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A THEOREM ON REPRODUCING KERNEL HILBERT SPACES OF PAIRS

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ABSTRACT. In this paper we study reproducing kernel Hilbert and Banach spaces of pairs. These are a generalization of reproducing kernel Hilbert spaces and, roughly speaking, consist of pairs of Hilbert (or Banach) spaces of functions in duality with respect to a sesquilinear form and admitting a left and a right reproducing kernel. We first investigate some properties of these spaces of pairs. It is then proved that to every function $K(z,w)$ analytic in z and w^* there is a neighborhood of the origin that can be associated with a reproducing kernel Hilbert space of pairs with left reproducing kernel $K(z,w)$ and right reproducing kernel $K(w,z)^*$.

1. Introduction. Hilbert spaces of functions with bounded point evaluations (reproducing kernel Hilbert spaces, the definition is recalled in the sequel) play an important role in a number of areas in analysis (see, e.g., [**7, 8, 10, 21**]). The special case of reproducing kernels of the form

$$K(z,w)=\frac{X(z)JX(w)^*}{\rho_w(z)}, \tag{1}$$

where

(a) J is a $\mathbf{C}^{m\times m}$ matrix subject to $J=J^*=J^{-1}$,

(b) X is a $\mathbf{C}^{k\times m}$ valued function meromorphic in Δ_+, where Δ_+ denotes either the open unit disk $\mathbf{D}$ or the open upper half plane $\mathbf{C}^+$,

and

(c) the function ρ is defined by

$$\rho_w(z)=\begin{cases}1-zw^* & \text{if } \Delta_+=\mathbf{D}\\ -2\pi i(z-w^*) & \text{if } \Delta_+=\mathbf{C}^+,\end{cases}$$

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is of particular interest. The associated reproducing kernel Hilbert spaces provide a convenient unifying framework for a number of problems in interpolation theory and lossless inverse scattering (see [**17, 4**], and for extensions to Pontryagin and Krein spaces [**1**] and [**5**]).

These spaces (called $\mathcal{B}(X)$ spaces in the above mentioned references) were first defined, for various choices of X and J, by L. de Branges and L. de Branges and J. Rovnyak (see [**10, 11, 12, 13**]), in their study of operator models.

The notion of Hilbert space extends naturally to the case of pairs of Hilbert spaces in duality with respect to a sesquilinear form and the notion of reproducing kernel space was extended in [**6**] to cover the case of spaces whose elements are pairs of functions endowed with a (in general, non-Hermitian) sesquilinear form. The paper [**6**] considered the finite dimensional case but an instance of an infinite dimensional vector space of functions which has a reproducing kernel with respect to a non-Hermitian inner product already appears in [**18**, section 7].

In the examples in [**6**] and [**18**], the reproducing kernel has non-Hermitian form

$$K(z,w) = \frac{X_L(z)JX_R(w)^*}{\rho_w(z)} \tag{2}$$

where J and ρ are as above and both X_L and X_R are $\mathbf{C}^{k\times m}$ valued and meromorphic in Δ_+.

Another example of function of the form (2) appears in [**9**], where transmission lines with left and right lattices are studied.

These examples suggest that the theory of $\mathcal{B}(X)$ spaces extends to the non-Hermitian case of pair of spaces with reproducing kernel of the form (2), such an extension giving a unifying framework for these (and other) examples.

The main result of this paper (Theorem 1) associates to any function $K(z,w)$ analytic in a neighborhood of the origin (in z and w^*) a reproducing kernel Hilbert space of pairs (the precise definition follows). These spaces of pairs should provide the tool to extend the theory of $\mathcal{B}(X)$ spaces to the non-Hermitian case. This extension will be presented elsewhere.

With an eye on future applications we do not restrict ourselves to functions of the form (2) but start, in Section 2, with arbitrary functions $K(z, w)$ and then study the analytic case.

We now define precisely some of the notions presented above.

