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ON THE DIOPHANTINE EQUATION $1 + x + y = z$

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ABSTRACT. In this paper all solutions to the equation $1 + x + y = z$, where x, y and z are positive integers such that xyz has the form $2^r 3^s 5^t$, with r, s and t nonnegative integers, are determined. This work extends earlier work of the authors and J.L. Brenner in the field of exponential Diophantine equations.

1. Introduction. In this paper we consider the equation

$$1 + x + y = z, \tag{1.1}$$

where x, y and z are positive integers such that xyz has the form $2^r 3^s 5^t$, for nonnegative integers r, s and t. This equation has the form

$$\Sigma x_i = 0, \tag{1.2}$$

where the primes dividing Πx_i are specified.

There has been little work done in general to solve such Diophantine equations. Some of these equations have an infinite number of trivial solutions. For example, the still-unsolved equation

$$1 + 2^a 3^b = 5^c + 2^d 3^e 5^f$$

has infinitely many solutions of the form $c = f = 0$, $a = d$, $b = e$. It is unknown whether such equations must have only a finite number of nontrivial solutions.

It follows from the work of Dubois and Rhin [**7**] and Schlickewei [**8**] that the related equation $p^a \pm q^b \pm r^c \pm s^d = 0$ has only finitely many solutions when p, q, r and s are distinct primes. However, their methods do not seem to apply when the terms in the equation are not powers of distinct primes.

The authors and J.L. Brenner [**1, 2, 4-6**] have recently developed techniques which solve such equations in some cases. These techniques

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involve careful consideration of the equation modulo a series of prime, prime power, and certain other moduli. These sequences of moduli can often be chosen so that conclusions or contradictions arise naturally immediately, that is, without calculation.

Such equations arise quite naturally in the character theory of finite groups. For, if G is a finite simple group and p is a prime dividing the order of G to the first power only, then the degrees $x_1, x_2, \dots, x_n$ of the ordinary irreducible characters in the principal p-block of G satisfy an equation of the form $\Sigma\delta_i x_i = 0$, $\delta_i = \pm 1$, where the primes dividing Πx_i are those in $|G|/p$. Much information concerning the group G can be obtained from the solutions to this degree equation. For example, one of the authors in [**3**] has used the solutions to the equation

$$1 + 2^a = 3^b 5^c + 2^d 3^e 5^f$$

to characterize the simple groups $L(2,7), U(3,3), L(3,4)$ and A_8.

In Section 2, below, all solutions to equation (1.1) are found such that xyz is divisible by at most two of the primes 2, 3 and 5. In Section 3 the solution of (1.1), where xyz has the form $2^r 3^s 5^t$, with r, s and t nonnegative integers, is completed. In Appendix A a complete list of solutions (x, y, z) to (1.1) is given. Appendix B is an order and smallest primitive root table for all of the moduli greater than 10 used in this paper.

We will use the symbol $\not\Vert$ to denote a contradiction. Also, all exponents appearing in this paper are assumed to be nonnegative integers.

Remark . We will be making extensive use of [**1**] in which the solutions to the equation $x + y = z$, where xyz has the form $2^r 3^s 5^t 7^u$, are obtained. We have found that there is an error in the proof of Lemma 3.4 of [**1**] which we will correct with the following lemma.

Lemma 1.1. *The solutions to* $3^a + 7^b = 2^c 5^d$ *in positive integers are* $(a, b, c, d) = (1, 1, 1, 1)$ *and* $(5, 1, 1, 3)$.

Proof. Let (a, b, c, d) be another solution. Consideration of our equation modulo 3 and modulo 7 produces the congruences $c \equiv d$

(mod 2) and $a \equiv 2c+5d \pmod 6$. Hence, from mod 4 and mod 5, $c = 1$, d and a are odd, $d \equiv 2-a \pmod 6$ and $(a, b) \equiv (1, 1)$ or $(3, 3) \pmod 4$. From mod 16 and mod 13 we conclude that $(a, b, d) \equiv (1, 1, 1)$ or $(5, 1, 3)$ (mod 12). In the first case, from mod 9, $a = 1$ and immediately from mod 43 and mod 49, $b = 1$, a contradiction. In the second case, from mod 43 and mod 49, $b = 1 (a > 5)$, and we thus have a contradiction mod 729 and mod 1459. □

The following lemma follows immediately from Lemma 1.1 and Theorem 3.5 of [**1**]. Note that a primitive solution to the equation $x + y = z$ is one such that g.c.d.$\{x, y\} = 1$.

Lemma 1.2. *There exist no primitive solutions to* $x + y = 2^i 3^j 5^k$, *where* $xy = 2^a 3^b 5^c 7^d$ $(x, y > 0)$, *for which* $i > 8$, $j > 4$, *or* $k > 4$.

Lemma 1.3. *There are no solutions to the equation* $1 + 2^i + 2^j = 3^k$ *with* $k > 4$.

Proof. This result follows immediately from Comment 8.035 on page 298 of [**6**]. □

2. The two prime case. In this section we determine all solutions of (1.1) for which xyz is divisible by at most two of the primes 2, 3 and 5. There are three natural cases to consider: $xyz = 2^u 3^v$, $xyz = 2^u 5^v$ and $xyz = 3^u 5^v$. In each of these cases, since xyz has the form $p^r q^s$, p, q prime, $p < q$, it follows from (1.1) that at least one of x, y, z is a power of p, and at least one of x, y, z is a power of q. Thus each of the above cases has three subcases of the forms

$$\begin{aligned} 1 + p^a + p^b q^c &= q^d, \\ 1 + q^a + p^b q^c &= p^d, \\ 1 + p^a + q^b &= p^c q^d, \qquad cd \neq 0. \end{aligned}$$

Observe that the exponents of the prime 2 in any of these equations must either both be zero or both be nonzero.

In each case the solutions to the above equations with all exponents on the right-hand side less than or equal to 10 were determined by computer. (These are precisely the solutions given below.) We now consider the three cases.

Case A. $xyz = 2^u 3^v$. (We have three theorems.)

Theorem 2.A.1. *The solutions to*

$$1 + 2^a + 2^b 3^c = 3^d \tag{2.A.1}$$

are given in Table 2.A.1.

TABLE 2.A.1. The solutions (a, b, c, d) to (2.A.1).

a	b	c	d
0	0	0	1
1	1	1	2
1	3	1	3
2	2	0	2
3	1	2	3
3	3	2	4
4	6	0	4
5	4	1	4
6	4	0	4
9	3	3	6

Proof. Let (a, b, c, d) be another solution. (Then $d > 10$.)

Lemma 2.A.1. (a) $a \geq 4$ (*and hence* $b \neq 0$).

(b) $c \leq 3$.

(c) $\text{Min}\,\{a, b\} \leq 4$.

