ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 2, Spring 1992

UNCONDITIONALLY CONVERGING AND COMPACT OPERATORS ON c_0

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ABSTRACT. It is shown that if Y is a Banach space, then c_0 imbeds in Y if and only if for every infinite dimensional Banach space X, there exists a noncompact operator $T : X \to Y$. In order to prove this, we first examine properties of operators on c_0, showing that if $T : c_0 \to X$ is a noncompact operator, then there exists a subspace Z of c_0 such that Z is isomorphic to c_0 and $T|_Z$ is an isomorphism.

The Josefson-Nissenzweig Theorem states that if X is an infinite dimensional Banach space, then there exists a weak*-null norm-1 sequence (x_n^*) in X^*. It is easy to see that, for such a sequence, $T : X \to c_0$, given by $T(x) = (x_n^*(x))$ is a noncompact operator. A natural problem then is to characterize the Banach spaces Y such that for every infinite dimensional Banach space X, there exists an operator $T : X \to Y$ such that T is noncompact. The goal of this paper is to show these Banach spaces are precisely those which contain isomorphic copies of c_0. Along the way we will examine properties of operators on c_0. Many properties of operators on c_0 can be determined by considering c_0 as a space of continuous functions on a locally compact Hausdorff space that vanish at infinity. For instance, one can modify the proof of Corollary IV.2.17 of [**3**] to get a corresponding result for c_0. The proof of this involves representing measures of such operators. Our investigation of these operators requires only a basic study of non relatively compact subsets of l^1.

All terms not defined in this paper can be found in [**2,3**]. If X is a Banach space, we denote the closed unit ball of X by B_X. Let X be a Banach space. Let (x_n^*) be a bounded sequence in X^* equivalent to

Received by the editors on February 1, 1989, and in revised form on July 12, 1989.
1980 AMS *Mathematics subject classification.* Primary 28B05, 46G10.
Keywords and phrases. Unconditionally converging operators, compact operators, Banach space.
Partially supported by a Faculty Research Grant from the University of North Texas.

(e_n^*), the unit vector basis of l^1. We define a normalized l^1-block of (x_n^*) to be a sequence (y_n^*) defined by

$$y_n^* = \sum_{i \in A_n} \alpha_i x_i^*,$$

where (A_n) is a sequence of pairwise disjoint finite subsets of $\mathbf{N}$ and $\sum_{i \in A_n} |\alpha_i| = 1$. Certainly (y_n^*) is also equivalent to (e_n^*).

Lemma 1. *Let V be a bounded nonrelatively compact subset of l^1. Then there exists $\varepsilon > 0$, sequences $\{x_n\}$ and $\{y_n\}$ in V, a sequence of positive numbers $\{\lambda_n\}$, and a normalized l^1-block $\{z_n\}$ of $\{e_n^*\}$ such that for every $n \in \mathbf{N}$*:

(1) $\|x_n - y_n\| \geq \varepsilon$,

(2) $\|z_n - \lambda_n(x_n - y_n)\| \leq 1/2$.

Remark. The two conditions above imply that $\lambda_n \leq 3/(2\varepsilon)$ for every $n \in \mathbf{N}$. Indeed, since $\|z_n\| = 1$, $\|\lambda_n(x_n - y_n)\| \leq 3/2$ by condition (2) and the desired result follows from condition (1).

Proof. Since V is not relatively compact, there exists $\varepsilon > 0$ and an infinite subset W of V such that $\|x - y\| > \varepsilon$ whenever $x, y \in W$, $x \neq y$. We proceed by induction. Suppose that x_j, y_j, z_j and λ_j have already been constructed for $1 \leq j \leq n-1$. Let $A_j \subset \mathbf{N}$ be the support of z_j. Thus,

$$U_n = \operatorname{span}\left\{e_i : i \in \bigcup_{j<n} A_j\right\}$$

is a finite dimensional subspace of c_0. Let T_n denote the restriction map from l^1 to U_n^*. Since V is bounded and T_n is a bounded linear operator with finite rank, we can cover $T_n(V)$ with finitely many balls B_i, $1 \leq i \leq p$, of radius $\varepsilon/12$. Since W is an infinite subset of V, we can choose x_n and y_n in W such that $T_n(x_n)$ and $T_n(y_n)$ are in the same ball, and hence $\|T_n(x_n) - T_n(y_n)\| \leq \varepsilon/6$. We can write:

$$x_n = x_n^1 + x_n^2 + x_n^3$$
$$y_n = y_n^1 + y_n^2 + y_n^3$$

with x_n^1, y_n^1 having support in $\cup_{j<n} A_j, x_n^2, y_n^2$ having support in some finite subset A_n of $\mathbf{N}$ such that A_n is disjoint from $\cup_{j<n} A_j$, and x_n^3, y_n^3

having support in $\mathbf{N}\setminus\cup_{j\le n}A_j$ such that $\|x_n^3\|\le\varepsilon/12$ and $\|y_n^3\|\le\varepsilon/12$. Note that:

$$\|x_n^1-y_n^1\|=\|T(x_n)-T(y_n)\|\le\varepsilon/6$$

and

$$\|x_n^2-y_n^2\|\ge\|x_n-y_n\|-\|x_n^1-y_n^1\|-\|x_n^3\|-\|y_n^3\|\ge 2\varepsilon/3.$$

In particular, $\|x_n^2-y_n^2\|\ne 0$. Thus, defining $\lambda_n=\|x_n^2-y_n^2\|^{-1}$ and $z_n=\lambda_n(x_n^2-y_n^2)$, then z_n has its support in A_n, $\|z_n\|=1$, and

$$\begin{aligned}\|z_n-\lambda_n(x_n-y_n)\|&=\lambda_n\|(x_n^2-y_n^2)-(x_n-y_n)\|\\&\le\lambda_n(\|x_n^1-y_n^1\|+\|x_n^3\|+\|y_n^3\|)\\&\le 3/(2\varepsilon)(\varepsilon/6+\varepsilon/12+\varepsilon/12)=1/2.\end{aligned}$$ □

Theorem 2. *Let X be a Banach space, and let $T:c_0\to X$ be a bounded linear operator. If T is not compact, then there exists a subspace Z of c_0 such that Z is isomorphic to c_0 and $T|_Z$ is an isomorphism.*

Proof. Let X be a Banach space, and let $T:c_0\to X$ be a noncompact operator. Thus, $T^*:X^*\to l^1$ is also noncompact and $V=T^*(B_{X^*})$ is a nonrelatively compact subset of l^1. By Lemma 1, there exists $\varepsilon>0$, sequences $\{u_n\}$ and $\{v_n\}$ in B_{X^*}, a sequence of positive numbers $\{\lambda_n\}$ and $\{z_n\}$, a normalized l^1 block of $\{e_n^*\}$ such that for every $n\in\mathbf{N}$:

(1) $\|T^*(u_n)-T^*(v_n)|\ge\varepsilon$

(2) $\|z_n-\lambda_n(T^*(u_n)-T^*(v_n)\|\le 1/2$.

Let us write $z_j=\sum_{i\in A_j}a_ie_i^*$. Let $P_j=\{i\in A_j:a_i>0\}$ and $N_j=\{i\in A_j:a_i<0\}$. Defining $x_j\in c_0$ by $x_j=\chi_{P_j}-\chi_{N_J}$ for every $j\in\mathbf{N}$. Clearly,

$$\langle z_i,x_j\rangle=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\ne j.\end{cases}$$

It is also clear that $Z=\{\sum t_ix_i;(t_i)\in c_0\}$ is isometrically isomorphic to c_0, as the x_i's have disjoint support. Moreover, if $(t_i)\in c_0$ and

$\|(t_i)\| = |t_j| = 1$, then

$$\begin{aligned}\left|T\Big(\sum t_i x_i\Big)\right| &= \sup_{x^*\in B_{X^*}} \left|\Big\langle x^*, T\Big(\sum t_i x_i\Big)\Big\rangle\right| \\ &\geq 1/2\left|\Big\langle u_j - v_j, T\Big(\sum t_i x_i\Big)\Big\rangle\right| \\ &= 1/2\left|\Big\langle T^* u_j - T^* v_j, \sum t_i x_i\Big\rangle\right| \\ &\geq (2\lambda_j)^{-1}\Big(\left|\Big\langle z_j, \sum t_i x_i\Big\rangle\right| \\ &\quad - \left|\Big\langle z_j - \lambda_j(T^* u_j - T^* v_j), \sum t_i x_i\Big\rangle\right|\Big) \\ &\geq (2\lambda_j)^{-1}(1 - 1/2) = (4\lambda_j)^{-1} \geq \varepsilon/8.\end{aligned}$$

Thus $T|_Z$ is an isomorphism. □

An operator $T : X \to Y$ is said to be *unconditionally converging* if $\sum T(x_n)$ is an unconditionally converging series in Y whenever $\sum x_n$ is weakly unconditionally Cauchy in X. (Recall that $\sum x_n$ is weakly unconditionally Cauchy if $\sum |x^*(x_n)|$ converges for every $x^* \in X^*$.) The standard example of an operator that is not unconditionally converging is the identity operator on c_0. Using the above theorem, we show that in fact every unconditionally converging operator on c_0 is compact.