Let B_L and B_R be two vector spaces over the complex numbers. The map $(x, y) \to [x, y]$ from $B_L \times B_R$ into $\mathbf{C}$ is called *bilinear* if it is linear in both variables and is called sesquilinear if it is linear in x and antilinear in y: for every choice of x in B_L, y_1, y_2 in B_R and α_1, α_2 in $\mathbf{C}$,

$$[x, \alpha_1 y_1 + \alpha_2 y_2] = \alpha_1^*[x, y_1] + \alpha_2^*[x, y_2].$$

Definition 1. Let B_L and B_R be two Banach spaces of $\mathbf{C}^n$ valued functions defined on some set Ω, with norm $|| \quad ||_L$ and $|| \quad ||_R$, respectively, and let $[\quad ,\quad]$ be a sesquilinear form on $B_L \times B_R$. Then $(B_L \times B_R, [\quad ,])$ is a *reproducing kernel Banach space of pairs* if there exists a pair (K^L, K^R) of $\mathbf{C}^{n\times n}$ valued functions defined on $\Omega \times \Omega$ and such that

(i) For every w in Ω and c in $\mathbf{C}^n$, the function $z \to K^R(z, w)c$ belongs to B_R and, for every f in B_L,

$$[f, K^R(\cdot, w)c] = c^* f(w). \tag{3}$$

(ii) For every w and c as in (i), the function $z \to K^L(z, w)c$ belongs to B_L and, for every g in B_R,

$$[K^L(\cdot, w)c, g] = g(w)^* c. \tag{4}$$

(iii) The continuous linear functionals from B_L (respectively, B_R) into $\mathbf{C}$ are exactly the maps of the form

$$\varphi(f) = [f, g_\varphi]$$
$$(\text{resp. } \varphi(g) = [g, f_\varphi]^*)$$

where g_φ (respectively, f_φ) spans B_R (respectively, B_L).

($\mathbf{C}^{n\times p}$ denotes the space of n rows p columns matrices with complex entries and $\mathbf{C}^n$ is short for $\mathbf{C}^{n\times 1}$. The adjoint of a matrix A is denoted by A^*; thus, w^* is the complex conjugate of the complex number w; the identity matrix of $\mathbf{C}^{n\times n}$ will be denoted by I_n).

The reproducing kernel property implies that g_φ and f_φ are uniquely determined, and from the uniform boundedness principle we see that (iii) implies:

(iv) There exists a constant $K < \infty$ such that, for every (f, g) in $B_L \times B_R$,

$$|[f, g]| \leq K||f||_L \cdot ||g||_R \tag{5}$$

(for bilinear forms, see, e.g., [**16**, p. 70]).

Condition (iii) is of a topological nature and permits identifying via the map $\varphi \to g_\varphi$ (respectively, $\varphi \to f_\varphi$) B_R (respectively, B_L) with the topological dual of B_L (respectively, B_R). For more information on pair of spaces in duality with respect to a sesquilinear or a bilinear form we refer to [**15, 20**]. When $B_L = B_R = B$ with norm $||\quad||_L = ||\quad||_R = ||\quad||$ and $[\quad,\quad]$ is an Hermitian form, the norm $||\quad||$ is called a Banach majorant when (iii) is in force while it is called an admissible majorant if (iv) is met (see [**14**]).

When the space B_L and B_R are Hilbert spaces, $(B_L \times B_R[\quad,\quad])$ will then be called a *reproducing kernel Hilbert space of pairs.*

When the spaces B_L and B_R are not necessarily Banach spaces and (iii) is not required, $(B_L \times B_R, [\quad,\quad])$ will be called a *reproducing kernel space of pairs.*

Lemma 1. *Let* $(B_L \times B_R, [\quad,\quad])$ *be a reproducing kernel space of pairs; then the pair of functions* (K^L, K^R) *is unique and satisfies the relationship*

$$K^L(z, w)^* = K^R(w, z). \tag{6}$$

The proof of this lemma is simple and will be omitted. The pair (K^L, K^R) will be called the *reproducing kernel* of $(B_L \times B_R, [\quad,\quad])$.

The following question is of interest: given a pair of $\mathbf{C}^{n\times n}$ valued function (K^L, K^R) satisfying (6), construct (if any) a reproducing kernel Banach space of pairs with reproducing kernel (K^L, K^R). The case of kernels defined for z, w in some open real interval and of class $\mathcal{C}^3$ was considered in [**3**]. The methods there relied on properties of compact operators. Here using different methods (close to those of [**2**]), we focus on the particular case where $K^L(z,w)$ is analytic in z and w^* in a neighborhood of the origin, and we will prove:

Theorem 1. *Let $K(z,w)$ be a $\mathbf{C}^{n\times n}$ valued function analytic in z and w^* in a neighborhood of the origin $\mathcal{V}$. Then there exists a reproducing kernel Hilbert space of pairs $(H_L \times H_R, [\quad,\quad])$ with reproducing kernel (K^L, K^R), where $K^L(z,w) = K(z,w)$ and $K^R(z,w) = K(w,z)^*$. The elements of H_L and H_R are analytic in a neighborhood $\mathcal{V}' \subset \mathcal{V}$ of the origin.*

The outline of the paper is as follows: In Section 2 we give some examples of reproducing kernel Banach spaces of pairs and discuss a number of properties of these pairs. In Section 3 we present results on operator ranges which are needed in Section 4, where Theorem 1 is proved.