Proof of Lemma . Part (a) follows from Lemma 1.2. For, if $a = 0, 1, 2$ or 3, (2.A.1) becomes (after cancellation) a primitive equation of the form of Lemma 1.2 for which $j > 8$. Define $s = c - d$, $t = b - a$. Suppose that (b) is false. Then $1 + 2^a \equiv 0 \pmod{81}$, so that $a \equiv 27 \pmod{54}$. Thus $2^b 3^s \equiv 1 \pmod{p}$, where $p = 19$ or 87211. Using the primitive roots 3 and 13 modulo the primes 19 and 87211, respectively, we have $7b+s \equiv 0 \pmod{18}$, $79135b+28155s \equiv 0 \pmod{87210}$. Thus $7b+s \equiv 0$, $7b+3s \equiv 0 \pmod 9$. It follows immediately that $b \equiv s \equiv 0 \pmod 9$. Thus $2+3^c \equiv \pm 3^c \pmod 7$, so that $(c,d) \equiv (3,0) \pmod 6$. Hence $2 \pm 3^3 \equiv \pm 1 \pmod{73}$. $\nparallel$. To prove (c), assume the contrary. Then, from mod 32, $d \equiv 0 \pmod 8$, $2^t 3^c \equiv -1 \pmod p$, $p = 5$ or 41. Using primitive roots 2 and 7 for the moduli 5 and 41, respectively, we have the system:

$$t - c \equiv 2 \pmod 4; \tag{2.A.1.1}$$

$$14t + 25c \equiv 20 \pmod{40}. \tag{2.A.1.2}$$

Considering this system modulo 4 we conclude that $(t,c) \equiv (2,0) \pmod 4$. Thus, by (b), $c = 0$. If $d \equiv 0 \pmod{16}$, then, from mod 17, $t \equiv 4 \pmod 8$, $4 \mid t$. $\nparallel$. Hence, $d \equiv 8 \pmod{16}$, and, considering cases, we find that (2.A.1) is false mod 17. □

To prove the theorem, observe that, by Lemma 1.3, $c \neq 0$. Hence, considering (2.A.1) mod 3, we conclude that a is odd, so that $a > 4$ and $b < 5$. If $c = 1$, then $1 + 2^a + 3 \cdot 2^b = 3^d$, and, by Lemma 1.2, $b = 2$, which produces a contradiction when we consider our equation mod 8. If $c = 2$, then $1+2^a+9\cdot 2^b = 3^d$ and we have a contradiction in each case using mod 16 and mod 5. (For, if we consider our equation mod 16, we conclude that either $b = 1$ and $d \equiv 1 \pmod 4$, $b = 3$ and $d \equiv 2 \pmod 4$, or $b = 4$ and $d \equiv 0 \pmod 4$. In the first case we have $19 + 2^a \equiv 3^d \equiv 3 \pmod 5$ so that $a \equiv 2 \pmod 4$. $\nparallel$. Similarly, in the next two cases we have $a \equiv 0 \pmod 4$. $\nparallel$.) Thus $c = 3$, $1 + 2^a + 27 \cdot 2^b = 3^d$. Using mod 8 we conclude that $b = 3$ or 4. From mod 9 and mod 7, $b \neq 4$. Thus, $217 + 2^a = 3^d$ and we have a final contradiction using mod 1024 and mod 257. For, from Table 2.A.1, $a > 9$ (or, alternately, since $d > 10$, certainly $a > 10$), so that $3^d \equiv 217 \pmod{1024}$, $d \equiv 134 \pmod{256}$, $217 + 2^a \equiv 3^{134} \equiv 42$, $2^a \equiv 82 \pmod{257}$. $\nparallel$. □

Theorem 2.A.2. *The solutions to*

$$1 + 3^a + 2^b 3^c = 2^d \tag{2.A.2}$$

are given in Table 2.A.2.

TABLE 2.A.2. The solutions (a, b, c, d) to (2.A.2).

a	b	c	d
0	1	0	2
0	1	1	3
1	2	0	3
1	2	1	4
2	1	1	4
2	1	3	6
3	2	0	5
3	2	2	6
5	2	1	8

Proof. Let (a, b, c, d) be another solution. (Then $d > 10$ and hence $b \neq 0$.)

Lemma 2.A.2. (a) $a \geq 3$. (b) *If a is even, then $b = 1$ and c is odd; if a is odd, then $b = 2$.* (c) $c \leq 2$.

Proof of Lemma . Part (a) follows from Lemma 1.2 and part (b) from mod 8. If $c \geq 3$, then, from mod 27, $d \equiv 0 \pmod{18}$. Thus, for $s = c - a$, $2^b 3^s \equiv -1 \pmod{p}$, $p = 7$ or 19. Using the primitive root 3 for mod 7 and mod 19, we conclude that $2b + s \equiv 3 \pmod{6}$, $7b + s \equiv 9 \pmod{18}$, so that $b \equiv 0 \pmod{6}$. $\not\parallel$. Hence, (c) is true. □

To prove the theorem, observe that if a is even, then $b = 1$ and $c = 1$, an impossibility by Lemma 1.2. Hence, a is odd, $b = 2$, $1 + 3^a + 4 \cdot 3^c = 2^d$. By Lemma 1.2, $c = 1$ or 2. In either case we have a contradiction (considering our equation) mod 1024 and mod 257. □

Theorem 2.A.3. *The solutions to*

$$1 + 2^a + 3^b = 2^c 3^d, \qquad cd \neq 0, \tag{2.A.3}$$

are given in Table 2.A.3.

TABLE 2.A.3. The solutions (a, b, c, d) to (2.A.3).

a	b	c	d
1	1	1	1
1	2	2	1
2	0	1	1
3	1	2	1
3	2	1	2
3	3	2	2
4	0	1	2
5	1	2	2

Proof. Let (a, b, c, d) be another solution. (Then c or $d > 10$.)

Lemma 2.A.3. (a) $a \geq 3$, $b \geq 3$.

(b) *If b is odd, then $c = 2$; if b is even, then $c = 1$ and d is even.*

(c) $a \equiv 9 \pmod{18}$.

Proof of Lemma . Part (a) follows from Lemma 1.2. Part (b) follows from mod 8. Part (c) follows from (a) and (b) using mod 27 (since $d > 10$). □

To prove the theorem, observe that $3^b \equiv 2^c 3^d \pmod{19}$ so that $b \equiv 7c + d \pmod{18}$. Thus, (from (b)) $c = 2$ and b and d are odd, so that we have a contradiction from mod 16 and mod 5. □

Case B. $xyz = 2^u 5^v$.

Theorem 2.B.1. *The solutions to*

$$1 + 2^a + 2^b 5^c = 5^d \tag{2.B.1}$$

are $(a, b, c, d) = (1, 1, 0, 1), (2, 2, 1, 2), (3, 4, 0, 2)$ *and* $(4, 3, 0, 2)$.

Proof. Let (a, b, c, d) be another solution. By Lemma 1.2, $a > 3$

$(b \neq 0)$. Also, from mod 8, $b \neq 1$. Further, if $b \geq 4$, then, from mod 16, $d \equiv 0 \pmod 4$ and, hence, $2^{a-b-9c} \equiv -1 \pmod{39}$. $\nmid$. Hence $b = 2$ or 3, and thus, by Lemma 1.2, $c \neq 0$, so that, from mod 5, $a \equiv 2 \pmod 4$. If $b = 2$, we have a contradiction from mod 8 and mod 3. Thus, $b = 3$ and we have a contradiction from mod 16, mod 3 and mod 13. □

Theorem 2.B.2. *The solutions to*

$$1 + 5^a + 2^b 5^c = 2^d \tag{2.B.2}$$

are $(a, b, c, d) = (0, 1, 0, 2), (1, 1, 0, 3), (1, 1, 1, 4), (1, 1, 3, 8)$ *and* $(3, 1, 0, 7)$.

