack{1\leq d\leq x^{1/r} \\ (d^r,h) \mid g}} \frac{\mu(d)}{d^{rk}}(d^r,h)^k + x^{k-1}0\left(\sum_{1\leq d\leq x^{1/r}}\left(\frac{g}{d^r h}\right)^{k-1}\right). \end{aligned} \tag{7}$$

Now by Lemma 2 the first term in (7) is

$$A\frac{x^k}{\zeta(rk)} - \frac{x^k}{h^k}\sum_{\substack{x^{1/r}<d\\(d^r,h)\,|\,g}}\frac{\mu(d)}{d^{rk}}(d^r,h)^k. \tag{8}$$

We now show that the second term in (8) is $0(x^{1/r})$.

$$\begin{aligned}\Bigg|\sum_{\substack{x^{1/r}<d\\(d^r,h)\,|\,g}}\frac{\mu(d)}{d^{rk}}(d^r,h)^k\Bigg| &\le g^k\sum_{x^{1/r}<d}\frac{1}{d^{rk}}\\ &\le g^k\int_{x^{1/r}-1}^{\infty}t^{-rk}dt = 0(x^{-k+1/r}).\end{aligned}$$

Now going back to the second term in (7) we have

$$\begin{aligned}O\bigg(\sum_{1\le d\le x^{1/r}}\Big(\frac{g}{d^r h}\Big)^{k-1}\bigg) &= O\bigg(\sum_{1\le d\le x^{1/r}}d^{-r(k-1)}\bigg)\\ &= \begin{cases}O(x^{1/r}) & \text{if } k=1\\ O(\log\,x) & \text{if } r(k-1)=1\\ O(1) & \text{if } r(k-1)\ge 2.\end{cases}\end{aligned}$$

Combining these results we have

$$Q(x;\bar{a},h,r,k) = A\frac{x^k}{\zeta(rk)} + O(x^{1/r}) + \begin{cases}O(x^{1/r}) & \text{if } k=1\\ O(x\log\,x) & \text{if } r(k-1)=1\\ O(x^{k-1}) & \text{if } r(k-1)\ge 2\end{cases}$$

The theorem now follows immediately. □

It should be mentioned that in special cases better error estimates are known; for example, in the case $k = 1$ and $r = 2$ see [**3**, **6**, **7**, **8**]. The error estimates given in Theorem 1 are sufficient for obtaining probabilistic and density results. In the next section we will see how Theorem 1 generalizes and ties together several earlier results.

4. Special cases and probabilistic interpretations. If we take $h = 1$ in Theorem 1, we have the following result of Benkoski [**1**].

Corollary 1. *The number of k-tuples of positive integers $\langle m_1, m_2, \dots, m_k\rangle$ with $1 \le m_i \le x$, $i = 1, 2, \dots, k$ which are relatively r-prime is given by*

$$\frac{x^k}{\zeta(rk)} + \begin{cases} O(x \log x) & \text{if } r = 1 \quad \text{and} \quad k = 2 \\ O(x^{1/r}) & \text{if } k = 1 \\ O(x^{k-1}) & \text{otherwise.} \end{cases}$$

If we further specialize to the case $r = 1$, we have the following result on the distribution of relatively prime integers which can be found in [**3**, **4**].

Corollary 2. *The number of k-tuples $\langle m_1, m_2, \dots, m_k\rangle$ of positive integers with $1 \le m_i \le x$, $i = 1, 2, \dots, k$ which are relatively prime is*

$$\frac{x^k}{\zeta(h)} + \begin{cases} O(x \log x) & \text{if } k = 2, \\ O(x^{k-1}) & \text{if } k > 2. \end{cases}$$

If we specialize Corollary 1 to the case $k = 1$, we have the following classical result on the distribution of r-free integers.

Corollary 3. *The number of r-free integers which do not exceed x is given by*

$$\frac{x}{\zeta(r)} + O(x^{1/r}).$$

If we specialize Theorem 1 to the case $k = 1$, we have the following result of E. Cohen and R.L. Robinson [**2**] on the distribution of r-free integers in residue classes.

Corollary 4. *If (a, h) is r-free, then the number of r-free integers not exceeding x and in the residue class of $a \pmod h$ is given by*

$$\frac{h^{r-1}}{\phi_r(h)} \frac{\phi^*(\omega_r((a,h)_*))}{\omega_r((a,h)_*)} \frac{x}{\zeta(r)} + O(x^{1/r}).$$

If we specialize Theorem 1 to the case $r = 1$, we have the following result on the distribution of relatively prime integers in residue classes.

Corollary 5. *If* $(a_1, a_2, \dots, a_k, h) = 1$*, then the number of* k*-tuples* $\langle m_1, m_2, \dots, m_k \rangle$ *of integers with* $1 \le m_i \le x$ *and* $m_i \equiv a_i \pmod{h}$*,* $i = 1, 2, \dots, k$ *which are relatively prime is given by*

$$\frac{1}{\phi_k(h)} \frac{x^k}{\zeta(k)} + \begin{cases} O(x \log x) & \text{if } k = 2, \\ O(x^{k-1}) & \text{if } k > 2. \end{cases}$$

Theorem 1 and each of the above corollaries have probabilistic interpretations. The following corollary is an example.