2. Examples and first properties. In the first two examples we relate the notions of reproducing kernel Banach space of pairs to more classical definitions.

Example 1. *Reproducing kernel Hilbert spaces* [**7**]. These correspond to the case where $B_L = B_R = B$ is a Hilbert space and $[\quad,\quad]$ coincides with the inner product in B. Then (iii) follows from the Riesz representation theorem and $K^L = K^R$.

Example 2. *Reproducing kernel Krein spaces.* These correspond to the case where $B_L = B_R = B$ is a Hilbert space with inner product $\langle\quad,\quad\rangle$ and linked to $[\quad,\quad]$ by:

$$[x, y] = \langle x, \sigma y\rangle$$

where σ is both unitary and self-adjoint.

In these examples, the sesquilinear form is Hermitian, $K^L = K^R = K$ and the space B (rather than the product $B \times B$) is called a reproducing kernel Hilbert (or Krein) space. Then (6) becomes

$$K(z,w) = K(w,z)^*. \tag{7}$$

In the case of a reproducing kernel Hilbert space, the function $K(z,w)$ is moreover positive (for every integer r, every $c_1,\dots,c_r$ in $\mathbf{C}^n$ and $w_1,\dots,w_r$ in Ω, $\sum_{i,j=1}^r c_i^* K(w_i,w_j)c_j \geq 0$) and there is a one-to-one correspondence between positive functions and reproducing kernel Hilbert spaces [**7, 21**]. In the case of reproducing kernel Krein spaces there is an onto (but not one-to-one) correspondence between difference of positive functions on Ω and reproducing kernel Krein spaces of $\mathbf{C}^n$ valued functions defined on Ω. For details and a nonuniqueness counterexample we refer to [**21**] which exposes the theory of reproducing kernel Hilbert spaces from a very general point of view. That paper and another nonuniqueness example are discussed in [**2**].

The nonuniqueness feature is even more present in the case of reproducing kernel Banach spaces of pairs, as illustrated by the next example.

Example 3. Let p and q be greater than 1 such that $1/p + 1/q = 1$, and let $B_L = H^p$ and $B_R = H^q$, the classical Hardy spaces of functions analytic in the unit disk. Then (H^p, H^q) endowed with the form

$$[f,g] = \frac{1}{2\pi}\int_0^{2\pi} g(e^{it})^* f(e^{it})\,dt$$

is a reproducing kernel Banach space of pairs with reproducing kernel (K^L, K^R) with $K^L(z,w) = 1/(1-zw^*)$.

Indeed, properties (i) and (ii) follow from Cauchy's formula (which still holds in H^p), and (iii) follows from the duality between H^p and H^q for the given choice of p and q.

Thus, there is a unique reproducing kernel Hilbert space with reproducing kernel $1/(1-zw^*)$ (namely, H^2) but a whole family of reproducing kernel Banach spaces of pairs with reproducing kernel $K^L(z,w) = K^R(z,w) = 1/(1-zw^*)$, as illustrated in Example 3.

The next example is taken from [**6**]. The spaces are finite dimensional, and thus the situation is much simpler: the topological requirements of (iii) are automatically met.

Example 4. Let B_L (respectively, B_R) be a finite dimensional space of $\mathbf{C}^n$ valued functions defined on a set Ω, of dimension N, and let $f_1, \dots, f_N$, (respectively $g_1, \dots, g_N$) be a basis of B_L (respectively, B_R). Let $[\quad,\quad]$ be a sesquilinear form defined on $B_L \times B_R$ and let G, the Gram matrix, be defined by

$$g_{ij} = [f_j, g_i]. \tag{8}$$

Then $(B_L \times B_R, [\quad,\quad])$ is a reproducing kernel Banach pair of spaces if and only if G is invertible. The reproducing kernel is given by

$$K^L(z,w) = \sum_{i,j=1}^{n} f_i(z)\gamma_{ij} g_j(w)^* \tag{9a}$$

and

$$K^R(z,w) = \sum_{i,j=1}^{n} g_j(z)\gamma_{ij}^* f_i(w)^* \tag{9b}$$

where γ_{ij} denotes the ij entry of the inverse G^{-1}.