Proof. Let (a, b, c, d) be another solution. (Then $d > 10$, $b \neq 0$.) From mod 4, $b = 1$, $1 + 5^a + 2 \cdot 5^c = 2^d$. Also, from Lemma 1.2, $a > 1$. If $c \geq 2$, then, from mod 25, $d \equiv 0 \pmod{20}$, and, using the primitive root 3 for mod 31, we have $20(a - c) - 24 \equiv 15 \pmod{30}$. $\nmid$. Hence, by Lemma 1.2, $c = 1$, $11 + 5^a = 2^d$, so that, using mod 25 and mod 31 we have a contradiction. □

Theorem 2.B.3. *The solutions to*

$$1 + 2^a + 5^b = 2^c 5^d, \qquad cd \neq 0, \tag{2.B.3}$$

are $(a, b, c, d) = (2, 1, 1, 1)$ *and* $(3, 0, 1, 1)$.

Proof. Let (a, b, c, d) be another solution. By Lemma 1.2, $a > 2$ and $b > 0$, and we have a contradiction considering our equation modulo 8, 5 and 3. □

Case C. $xyz = 3^u 5^v$.

Theorem 2.C.1. *The solutions to*

$$1 + 3^a + 3^b 5^c = 5^d \tag{2.C.1}$$

are $(a, b, c, d) = (0, 1, 0, 1), (1, 0, 0, 1)$ *and* $(2, 1, 1, 2)$.

Proof. Let (a,b,c,d) be another solution. By Lemma 1.2, $a>1$. Also, if $c=0$, by Lemma 1.2, $b\neq 0$ and we have a contradiction mod 3 and mod 8. Thus, $c\neq 0$, so that, from mod 5, $a\equiv 2 \pmod 4$. If $b\geq 2$, using the moduli 9, 8 and 7 we produce a contradiction. If $b=0$, we have a contradiction mod 4. Thus, $b=1$, $1+3^a+3\cdot 5^c=5^d$. From Lemma 1.2, $c>1$. Considering our equation mod 25 we have $a\equiv 10 \pmod{20}$, whence, using the primitive root 7 mod 1181, $177+914(c-d)\equiv 0 \pmod{1180}$. $\nparallel$. □

Theorem 2.C.2. *The solutions to*

$$1+5^a+3^b5^c=3^d \tag{2.C.2}$$

are $(a,b,c,d)=(0,0,0,1),(0,0,2,3),(1,1,0,2),(1,1,2,4)$ *and* $(2,0,0,3)$.

Proof. Let (a,b,c,d) be another solution. By Lemma 1.2, $a>1$ so that, if $c>1$, from mod 25 and mod 11 (using the primitive root 2 for mod 11), $d\equiv 0 \pmod{20}$, $4(c-a)+8b\equiv 5 \pmod{10}$. $\nparallel$. If $c=0$, by Lemma 1.2, $b>1$, and considerations modulo 9, 7 and 13 yield a contradiction. Thus, $c=1$, so that, again from Lemma 1.2, $b\neq 0$, and we have a contradiction mod 3, mod 5 and mod 8. □

Theorem 2.C.3. *The solutions to*

$$1+3^a+5^b=3^c5^d, \qquad cd\neq 0, \tag{2.C.3}$$

are $(a,b,c,d)=(2,1,1,1)$ *and* $(2,3,3,1)$.

Proof. Let (a,b,c,d) be another solution. By Lemma 1.2, $a>2$ and $b>1$. From mod 3, mod 5 and mod 8, a is even while b,c and d are odd. If $d>1$, then, from mod 25 and mod 1181, $a\equiv 10 \pmod{20}$, $914(d-b)+177c\equiv 0 \pmod{1180}$ (using primitive root 7 mod 1181), an impossibility mod 2. Thus, $d=1$, $1+3^a+5^b=5\cdot 3^c$ $(c>10)$. From mod 27, $b\equiv 9 \pmod{18}$. From mod 7, $c-a+5\equiv 0 \pmod 6$. Thus, from mod 13 we have $(a,c)\equiv(2,3) \pmod 6$. This produces a final contradiction mod 19 and, thus, the theorem is proven. □

3. The general case. In this section we complete the solution of (1.1). Since, in Section 2, we have determined all such solutions with

xyz divisible by at most two of the primes 2, 3 and 5, we need only (from mod 2) consider the following 12 forms of (1.1):

(3.A)
$$1+3^a5^b+2^c3^d5^e=2^f, \qquad cf\neq 0,\ b+e\neq 0,\ a+d\neq 0;$$
(3.B)
$$1+2^a5^b+2^c3^d5^e=3^f, \qquad acf\neq 0,\ b+e\neq 0;$$
(3.C)
$$1+2^a3^b+2^c3^d5^e=5^f, \qquad acf\neq 0,\ b+d\neq 0;$$
(3.D)
$$1+5^a+2^b3^c5^d=2^e3^f, \qquad bef\neq 0,\ a+d\neq 0;$$
(3.E)
$$1+2^a5^b+3^c5^d=2^e3^f, \qquad acef\neq 0,\ b+d\neq 0;$$
(3.F)
$$1+3^a+2^b3^c5^d=2^e5^f, \qquad bef\neq 0,\ a+c\neq 0;$$
(3.G)
$$1+2^a3^b+3^c5^d=2^e5^f, \qquad adef\neq 0,\ b+c\neq 0;$$
(3.H)
$$1+2^a+2^b3^c5^d=3^e5^f, \qquad abef\neq 0;$$
(3.I)
$$1+2^a3^b+2^c5^d=3^e5^f, \qquad abcdef\neq 0;$$
(3.J)
$$1+3^a+2^b5^c=2^d3^e5^f, \qquad bdef\neq 0;$$
(3.K)
$$1+5^a+2^b3^c=2^d3^e5^f, \qquad bdef\neq 0;$$
(3.L)
$$1+2^a+3^b5^c=2^d3^e5^f, \qquad abcdef\neq 0.$$

The solutions to these 12 equations are determined in the 12 theorems below. In each case we first prove a lemma giving elementary properties of the equation, including (except in case E, G and I) bounds on some of the exponents. Then we examine the equation with computer assistance modulo several moduli to reduce the number of possibilities to sextuples $(a,b,c,d,e,f) \pmod{(m_1,m_2,m_3,m_4,m_5,m_6)}$, where $m_i = 5400$ or 10800 for each i. There are four main programs used, which we refer to as AHL, BFJ, CDK and EGI. Each of these programs is used for the three forms of (1.1) designated in its title (with the exception of (3.L), which is trivial). Each program has 16 major sections which consider solutions of (1.1) relative to the moduli in the set $M = \{217, 671, 13, 41, 241, 17, 73, 181, 703, 601, 151, 401, 101, 271, 109, 433\}$ (in the given order). Further, various sections of these programs refer to conditions involving the moduli in the set $N = \{2, 4, 8, 16, 32, 64, 3, 9, 27, 81, 5, 25, 125\}$. The sets M and N were chosen in such a fashion that the sextuples resulting from consideration of (1.1) relative to these moduli all correspond to actual solutions to the equation. Some of the solutions to (1.1) are completely determined by the moduli in M and N. For the other solutions, some of the exponents a, b, c, d, e, f are determined. These are denoted by an adjacent asterisk $(*)$. The undetermined exponents are then established with modest (or no) calculation, using the moduli listed in the order table (Appendix B).

Theorem 3.A. *The solutions to* (3.A) *are given in* Table 3.A.1.