Corollary 3. *Let X be a Banach space, and let $T : c_0 \to Y$ be a bounded linear operator. Then T is unconditionally converging if and only if T is compact.*

Proof. Certainly every compact operator is unconditionally converging. The converse follows from Theorem 2. □

We now prove the main result of this paper.

Theorem 4. *Let Y be a Banach space. Then the following are equivalent*:

(i) c_0 *imbeds isomorphically into* Y.

(ii) *For every infinite dimensional Banach* X *there exists a bounded linear operator* $T : X \to Y$ *such that* T *is noncompact.*

(iii) *There exists a bounded linear operator* $T : c_0 \to Y$ *such that* T *is noncompact.*

Proof. By the Josefson-Nissenzweig theorem, if X is an infinite dimensional Banach space, then there exists a noncompact bounded linear operator $T : X \to c_0$. Hence, if $\iota : c_0 \to Y$ is an imbedding, then clearly $\iota T : X \to Y$ is noncompact. Thus (i) $\Rightarrow$ (ii).

Certainly, (ii) $\Rightarrow$ (iii). Now suppose that there exists an operator $T : c_0 \to Y$ which is noncompact. Thus, by Theorem 2, there exists a subspace Z of c_0 such that $T(Z)$ is isomorphic to c_0. Therefore, c_0 imbeds in Y. $\square$

It seems appropriate to conclude this paper with the following theorem from [**1**] regarding noncompact operators and l^1 as a comparison to Corollary 4.

Theorem 5. *Let* X *be a Banach space. Then the following are equivalent*:

(i) l^1 *is complemented in* X.

(ii) *For every infinite dimensional Banach space* Y *there exists a bounded linear operator* $T : X \to Y$ *such that* T *is noncompact.*

(iii) *There exists a bounded linear operator* $T : X \to l^1$ *such that* T *is noncompact.*

Proof. (i) $\Rightarrow$ (ii). Let Y be an infinite dimensional Banach space and Z any infinite dimensional separable closed subspace of Y. It is well known that there exists a continuous linear surjection $S : l^1 \to Z$ [**2**, p. 73]. Let $P : X \to l^1$ be a projection, and let $\iota : Z \to Y$ be the inclusion operator. Clearly, $\iota \circ S \circ P : X \to Y$ is noncompact.

The fact that (ii) $\Rightarrow$ (iii) is clear. We now show that (iii) $\Rightarrow$ (i). Suppose that $T : X \to l^1$ is noncompact. Hence, $T^* : l^\infty \to X^*$ is noncompact and weak*-weak* continuous. Since B_{c_0} is weak* dense in B_{l^∞}, $T^*(B_{c_0})$ is weak* dense in $T^*(B_{l^\infty})$. Hence, $T^*(B_{c_0})$ is not relatively compact in X^*. Hence, by Theorem 4, there is a subspace Z of c_0 such that Z is isomorphic to c_0 and $T|_Z$ is an isomorphism. This implies that X^* contains an isomorphic copy of c_0, which is equivalent [**2**, p. 48, Theorem 10] to l^1 being complemented in X. □

Acknowledgments. The author would like to express her thanks to the referee for many helpful suggestions, especially for simplifying the proof of Theorem 2.

REFERENCES

1. E.M. Bator, P.W. Lewis and D. Race, *Some connections between Pettis integration and operator theory*, Rocky Mountain J. Mathematics **17** (1987), 683–695.

2. J. Diestel, *Sequences and series in Banach spaces*, Graduate Texts Mathematics **92**, Springer-Verlag, New York, 1984.

3. J. Diestel and J.J. Uhl, Jr., *Vector measures*, Math. Surveys **15**, Amer. Math. Soc., Providence, 1977.

4. H. Rosenthal, *A characterization of Banach spaces containing* l^1, Proc. Nat. Acad. Sci. **71** (1974), 2411–2413.

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