The next result deals with reproducing kernel spaces of pairs (without the requirements (iii)).

Theorem 2. *Let (K^L, K^R) be a pair of $\mathbf{C}^{n\times n}$ valued functions satisfying* (6)*, and let V_L, V_R be defined by*:

$$V_L = \text{linear span}\,\{K^L(\cdot, w)c, w \in \Omega, c \in \mathbf{C}^n\} \tag{10a}$$

$$V_R = \text{linear span}\,\{K^R(\cdot, \nu)d, \nu \in \Omega, d \in \mathbf{C}^n\} \tag{10b}$$

and let on $V_L \times V_R$ a sesquilinear form $[\quad,\quad]_0$ be defined by

$$[K^L(\cdot, w)c, K^R(\cdot, \nu)d]_0 = d^* K^L(\nu, w)c. \tag{11}$$

Then $(V_L \times V_R, [\quad,\quad]_0)$ is a reproducing kernel space of pairs with reproducing kernel (K^L, K^R). Moreover, any reproducing kernel Banach space of pairs $(B_L \times B_R, [\quad,\quad])$ with reproducing kernel (K^L, K^R) will contain isometrically $(V_L \times V_R, [\quad,\quad]_0)$, that is,

$$V_L \subset B_L, \qquad V_R \subset B_R \tag{12}$$

and

$$[f,g]_0 = [f,g]$$

for $f, g \in V_L \times V_R$.

We omit the proof of this result and just mention that the main step is to verify that $[\quad,\quad]_0$ is well-defined.

In general the spaces V_L and V_R have no nice topological structure.

Theorem 3. *Let* $(B_L \times B_R, [\quad,\quad])$ *be a reproducing kernel Banach space of pairs and let* V_L, V_R *be as in* (10). *Then* V_L (*respectively,* V_R) *is dense in* B_L (*respectively,* B_R).

Proof. Let $||\quad||_L$ denote the norm of B_L and let $\overline{V}_L$ be the closure of V_L in B_L in the $||\quad||_L$ norm. If $\overline{V}_L \neq B_L$, by the Hahn-Banach theorem we can find a continuous linear functional φ which is nonzero and vanishes on $\overline{V}_L$. By (iii), $\varphi(f) = [f, g_\varphi]$ for some nonzero g_φ in B_R. Setting $f = K^L(\cdot, w)c$ we obtain $g_\varphi = 0$, a contradiction. Hence, $\overline{V}_L = B_L$ and, similarly, $\overline{V}_R = B_R$. □

The case where B_L and B_R are Hilbert spaces is of special interest. Then, under certain hypotheses (which will be satisfied in the applications in Sections 3 and 4), conditions (iii) and (iv) are in fact equivalent. Let us suppose (iv) is in force. By the Riesz representation theorem, there exist two linear operators G_L (from B_L into B_R) and G_R (from B_R into B_L) such that, for every (f,g) in $B_L \times B_R$,

$$[f,g] = \langle f, G_R g\rangle_L = \langle G_L f, g\rangle_R \tag{13}$$

(where $\langle\quad,\quad\rangle_i$ denotes the inner product of B_i, $i = L, R$).

We claim that G_R and G_L are bounded operators. Indeed,

$$||G_R g||_L = \sup_{f \in B_L} \frac{|\langle f, G_R g\rangle_L|}{\langle f, f\rangle_L^{1/2}} = \sup \frac{|[f,g]|}{\langle f, f\rangle_L^{1/2}}$$

and hence, using (5), we obtain $||G_R g||_L \leq K||g||_R$, $g \in B_R$; thus,

$$||G_R|| \leq K.$$

Similarly, $||G_L|| \leq K$.

Equation (13) expresses the fact that G_R and G_L are adjoint operators. Moreover, they have a kernel reduced to $\{0\}$. Indeed, let g be in B_R such that $G_R g = 0$. The choice of $f = K^L(\cdot, w)c$ in (13) leads to

$$g(w)^* c = [K^L(\cdot, w)c, g] = \langle K^L(\cdot, w)c, G_R g\rangle_L = 0,$$

and hence $g = 0$, so that $\operatorname{Ker} G_R = \{0\}$ and, similarly, $\operatorname{Ker} G_L = \{0\}$.