TABLE 3.A.1. The solutions (a, b, c, d, e, f) to (3.A).

a	b	c	d	e	f
0	0	1	1	1	5
0	2	1	1	0	5
0	2	1	5	0	9
1	0	2	1	1	6
1	1	4	0	0	5
1	1	4	1	0	6
1	1	4	1	1	8
1	2	2	2	1	8
1	3	3	4	0	10
2	1	1	2	0	6
2	1	1	4	2	12
2	2	1	1	1	8
3	0	2	0	2	7
3	1	3	1	1	8
3	3	4	2	1	12
5	1	6	2	1	12
6	1	1	2	2	12

Proof. Let (a,b,c,d,e,f) be another solution. From a computer search, $f > 12$.

Lemma 3.A. (a) (i) *If $c=1$ and d is odd, then a and b are even.*

(ii) *If $c=1$ and d is even, then a is even and b is odd.*

(iii) *If $c=2$, then a is odd and b is even.*

(iv) *If $c \geq 3$, then a and b are odd.*

(v) *If $c \geq 4$, then $a \equiv b \equiv 1$ or $3 \pmod 4$.*

(vi) *If $c \geq 5$, then $(a,b) \equiv (1,5),(5,1),(3,7)$ or $(7,3) \pmod 8$.*

(vii) *If $c \geq 6$, then $(a,b) \equiv (1,13),(3,7),(5,1),(7,11),(9,5),(11,15),(13,9)$, or $(15,3) \pmod{16}$.*

(b) *If $b,e \geq 1$, then $f \equiv 0 \pmod 4$.*

(c) $\text{Min}\,\{b,e\} \leq 1$.

(d) $\text{Min}\,\{a,d\} \leq 2$.

Proof of Lemma . Part (a) follows easily from the moduli 8, 16, 32 and 64. (b) follows from mod 5. Define $u = d-a$, $v = e-b$. Suppose that (c) is false. Then, from mod 25, $f \equiv 0 \pmod{20}$, so that $2^f \equiv 1 \pmod p$, $p = 31$ or 41. Hence, $2^c 3^u 5^v \equiv -1 \pmod p$. Using the primitive roots 3 and 6 for the moduli 31 and 41, respectively, we conclude that $24c + u + 20v \equiv 15 \pmod{30}$, $26c + 15u + 22v \equiv 20 \pmod{40}$, which is absurd mod 2. Thus, (c) is true. If (d) is false, then consideration of (3.A) mod 27 yields $f \equiv 0 \pmod{18}$. Thus, $2^f \equiv 1 \pmod q$, where $q = 7$, 19 or 73. Using the primitive roots 2, 3 and 5 for the moduli 19, 7 and 73, respectively, we have the following congruences:

$$c + 13u + 16v \equiv 9 \pmod{18}, \tag{3.A.1}$$

$$2c + u + 5v \equiv 3 \pmod 6, \tag{3.A.2}$$

$$8c + 6u + v \equiv 36 \pmod{72}. \tag{3.A.3}$$

If we consider the system (3.A.1)–(3.A.3) modulo 6, we find $v \equiv c \equiv 0$ and $u \equiv 3 \pmod 6$. In particular, $c > 5$. From (a), a and b are odd, so that e is also odd. Thus, from (b), $f \equiv 0 \pmod 4$ so that in fact $f \equiv 0 \pmod{36}$, $2^f \equiv 1 \pmod q$, $q = 13$ or 37. Using the primitive root 2 mod 13 and mod 37 we obtain the system $c+26u+23v \equiv 18 \pmod{36}$, $c+4u+9v \equiv 6 \pmod{12}$. Thus $2(u+v) \equiv 0 \pmod 4$ so that $u \equiv v \pmod 2$. $\nparallel$. This proves (d). □

From Program AHL, the lemma, and the moduli in N, we have six cases, which are listed in Table 3.A.2. We consider these cases separately.

TABLE 3.A.2. $(a, b, c, d, e, f) \pmod{(10800, 10800, 5400, 10800, 10800, 5400)}$.

	a	b	c	d	e	f
1.	0*	2*	1*	5	0*	9
2.	1*	3	3*	4	0*	10
3.	2*	1*	1*	4	2*	12
4.	3*	3	4*	2*	1*	12
5.	5	1*	6	2*	1*	12
6.	6	1*	1*	2*	2*	12

Case 1. $13+3^d = 2^g$, where $g = f-1$, $(d, g) \equiv (5, 8) \pmod{(10800, 5400)}$ $(g > 8)$. Here we have a contradiction mod 1024 and mod 257.

Case 2. $1+3\cdot 5^b+8\cdot 3^d = 2^f$, $(b, d, f) \equiv (3, 4, 10) \pmod{(10800, 10800, 5400)}$. From mod 163, $(d, f) \equiv (4, 10) \pmod{162}$. Immediately from mod 243, $d = 4$, $649+3\cdot 5^b = 2^f$. Hence, from mod 625 and mod 751, we have contradiction.

Case 3. $23+25\cdot 3^d = 2^g$, $g = f-1$, $(d, g) \equiv (4, 11) \pmod{(10800, 5400)}$. Without calculation, from the moduli 811, 163 and 243, $d = 4$. $\nparallel$.

Case 4. $721+27\cdot 5^b = 2^f$, $(b, f) \equiv (3, 12) \pmod{(10800, 5400)}$. Here we have a contradiction mod 625 and mod 751.

Case 5. $1+5\cdot 3^a+45\cdot 2^c = 2^f$, $(a, c, f) \equiv (5, 6, 12) \pmod{(10800, 5400, 5400)}$. From mod 577, $(c, f) \equiv (6, 12) \pmod{144}$, so that, immediately, from mod 257 and mod 128, $c = 6$, $2881+5\cdot 3^a = 2^f$. Hence, immediately from the moduli 1621, 9721, 487 and 729, we conclude that $a = 5$. $\nparallel$.

Case 6. $451 + 5 \cdot 3^a = 2^f$, $(a, f) \equiv (6, 12)$ (mod $(10800, 5400)$). Without calculation, from the moduli 729, 1459, 17497 and 2187, we conclude that $a = 6$. $\not\parallel$. Thus, the proof of Theorem 3.A is complete. □

Theorem 3.B. *The solutions to* (3.B) *are given in* Table 3.B.1.

TABLE 3.B.1. The solutions (a, b, c, d, e, f) to (3.B).

a	b	c	d	e	f
1	0	4	1	1	5
1	1	4	0	0	3
1	2	1	1	1	4
1	2	6	1	0	5
1	4	11	2	0	9
2	1	1	1	0	3
2	1	2	1	1	4
3	0	4	2	1	6
3	1	3	0	1	4
4	0	1	0	1	3
4	1	1	4	0	5
4	1	3	4	0	6
4	1	4	4	1	8
5	1	8	0	2	8
5	2	7	2	1	8
5	3	9	0	1	8
7	0	3	1	2	6
8	2	5	0	1	8
9	1	5	0	3	8
10	1	5	2	1	8
12	3	4	5	1	12

Proof. Let (a, b, c, d, e, f) be another solution. From a computer search, $f > 12$.

Lemma 3.B. (a) (i) *If* $d \equiv 0$*, then* $a + b$ *is odd.*

(ii) *If* $d \geq 2$*, then* $a - b \equiv 3 \pmod 6$.

(iii) *If* $d \geq 3$*, then* $a + 5b \equiv 9 \pmod{18}$.

(iv) *If* $d \geq 4$*, then* $a + 23b \equiv 27 \pmod{54}$.

(b) $be \neq 0$ *if and only if* $f \equiv 0 \pmod 4$.