Corollary 6. *Assume* $(a_1, a_2, \dots, a_k, h) = 1$*. If* k *positive integers are chosen such that the* i*th integer is chosen at random from the residue class of* $a_i \pmod{h}$*, then the (limiting) probability that these* k *integers are relatively prime is*

$$\frac{1}{\zeta(k)} \prod_{p \mid h} (1 - p^{-k})^{-1}.$$

Proof. If the ith integer is chosen between 1 and x and in the residue class of $a_i \pmod{h}$, then there are $[x/h]$ ways that integer can be chosen. Hence the limiting probability of the corollary is

$$\lim \frac{Q(x; \bar{a}, h, 1, k)}{[x/h]^k} = \frac{h^k}{\phi_k(h)} \frac{1}{\zeta(k)} = \frac{1}{\zeta(k)} \prod_{p \mid h} (1 - p^{-k})^{-1}.$$

It should be noted that the probability given in Corollary 6 is only dependent on the distinct prime factors of h.

5. Equidistribution. In this section a_i will denote both the integer a_i and the residue class of $a_i \pmod{h}$ and $\bar{a} = \langle a_i, a_2, \dots, a_k \rangle$ will denote both the k-tuple of integers and the k-tuple of residue classes $\pmod{h}$; the context will make clear which is intended. A

k-tuple $\langle a_1, a_2, \dots, a_k \rangle$ of residue classes (mod h) will be called *r-admissible* if $(a_1, a_2, \dots, a_k, h)_r = 1$. From Theorem 1, these are the k-tuples of residue classes for which there exist k-tuples of integers $\langle m_1, m_2, \dots, m_k \rangle$ with $m_i \equiv a_i \pmod{h}$ and $(m_1, m_2, \dots, m_k)_r = 1$. Let $\Phi(h; r, k)$ denote the number of r-admissible k-tuples of residue classes (mod h).

Theorem 2.

$$\Phi(h; r, k) = h^k \prod_{p^r \mid h} (1 - p^{-rk})$$

Proof. Note first that $(a_1, a_2, \dots, a_k, h)_r = d^r$ if and only if $(a_1/d^r, a_2/d^r \dots, a_k/d^r, h/d^r)_r = 1$. Hence

$$h^k = \sum_{d^r \mid h} \Phi(h/d^r; r, k).$$

Now applying (a form of) the Möbius inversion formula we have

$$\Phi(h; r, k) = \sum_{d^r \mid h} \mu(d)(h/d^r)^k = h^k \sum_{d^r \mid h} \mu(d)/d^{rk}.$$

Using the fact that for f a multiplicative function,

$$\sum_{d^r \mid h} \mu(d) f(d) = \prod_{p^r \mid h} (1 - f(p)),$$

the theorem follows immediately. □

The *density* of the k-tuples of relatively r-prime integers in the k-tuple of residue classes $\bar{a}$, denoted by $\delta(\bar{a}, h, r, k)$, is defined by

$$\delta(\bar{a}, h, r, k) = \lim_{x \to \infty} \frac{Q(x; \bar{a}, h, r, k)}{x^k}$$

From this definition and Theorem 1, we have the following result.

Lemma 3.

$$\delta(\bar{a}, h, r, k) = \begin{cases} A/\zeta(rk) & \text{if } \bar{a} \text{ is } r\text{-admissible} \\ 0 & \text{if } \bar{a} \text{ is not } r\text{-admissible} \end{cases}$$

where A is as given in Theorem 1.

The relative density of the k-tuples of relatively r-prime integers in the k-tuple of residue classes $\bar{a}$, denoted by $\delta^*(\bar{a}, h, r, k)$, is defined by

$$\delta^*(\bar{a}, h, r, k) = \lim_{x\to\infty} \frac{Q(x; \bar{a}, h, r, k)}{Q(x; \bar{a}, 1, r, k)}.$$

From the definitions of $\delta(\bar{a}, h, r, k)$ and $\delta^*(\bar{a}, h, r, k)$ and Theorem 1, we have the following relationship:

$$\delta^*(\bar{a}, h, r, k) = \zeta(rk)\delta(\bar{a}, h, r, k).$$

The k-tuples of relatively r-prime integers are said to be *equidistributed* $\pmod h$ if the relative density $\delta^*(\bar{a}, h, r, k)$ has the same value for each r-admissible k-tuple $\bar{a}$ of residue classes $\pmod h$.

Since there are $\Phi(h; r, k)$ r-admissible k-tuples of residue classes, we have the following result.

Lemma 4. *The k-tuples of relatively r-prime integers are equidistributed* $\pmod h$ *if and only if* $\delta^*(\bar{a}, h, r, k) = 1/\Phi(h; r, k)$ *for every r-admissible $\bar{a}$.*

A positive integer h is said to be r-full if whenever a prime p divides n, p^r also divides n. We are now ready to state the main result of this section which generalizes a result of Cohen and Robinson [**2**].

Theorem 3. *The k-tuples of relatively r-prime integers are equidistributed* $\pmod h$ *if and only if h is r-full.*

Proof. By Lemmas 2 and 3 and Theorems 1 and 2, equidistribution $\pmod h$ occurs if and only if

$$\frac{\phi^*((\omega_r((g,h)_*))^k)}{(\omega_r((g,h)_*))^k} = \prod_{\substack{p \mid h \\ p^r \nmid h}} (1 - p^{-rk}) \tag{9}$$

for every r-admissible $\bar{a} = \langle a_1, a_2, \dots, a_k\rangle$ where $g = (a_1, a_2, \dots, a_k)$. Now if h is r-full and $\bar{a}$ is r-admissible, $(g, h)_* = 1$ and hence the left hand side of (9) is 1. Also the right hand side of (9) is 1 since it is the empty product. Conversely assume (9) holds for every r-admissible $\bar{a}$. Taking $\bar{a} = (1, 1, \dots, 1)$, the left hand side of (9) is 1. Hence the right hand side of (9) must be the empty product and hence h is r-full. □

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