Proposition 1. *If B_L and B_R are Hilbert spaces and G_R is invertible, then (iii) and (iv) are equivalent.*

Proof. Let φ be a bounded, linear functional on B_L. Then, by the Riesz representation theorem,

$$\varphi(f) = \langle f, h_\varphi\rangle_L$$

for a unique h_φ in B_L, which can be rewritten as

$$\varphi(f) = [f, g_\varphi]$$

with $g_\varphi = G_R^{-1} h_\varphi$, hence the result. □

3. Operator ranges in Hilbert spaces. In this section we gather a number of results on operator ranges which will be needed in the proof of Theorem 1. In the whole section, H denotes a Hilbert space with norm $||\ \ ||$ and inner product $\langle, \rangle$. We first review the polar decomposition of an operator T in $L(H)$.

Lemma 1. *Let T be a bounded linear operator in $L(H)$. Then T can be written as $T = UP$ where P is a positive operator and U is a partial isometry. More precisely, $P = \sqrt{T^*T}$, the positive square root of T^*T and U is uniquely determined by the condition $\operatorname{Ker} U = \operatorname{Ker} P$. For this U, we have $P = U^*T$.*

For a proof of this result, see [**19**]. The decomposition $T = UP$ where $\operatorname{Ker} U = \operatorname{Ker} P$ is called the polar decomposition of T. We note that $P = U^*T$ implies that $PU^* = U^*TU^*$ and, hence,

$$T^* = U^*TU^*. \tag{14}$$

Moreover, if $u \in \operatorname{Ker} T$, $Tu = 0$, then $Pu \in \operatorname{Ker} U$, hence $Pu \in \operatorname{Ker} P$ and so, $P^2u = 0$ which implies $Pu = 0$ since P is a positive operator.

Let T now be in $L(H)$ with polar decomposition $T = UP$. We introduce two spaces H_L and H_R as follows:

$$H_L = \{F = U\sqrt{P}u, u \in H;\ ||F||_L = ||(I-\pi)u||\}$$

and

$$H_R = \{G = \sqrt{P}u, u \in H;\ ||G||_R = ||(I-\pi)u||\}$$

where π denotes the orthogonal projection with range the kernel of P.

Lemma 2. *The spaces H_L and H_R respectively endowed with $||\quad||_L$ and $||\quad||_R$ are Hilbert spaces. Moreover,* $\operatorname{Ran} T$ *(respectively,* $\operatorname{Ran} T^*$*) is dense in H_L (respectively, H_R), and $||Tu||_L = ||\sqrt{P}u||$, $||T^*u||_R = ||\sqrt{P}U^*u||$ for $u \in H$.*

Proof. We begin with the case of H_L and first check that $||\quad||_L$ is well defined. If $F = U\sqrt{P}u_1 = U\sqrt{P}u_2$ for two different elements u_1 and u_2 in H, then $\sqrt{P}(u_1 - u_2) \in \operatorname{Ker} U = \operatorname{Ker} P^{1/2}$ and so $P(u_1 - u_2) = 0$, i.e., $\pi(u_1 - u_2) = (u_1 - u_2)$ and hence $u_1 - \pi u_1 = u_2 - \pi u_2$. In particular, these two elements have the same norm and $||F||_L$ is well defined.

To prove that $||\quad||_L$ is a norm, the only difficulty is to show that $F = 0$ if and only if $||F||_L = 0$.

Indeed, if $F = U\sqrt{P}u \in H_L$ is such that $||F||_L = 0$, then $\pi u = u$ and so $u \in \operatorname{Ker} P^{1/2}$, hence $F = 0$. Conversely, if $F = U\sqrt{P}u = 0$, then $\sqrt{P}u \in \operatorname{Ker} U = \operatorname{Ker} P$ so that $P^{3/2}u = 0$; hence, $P^{1/2}u = 0$, $u = \pi u$ and $||F||_L = 0$. It is then easy to check that $||\quad||_L$ is a quadratic norm with associated inner product

$$\langle F_1, F_2 \rangle_L = \langle (I-\pi)u_1, u_2 \rangle$$

(with $F_i = U\sqrt{P}u_i$, $i = 1, 2$). Thus, $(H_L, \langle , \rangle_L)$ is a pre-Hilbert space. To prove that it is complete, let $(F_n) = (U\sqrt{P}u_n)$ be a Cauchy sequence in H_L. Then $(I-\pi)u_n$ is a Cauchy sequence in H and converges to an element h in H such that $\pi h = 0$. Thus, F_n converges in the $||\quad||_L$ norm to $U\sqrt{P}h$.