(c) *If* $a, c > 3$*, then* $f \equiv 0 \pmod 4$*; Thus if* $be = 0$*, then* $\min\{a, c\} \leq 3$.

(d) Min $\{b, e\} \leq 1$.

(e) Min $\{a, c\} \leq 5$.

Proof of Lemma . Part (a) follows from consideration of (3.B) modulo 3, 9, 27 and 81. Parts (b) and (c) follow from mod 5 and mod 16. Define $s = c - a$ and $t = e - b$. To prove (d), suppose the contrary so that, from mod 25, $f \equiv 0 \pmod{20}$. Thus, $2^s 3^d 5^t \equiv -1 \pmod p$, where $p = 11$ or 61. Using the primitive root 2 for mod 11 and mod 61, we have $s + 8d + 4t \equiv 5 \pmod{10}$, $s + 6d + 22t \equiv 30 \pmod{60}$, a contradiction mod 2. To prove (e), let $a, c \geq 6$ so that, from mod 64, $f \equiv 0 \pmod{16}$, $2^s 3^d 5^t \equiv -1 \pmod p$, $p = 17$, 41 or 193. Using primitive roots 3, 7 and 5 modulo 17, 41 and 193, respectively, we have

$$14s + d + 5t \equiv 8 \pmod{16}, \tag{3.B.1}$$

$$14s + 25d + 18t \equiv 20 \pmod{40}, \tag{3.B.2}$$

$$34s + 84d + t \equiv 96 \pmod{192}. \tag{3.B.3}$$

We consider the system (3.B.1)–(3.B.3) mod 8 and obtain $(s, d, t) \equiv (2, 0, 4) \pmod{(4, 8, 8)}$. Hence, by (c) and (d), $b \equiv e \equiv 1 \pmod 4$, b or $e = 1$, $c \equiv a + 2 \pmod 4$. Also, from (3.B.3), we have

$$t \equiv 2s \pmod{12}. \tag{3.B.4}$$

Observe that if $3 \mid f$, then, from mod 7 (using primitive root 3), $2s + d + 5t \equiv 3 \pmod 6$, a contradiction mod 2. Hence, $3 \nmid f$. Suppose that $d \neq 0$. Then a is even, say $a = 2k$. Write $c = 4j + a + 2$, $r = j + d$. Then $4^k 5(1 + 4 \cdot 3^r) \equiv 2$ or $8 \pmod{13}$, an absurdity since $5(1 + 4 \cdot 3^r) \in \langle 4 \rangle \cup \{0\}$, 2, $8 \notin \langle 4 \rangle$. Thus, $d = 0$ and a is odd. Easily, $(a, c, f) \equiv$

(7, 9, 1), (9,7,1), (9,11,2) or (11, 9, 2) (mod (12, 12, 3)). Thus, using mod 7 we conclude that $(a, b, c, e, f) \equiv (7, 9, 9, 1^*, 4)$, $(9, 1^*, 7, 9, 4)$, $(9, 1^*, 11, 9, 2)$ or $(11, 9, 9, 1^*, 2)$ (mod (12, 12, 12, 12, 6)). In the first two cases we have an absurdity mod 9; in the last two cases (3.B.4) does not hold. □

From Program BFJ, the lemma and the moduli in N we determine 13 cases, which are listed in Table 3.B.2.

TABLE 3.B.2. (a, b, c, d, e, f) (mod (5400, 10800, 5400, 10800, 10800, 10800)).

	a	b	c	d	e	f
1.	1*	2*	6	1*	0*	5
2.	1*	4	11	2*	0*	9
3.	4*	1*	1*	4	0*	5
4.	4*	1*	3*	4	0*	6
5.	4*	1*	4*	4	1*	8
6.	5*	1*	8	0*	2*	8
7.	5*	2*	7	2*	1*	8
8.	5*	3	9	0*	1*	8
9.	7	0*	3*	1*	2*	6
10.	8	2*	5*	0*	1*	8
11.	9	1*	5*	0*	3	8
12.	10	1*	5*	2*	1*	8
13.	12	3	4*	5	1*	12

We dispense with these quickly.

Case 1. $17 + 2^c = 3^g$, $(c, g) \equiv (6, 4)$ (mod (5400, 10800)), $g = f - 1$ $(g > 4)$. Here we are finished by Lemma 1.3.

Case 2. $1 + 2 \cdot 5^b + 9 \cdot 2^c = 3^f$, $(b, c, f) \equiv (4, 11, 9)$ (mod (10800, 5400, 10800)). Without calculation, from the moduli 251, 625 and 751 we have $(b, c, f) \equiv (4, 11, 9)$ (mod 4500). Thus, from mod 3125 and mod 22501 we have $b = 4$ and $139 + 2^c = 3^g$ $(g = f - 2)$. This produces a contradiction mod 6561 and mod 17497.

Cases 3, 4 *and* 5. Here $81 + 2^c 3^d 5^e = 3^f$ $(f \gg 12)$, which is clearly a contradiction by Lemma 1.2.

Cases 6 *and* 10. $161 + 25 \cdot 2^x = 3^f$, $(x, f) \equiv (8, 8) \pmod{(5400, 10800)}$, $x = a$ or c $(x \neq 8)$. Here, mod 1024 and mod 257 produce a contradiction.

Case 7. $89 + 5 \cdot 2^c = 3^g$, $(c, g) \equiv (7, 6) \pmod{(5400, 10800)}$, $g = f - 2$ $(c \neq 7)$. Here also we have a contradiction from mod 1024 and mod 257.

Cases 8 *and* 11. $1 + 32 \cdot 5^x + 5 \cdot 2^y = 3^f$, $(x, y, f) \equiv (3, 9, 8)$ $\pmod{(10800, 5400, 10800)}$, $(x, y) = (b, c)$ or (e, a). Without calculation, from mod 251 and mod 625, we conclude that $x = 3$, $4001 + 5 \cdot 2^y = 3^f$ $(y \neq 9)$, which produces a contradiction mod 1024 and mod 257.

Case 9. $601 + 2^a = 3^f$, $(a, f) \equiv (7, 6) \pmod{(5400, 10800)}$ $(a \neq 7)$. Again, we have a contradiction from mod 1024 and mod 257.

Case 12. $1441 + 5 \cdot 2^a = 3^f$, $(a, f) \equiv (10, 8) \pmod{(5400, 10800)}$ $(a \neq 10)$. Here we have a contradiction from mod 32768 and mod 40961.

Case 13. $1 + 2^a 5^b + 80 \cdot 3^d = 3^f$, $(a, b, d, f) \equiv (12, 3, 5, 12)$ $\pmod{(5400, 10800, 10800, 10800)}$. Using mod 251 and mod 625, we conclude that $b = 3$, $1 + 125 \cdot 2^a + 80 \cdot 3^d = 3^f$. If $d > 5$, from mod 729 and mod 487, we have an absurdity. Hence, $19441 + 125 \cdot 2^a = 3^f$ $(a \neq 12)$ and a final contradiction is produced from mod 32768 and mod 40961. Thus, Theorem 3.B is proven. $\square$

Theorem 3.C. *The solutions to* (3.C) *are given in* Table 3.C.1.

TABLE 3.C.1. The solutions (a, b, c, d, e, f) to (3.C).

a	b	c	d	e	f
1	1	1	2	0	2
1	2	1	1	0	2
1	7	1	2	4	6
2	0	3	1	1	3
2	1	2	1	0	2
2	3	4	0	0	3
2	4	2	1	2	4
3	1	2	0	2	3
3	1	3	1	2	4
3	2	6	5	0	6
4	0	2	3	0	3
4	1	6	2	0	4
4	2	5	1	1	4
4	3	6	1	0	4
6	0	2	1	1	3
6	1	4	3	0	4
6	2	4	1	0	4
6	5	3	2	0	6
7	1	4	1	1	4
9	3	3	2	2	6

Proof. Let (a, b, c, d, e, f) be another solution. From computer data, $f > 6$.

Lemma 3.C. (a) *If* $e > 0$, *then* $a - b \equiv 2 \pmod 4$; *if* $e > 1$, *then* $a + 7b \equiv 10 \pmod{20}$; *if* $e > 2$, *then* $a + 7b \equiv 50 \pmod{100}$.

(b) $\text{Min}\,\{b, d\} \leq 2$.

(c) $\text{Min}\,\{a, c\} \leq 4$.

Proof of Lemma . Part (a) follows easily from the moduli 5, 25 and 125. Define $s=c-a$, $t=d-b$. To prove (b), assume the contrary, so that, from mod 27, $f\equiv 0 \pmod{18}$. Hence, $5^f\equiv 1 \pmod{p}$, where $p=19$, 829 or 5167, so that $2^s3^t5^e\equiv -1 \pmod{p}$. Using the primitive roots 2, 2 and 6 for the moduli 19, 829 and 5167, respectively, we have the following system:

$$s+13t+16e\equiv 9 \pmod{18}, \tag{3.C.1}$$

$$s+376t+92e\equiv 414 \pmod{828}, \tag{3.C.2}$$

$$1086s+4081t+3157e\equiv 2583 \pmod{5166}. \tag{3.C.3}$$

Considering this system mod 18, we easily conclude that $(s,t,e)\equiv(0,9,0) \pmod{18}$. Further, from (3.C.2), $s\equiv 2 \pmod{4}$. Thus $c\equiv a+18 \pmod{36}$. We now have an absurdity from mod 37 and mod 13. To prove (c), assume the contrary, so that, from mod 32, $f\equiv 0 \pmod{8}$, $2^s3^t5^e\equiv -1 \pmod{p}$, $p=13$ or 313. Using primitive roots 2 and 10 we have $s+4t+9e\equiv 6 \pmod{12}$ and $274s+56t+39e\equiv 156 \pmod{312}$. Considering this system mod 4 and mod 3, we conclude that

$$e\equiv 0 \pmod{4}, \qquad s\equiv 6 \pmod{12}, \qquad t\equiv 0 \pmod{3}. \tag{3.C.4}$$

From mod 3, $bd\neq 0$. Further, $b=1$ or $d=1$. For, if $b,d\geq 2$, then, from mod 9, we conclude that $f\equiv 0 \pmod{12}$, $5^f\equiv 1 \pmod{601}$, and, using primitive root 7, $432s+304t+550e\equiv 300 \pmod{600}$, a contradiction mod 8 (by (3.C.4)). Note that, from mod 64 and mod 17,

$$t\equiv 0 \pmod{12}, \quad \text{for } a,c\geq 6. \tag{3.C.5}$$

For, in this case, $f\equiv 0 \pmod{16}$, and, using the primitive root 3 mod 17 we have $14s+t+5e\equiv 8 \pmod{16}$, $t\equiv 0 \pmod{4}$. Suppose that $e=0$, $1+2^a3^b+2^c3^d=5^f$. If $a,c\geq 6$, from (3.C.5), (3.C.4), we have an immediate contradiction mod 5. Hence, w.l.o.g., let $a=5$. Then $c=3 \pmod{4}$, and, from mod 5, $b=1$, $d\equiv 0 \pmod{4}$, which provides a contradiction mod 9 and mod 7. Hence, $e\neq 0$, and from part (a), $a+7b\equiv 10 \pmod{20}$. Suppose that $b=1$. Then $a\equiv 3 \pmod{20}$.

If $d > 1$, then, from mod 9, $f \equiv 2 \pmod 6$, an easy contradiction mod 31. If $d = 1$, from mod 9, $f \equiv 4 \pmod 6$, and, with a little effort we have a contradiction from the moduli 31, 7, 61 and 41. Hence, $b > 1$, $d = 1$. If $a, c \geq 6$, then, from (3.C.5), $b \equiv 1 \pmod{12}$, $(a, c) \equiv (3, 1) \pmod 4$. From the moduli 9, 7, 13, 27 and 37 we then have another contradiction. Thus, a or $c = 5$. In the first case $(b, c) \equiv (3, 3) \pmod 4$, which yields a contradiction mod 41. In the second case, a, and hence b, are odd, $b \equiv 1 \pmod 6$, and we have a final contradiction mod 9 and mod 31. □

From Program CDK, the lemma, and the moduli in N, we determine 11 cases, which are listed in Table 3.C.2.

TABLE 3.C.2. $(a, b, c, d, e, f) \pmod{(5400, 10800, 5400, 10800, 10800, 10800)}$.

	a	b	c	d	e	f
1.	1*	7	1*	2*	4	6
2.	2*	4	2*	1*	2*	4
3.	3*	2*	6	5	0*	6
4.	4*	1*	6	2*	0*	4
5.	4*	3*	6	1*	0*	4
6.	6	0*	2*	1*	1*	3
7.	6	1*	4*	3*	0*	4
8.	6	2*	4*	1*	0*	4
9.	6	5	3*	2*	0*	6
10.	7	1*	4*	1*	1*	4
11.	9	3*	3*	2*	2*	6

Case 1. $1 + 2 \cdot 3^b + 18 \cdot 5^e = 5^f$, $(b, e, f) \equiv (7, 4, 6) \pmod{10800}$. From mod 625 and mod 751, we conclude that $(b, e, f) \equiv (7, 4, 6) \pmod{54000}$. If $e \neq 4$, we have an absurdity from mod 3125 and mod 22501. Hence, $11251 + 2 \cdot 3^b = 5^f$. Without calculation, from the moduli 2187, 58321, 17497 and 6561, $b = 7$. $\nparallel$.

Case 2. $301 + 4 \cdot 3^b = 5^f$, $(b, f) \equiv (4, 4) \pmod{10800}$. Immediately from mod 1621 and mod 243, $b = 4$. $\nparallel$.

Cases 3 *and* 9. $73+2^x 3^y = 5^f$, $(x, y, f) \equiv (6, 5, 6) \pmod{(5400, 10800, 10800)}$, $(x, y) = (a, b)$ or (c, d). If $x \neq 6$, we have a contradiction from mod 256 and mod 577. Hence, $73 + 64 \cdot 3^y = 5^f$ $(y \neq 5)$ and, without calculation, we have a contradiction from the moduli 487, 9721 and 729.

Cases 4 *and* 8. $49 + 9 \cdot 2^x = 5^f$, $(x, f) \equiv (6, 4) \pmod{(5400, 10800)}$, $x = a$ or c $(f > 4)$. By Lemma 1.2, we have an absurdity.

Cases 5 *and* 7. $433 + 3 \cdot 2^x = 5^f$, $(x, f) \equiv (6, 4) \pmod{(5400, 10800)}$, $x = a$ or c $(x \neq 6)$. Easily from mod 128, mod 97, we have an absurdity.

Case 6. $61 + 2^a = 5^f$, $(a, f) \equiv (6, 3) \pmod{(5400, 10800)}$ $(a \neq 6)$. Here again we have a contradiction from mod 128, mod 97.