We now prove that $\operatorname{Ran} T$ is dense in H_L. From the polar decomposition it is plain that $\operatorname{Ran} T \subset H_L$. Let now $F = U\sqrt{P}u$ be orthogonal to $\operatorname{Ran} T$. Then, for every v in H,

$$\begin{aligned} 0 &= \langle F, Tv\rangle_L \\ &= \langle U\sqrt{P}u, U\sqrt{P}\sqrt{P}v\rangle_L \\ &= \langle (I-\pi)u, \sqrt{P}v\rangle \\ &= \langle u, \sqrt{P}v\rangle \end{aligned}$$

where we have used the identity

$$(I-\pi)\sqrt{P} = \sqrt{P} \tag{15}$$

and so, $\sqrt{P}u = 0$ and $F = 0$.

Finally, if $u \in H$, $||Tu||_L = ||\sqrt{P}u||$ follows also from (15). The claims on H_R are proved similarly. □

On $\operatorname{Ran} T \times \operatorname{Ran} T^*$ we define a sesquilinear form by

$$[Tu, T^*v]_T = \langle Tu, v\rangle. \tag{16}$$

If $Tu_1 = Tu_2$ and $T^*v_1 = T^*v_2$, we have

$$\langle Tu_1, v_1\rangle = \langle Tu_2, v_2\rangle$$

and, hence, $[\ ,\ ]$ is well-defined. Moreover,

$$\begin{aligned} |[Tu, T^*v]_T &= |\langle Tu, v\rangle| \\ &= |\langle \sqrt{P}u, \sqrt{P}U^*v\rangle| \\ &\le ||\sqrt{P}u|| \cdot ||\sqrt{P}U^*v||, \end{aligned}$$

i.e., using Lemma 1,

$$|[Tu, T^*v]_T| \le ||Tu||_L \cdot ||T^*v||_R. \tag{17}$$

The form $[\ ,\ ]$ thus admits a unique continuous extension to the product $H_L \times H_R$. We claim that the associated operators G_L and

G_R (defined in Section 2) are invertible. Indeed, from the sequence of equalities,

$$\begin{aligned}[Tu, T^*v]_T &= \langle Tu, v\rangle \\ &= \langle Pu, U^*v\rangle \\ &= \langle \sqrt{P}u, \sqrt{P}U^*v\rangle \\ &= \langle U\sqrt{P}\sqrt{P}u, U\sqrt{P}\sqrt{P}U^*v\rangle_L \\ &= \langle Tu, U\sqrt{P}\cdot\sqrt{P}U^*v\rangle_L\end{aligned}$$

we obtain $G_R(T^*v) = TU^*v$ which implies that $U^*G_RT^*v = U^*TU^*v$.

Using (14), we get $U^*G_RT^* = T^*$, that is,

$$U^*G_R = I \tag{18}$$

on $\operatorname{Ran} T^*$.

Similarly, we have $G_LTu = Pu$, $u \in H$, which implies that $UG_LT = T$ and, hence,

$$UG_L = I \tag{19}$$

on $\operatorname{Ran} T$.

Since $\operatorname{Ran} T$ (respectively, $\operatorname{Ran} T^*$) is dense in H_L (respectively, H_R) we deduce that (18) and (19) hold on H and so G_L and G_R are invertible.

We conclude this section with the following lemma:

Lemma 3. *Let T be in $L(H)$ and H_L, H_R, $[\quad,\quad]_T$ associated to T as above. We suppose that H is a reproducing kernel Hilbert space of $\mathbf{C}^n$ valued functions defined on a set Ω with reproducing kernel $k(z,w)$. Then $(H_L \times H_R, [\quad,\quad]_T)$ is a reproducing kernel Hilbert space of pairs with reproducing kernel (K^L, K^R) defined by*

$$\begin{aligned}K^L(z,w)c &= (Tk(\cdot,w)c)(z)\\ K^R(z,w)c &= (T^*k(\cdot,w)c)(z)\end{aligned}$$

where $w \in \Omega$ and $c \in \mathbf{C}^n$.