Case 10. $241+3 \cdot 2^a = 5^f$, $(a, f) \equiv (7, 4) \pmod{(5400, 10800)}$ $(a \neq 7)$. From mod 1024 and mod 257, we have a contradiction.

Case 11. $1801 + 27 \cdot 2^a = 5^f$, $(a, f) \equiv (9, 6) \pmod{(5400, 10800)}$ $(a > 9)$. From mod 1024 and mod 257, we have a final contradiction. Thus, Theorem 3.C is proven. □

Theorem 3.D. *The solutions to* (3.D) *are given in* Table 3.D*.*

TABLE 3.D. The solutions (a,b,c,d,e,f) to (3.D).

a	b	c	d	e	f
0	1	0	1	2	1
0	5	0	1	1	4
1	1	1	0	2	1
1	1	1	1	2	2
1	1	2	0	3	1
1	1	2	1	5	1
1	2	1	0	1	2
1	4	1	0	1	3
1	5	1	1	1	5
2	1	0	1	2	2
3	1	2	0	4	2
3	1	2	1	3	3
3	1	2	2	6	2
3	1	4	0	5	2
3	2	2	0	1	4
3	3	2	1	1	5

Proof. Let (a,b,c,d,e,f) be another solution. From computer data, either $e>9$ or $f>6$.

Lemma 3.D. (a) $a \neq 0$, 1 *or* 3.

(b) *If* $d \neq 0$, *then* $e \equiv f \pmod 4$; *if* $d \geq 2$, *then* $e+7f \equiv 0 \pmod{20}$; *if* $a,d \geq 3$, *then* $e+7f \equiv 0 \pmod{100}$.

(c) $b=1$ *or* $e=1$.

(d) $\text{Min}\,\{c,f\} \leq 2$.

Proof of Lemma . Part (a) follows from Lemma 1.2. (For, in each of these cases, after cancellation, we have a primitive equation of the form $x+y=2^g3^h$, where $g>8$ or $h>4$.) Part (b) follows from (a) and the moduli 5, 25 and 125, while part (c) follows from mod 4. To prove (d), assume the contrary. Define $s = b - e$, $t = c - f$. Suppose first that $c, f \geq 4$, so that, from mod 81, $a \equiv 27 \pmod{54}$, $1 + 5^a \equiv 0 \pmod{p}$, $2^s 3^t 5^d \equiv 1 \pmod{p}$, where $p = 7$, 163, 487 or 5167. Using the primitive roots 3, 2, 3 and 6 for these primes (in order) we obtain

$$2s + t - d \equiv 0 \pmod{6}; \tag{3.D.1}$$

$$s + 101t + 15d \equiv 0 \pmod{162}; \tag{3.D.2}$$

$$238s + t + 99d \equiv 0 \pmod{486}; \tag{3.D.3}$$

$$1086s + 4081t + 3157d \equiv 0 \pmod{5166}. \tag{3.D.4}$$

In particular, from (3.D.2)–(3.D.4), we obtain the system $s+11t+15d \equiv 0$, $4s + t + 9d \equiv 0$, $6s + 13t + 7d \equiv 0 \pmod{18}$. It follows easily that $s \equiv 0$, $t \equiv d \equiv 0$ or $9 \pmod{18}$. In the first case we have an immediate contradiction mod 19. In the second case, from (c) and mod 19, $(c, f) \equiv (2, 11) \pmod{18}$, so that we have a contradiction from mod 13 and mod 37. Thus either c or f is equal to 3, and, from mod 27, $a \equiv 9 \pmod{18}$. From (3.D.4), (3.D.1), we have

$$t + d \equiv 0 \pmod{6}, \qquad s + t \equiv 0 \pmod{3}. \tag{3.D.5}$$

Clearly $(e, f) \neq (1, 3)$ and there thus remain three cases: $(b, c) = (1, 3)$, $(c, e) = (3, 1)$ and $(b, f) = (1, 3)$, which we consider separately.

Case 1. $1 + 5^a + 54 \cdot 5^d = 2^e 3^f$. Clearly, $d \neq 0$, $e \equiv f \pmod{2}$, and, from (3.D.5), $d \not\equiv f \pmod{2}$ and $e + f \equiv 1 \pmod{3}$. Hence, from mod 16, $e = 2$, $f \equiv 2 \pmod{6}$, d is odd and $a \equiv 1 \pmod{4}$. Easily then we have a contradiction mod 13.

Case 2. $1 + 5^a + 27 \cdot 2^b 5^d = 2 \cdot 3^f$. From (3.D.5), $b - f \equiv 1 \pmod{3}$, $d - f \equiv 3 \pmod{6}$. Thus, if $d = 0$, $f \equiv 3 \pmod{6}$, $b \equiv 1 \pmod{3}$, and we have a contradiction mod 5 and mod 13. Hence, $d \neq 0$, $f \equiv 1 \pmod{4}$ and, from mod 13, mod 16, we conclude that $(b, d, f) \equiv (0, 2, 5) \pmod{6}$, a contradiction mod 19.

Case 3. $1 + 5^a + 2 \cdot 3^c 5^d = 27 \cdot 2^e$. Here, from (3.D.5), $c + d \equiv 3 \pmod{6}$, $e \equiv c + 1 \pmod{3}$. If $d = 0$, then $(c, e) \equiv (3, 1) \pmod{(6, 3)}$, which produces a contradiction mod 5 and mod 13. Thus, $d \neq 0$, $e \equiv 3 \pmod{4}$, $e \equiv 4c + 7 \pmod{12}$ ($c \not\equiv d \pmod{2}$). Clearly, $e > 4$, so that, from mod 16, $a \equiv 1 \pmod{4}$, c is even and d is odd. Thus, we have an easy contradiction mod 13. □

From Program CDK, the lemma, and the moduli in N, we determine three cases: $(a, b, c, d, e, f) \equiv (3, 1^*, 2^*, 2^*, 6, 2^*)$, $(3, 2^*, 2^*, 0^*, 1^*, 4)$ or $(3, 3^*, 2^*, 1^*, 1^*, 5)$ (mod $(10800, 5400, 10800, 10800, 5400, 10800)$). In the first case $451 + 5^a = 9 \cdot 2^e$ $(e \neq 6)$, and we have a contradiction from mod 1024 and mod 257. In the second case $37 + 5^a = 2 \cdot 3^f$ $(f \neq 4)$ and the moduli 243 and 1621 produce a contradiction. In the last case, $361 + 5^a = 2 \cdot 3^f$ so that immediately from mod 251 and mod 625, $a = 3$. $\nparallel$. Thus, Theorem 3.D is proven. □

Theorem 3.E. *The solutions to* (3.E) *are given in* Table 3.E.1.

TABLE 3.E.1. The solutions (a, b, c, d, e, f) to (3.E).

a	b	c	d	e	f
1	0	1	1	1	2
1	0	2	1	4	1
1	2	1	0	1	3
1	2	2	1	5	1
1	4	2	1	4	4
2	1	1	0	3	1
2	1	1	1	2	2
2	1	1	2	5	1
2	1	3	0	4	1
2	3	1	2	6	2
3	0	1	1	3	1
3	0	1	3	7	1
3	0	2	1	1	3
3	0	3	1	4	2
3	2	1	1	3	3
3	2	1	3	6	2
4	1	1	1	5	1

TABLE 3.E.1. (Continued)

a	b	c	d	e	f
4	1	3	0	2	3
4	1	3	1	3	3
4	1	3	3	7	3
4	1	4	0	1	4
4	1	4	1	1	5
4	1	5	0	2	4
4	1	5	1	4	4
5	0	1	1	4	1
5	0	1	2	2	3
6	1	1	0	2	4
6	1	5	1	9	1
6	3	5	1	10	2
7	0	1	1	4	2
8	1	1	1	4	4
9	0	3	1	3	4
9	0	3	3	4	5
9	0	5	1	6	3

Proof. Let (a, b, c, d, e, f) be another solution. From computer data, $e > 10$ or $f > 5$.