Proof. The claims follow only from the equalities

$$[Tu, K^R(\cdot, w)c]_T = \langle Tu, k(\cdot, w)c\rangle = c^*(Tu)(w)$$

and

$$[K^L(\cdot, w)c, T^*v]_T = \langle k(\cdot, w)c, T^*v\rangle = (T^*v)(w)^*c. \qquad \square$$

4. Proof of Theorem 1. In this section H_n^2 denotes the Hilbert space of $n \times 1$ column vectors with entries in H^2, the classical Hardy space of the circle, with norm

$$\left\| \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\|_{H_n^2}^2 = \sum_{i=1}^n ||f_i||_{H^2}^2.$$

The space H_n^2 is a reproducing kernel Hilbert space with reproducing kernel

$$k(z, w) = \frac{I_n}{1 - zw^*}.$$

To prove Theorem 1, we first suppose that the function $K(z, w)$ is analytic in z and w^* for z and w^* of modulus less than r with $r > 1$. The map $f \to Tf$,

$$(Tf)(z) = \frac{1}{2\pi}\int_0^{2\pi} K^L(z, e^{it})f(e^{it})\, dt$$

is then a bounded operator from H^2 into itself, with adjoint operator

$$(T^*f)(z) = \frac{1}{2\pi}\int_0^{2\pi} K^R(z, e^{it})f(e^{it})\, dt$$

(with $K^L(z, w) = K(z, w)$ and $K^R(z, w) = K(w, z)^*$).

Using Lemma 2 we construct a pair (H_L, H_R) of Hilbert spaces and define on $H_L \times H_R$ the sesquilinear form $[\quad , \quad]_T$ defined in (16). It satisfies inequality (17). By the analysis of Section 3 the Gram operator G_L is invertible and, thus, by Proposition 1, (iii) of Definition

1 is in force. Finally, Lemma 3 implies that $(H_L \times H_R, [\quad,\quad]_T)$ is a reproducing kernel Hilbert space of pairs with reproducing kernel

$$((Tk(\cdot, w)c)(z), (T^*k(\cdot, w)c)(z)),$$

that is, using Cauchy's formula,

$$(K^L(z, w)c, K^R(z, w)c).$$

We now suppose that the function $K(z, w)$ is analytic in z and w^* for z and w^* in some neighborhood $\mathcal{V}$ of the origin. Then $\mathcal{V}$ contains a ball B of radius r, $r < 1$, and for $\rho < r$ the function $K_\rho(z, w) = K(\rho z, \rho w)$ is analytic in z and w^* for z and w^* of modulus less than r/ρ. Since $r/\rho > 1$ we are in the case of the first part of the proof and there exists a reproducing kernel Hilbert space of pairs $(H_{L,\rho} \times H_{R,\rho}, [\quad,\quad]_\rho)$ with reproducing kernel pair $(K_\rho^L(z, w), K_\rho^R(z, w))$ where $K_\rho^L(z, w) = K_\rho(z, w)$ and $K_\rho^R(z, w) = K_\rho^L(w, z)^*$.

Now let $(H_L \times H_R, [\quad,\quad])$ be defined by

$$H_i = \{f : f(z) = F(z/\rho), F \in H_{i,\rho}\} \qquad i \in L, R$$

and

$$[f, g] = [F, G]_\rho$$

(where $G(z/\rho) = g(z)$, $G \in H_{R,\rho}$).

The function $z \to K(z, \rho w)c$ belongs to H_L and, for every f in H_L,

$$[f, K(\cdot, \rho w)] = [F, K(\rho\cdot, \rho w)c]_\rho = c^*F(\rho w) = c^*f(w),$$

and, similarly, the function $z \to K(\rho w, z)^*c$ belongs to H_L and for every g in H_R

$$[K(\rho w, \cdot)^*c, g] = [K(\rho w, \rho\cdot)^*c, G]_\rho = G(\rho w)^*c = g(w)^*c.$$

So, the pair $(H_L \times H_R, [\quad,\quad])$ is a reproducing kernel Hilbert space of pairs with reproducing kernel (K^L, K^R).

Finally, we note that the elements of H_L and H_R are by construction analytic in $\mathcal{V}' = \{z, |z| < \rho\}$ and $\mathcal{V}' \subset \mathcal{V}$ which concludes the proof of Theorem 1. $\square$

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