Lemma 3.E. (a) *If $a, e \geq 2$, then c is odd; if $a, e \geq 3$, then c and d are odd; if $a, e \geq 4$, then $c \equiv d \equiv 1$ or $3 \pmod 4$; if $a, e \geq 5$, then $(c, d) \equiv (1, 5)$, $(3, 7)$, (5.1) or $(7.3) \pmod 8$; if $a, e \geq 6$, then $(c, d) \equiv (1, 13)$, $(3, 7)$, $(5, 1)$, $(7, 11)$, $(9, 5)$, $(13, 9)$ or $(15, 3) \pmod{16}$.*

(b) *If $b, d \geq 1$, then $e \equiv f \pmod 4$; if $b, d \geq 2$, then $e + 7f \equiv 0 \pmod{20}$; if $b, d \geq 3$, then $e + 7f \equiv 0 \pmod{100}$.*

(c) *$a + b$ is odd; if $c, f \geq 2$, then $a - b \equiv 3 \pmod 6$; if $c, f \geq 3$, then $a + 5b \equiv 9 \pmod{18}$; if $c, f \geq 4$, then $a + 23b \equiv 27 \pmod{54}$.*

Proof of Lemma . These assertions follow easily from the moduli 4, 8, 16, 32, 64, 5, 25, 125, 3, 9, 27, 81. □

From Program EGI, the lemma, and the moduli in N, we find that there are 19 possibilities. In the following ten cases (since e or $f > 5400$) we are clearly finished by Lemma 1.2: $(a,b,c,d,e,f) \equiv (3^*,0^*,1^*,3,7,1^*)$, $(4^*,1^*,1,1^*,5^*,1)$, $(4^*,1^*,3,0^*,2^*,3)$, $(4^*,1^*,3,1^*,3^*,3)$, $(4^*,1^*,3,3,7,3)$, $(4^*,1^*,4,0^*,1^*,4)$, $(4^*,1^*,4,1^*,1^*,5)$, $(4^*,1^*,5,0^*,2^*,4)$, $(6,1^*,1^*,0^*,2^*,4)$ or $(8,1^*,1^*,1^*,4^*,4)$. The remaining possibilities are listed in Table 3.E.2 and are considered separately below.

TABLE 3.E.2. (a,b,c,d,e,f) (mod (5400, 10800, 10800, 10800, 5400, 10800)).

	a	b	c	d	e	f
1.	1*	4	2*	1*	4*	4
2.	2*	3	1*	2*	6	2*
3.	3*	2*	1*	3	6	2*
4.	4	1*	5	1*	4	4
5.	6	1*	5	1*	9	1*
6.	6	3	5	1*	10	2*
7.	9	0*	3*	1*	3*	4
8.	9	0*	3*	3	4*	5
9.	9	0*	5	1*	6	3*

Case 1. $23 + 5^b = 8 \cdot 3^f$, $(b,f) \equiv (4,4) \pmod{10800}$ $(f > 5)$. Immediately from mod 1621, $b \equiv 4 \pmod{1620}$, so that, from mod 243, $f = 4$. ∦.

Case 2. $19 + 5^b = 9 \cdot 2^g$, $g = e - 2$, $(b,g) \equiv (3,4)$ (mod (10800, 5400)) $(g > 4)$. Here we have an immediate contradiction mod 32.

Case 3. $67 + 5^d = 3 \cdot 2^e$, $(d,e) \equiv (3,6)$ (mod (10800, 5400)) $(e > 6)$. Without calculation, from mod 1601, mod 97 and mod 128 we have $e = 6$. ∦.

Case 4. $1 + 5 \cdot 2^a + 5 \cdot 3^c = 2^e 3^f$, $(a, c, e, f) \equiv (4, 5, 4, 4) \pmod{(5400, 10800, 5400, 10800)}$. From mod 97, $(a, e) \equiv (4, 4) \pmod{48}$. From mod 1601, $(c, f) \equiv (5, 4) \pmod{1600}$, $\pmod{64}$, so that, from mod 128, $a = e = 4$, and hence $1 + 5 \cdot 3^g = 16 \cdot 3^h$, where $g = c - 4$, $h = f - 4 > 0$, a contradiction mod 3.

Case 5. $1+5\cdot 2^a+5\cdot 3^c = 3\cdot 2^e$, $(a, c, e) \equiv (6, 5, 9) \pmod{(5400, 10800, 5400)}$. From mod 97, $(a, e) \equiv (6, 9) \pmod{48}$, so that immediately, from mod 1601, $c \equiv 5 \pmod{1600}$, and, from mod 128, $a = 6$, $107 + 5 \cdot 3^g = 2^e$, $g = e - 1$, an immediate contradiction mod 1621 and mod 243.

Case 6. $1 + 2^a 5^b + 5 \cdot 3^c = 9 \cdot 2^e$, $(a, b, c, e) \equiv (6, 3, 5, 10) \pmod{(5400, 10800, 10800, 5400)}$ $(e \gg 10)$. Suppose that $a \neq 6$. Then, from mod 256, we conclude that $c \equiv 1205 \pmod{1600}$ so that we have a contradiction mod 1601. Thus, $1 + 64 \cdot 5^b + 5 \cdot 3^c = 9 \cdot 2^e$. Immediately from mod 243, mod 811 and mod 163, $b \equiv 3 \pmod{162}$, $c \equiv 5 \pmod{810}$, $\pmod{1620}$, $e \equiv 10 \pmod{162}$. Thus, if $c \neq 5$, we have a contradiction mod 729 and mod 9721. Hence, $19+5^b = 9\cdot 2^g$, $g = e-6$, which forces $g = 4$, from mod 32. $\not\parallel$.

Case 7. $17+2^g = 3^f$ $(f > 5)$, $g = a-3$. Here we have a contradiction by Lemma 1.3.

Case 8. $1+2^a+27\cdot 5^d = 16\cdot 3^f$, $(a, d, f) \equiv (9, 3, 5) \pmod{(5400, 10800, 10800)}$ $(f > 5)$. Immediately, from mod 243 and mod 163, $(a, f) \equiv (9, 5) \pmod{162}$. Also, from mod 729, $a \equiv 171 \pmod{486}$. Thus, we have a contradiction mod 487.

Case 9. $1+2^a+5\cdot 3^c = 27\cdot 2^e$, $(a, c, e) \equiv (9, 5, 6) \pmod{(5400, 10800, 5400)}$, $(e > 6)$. From mod 97, $(a, e) \equiv (9, 6) \pmod{48}$, $\pmod{400}$, so that, with no effort, from mod 1601 and mod 128 we have a final contradiction. Thus, Theorem 3.E is proven. □

Theorem 3.F. *The solutions to* (3.F) *are given in* Table 3.F.1.

TABLE 3.F.1. The solutions (a, b, c, d, e, f) to (3.F).

a	b	c	d	e	f
0	1	2	0	2	1
0	4	1	0	1	2
1	1	1	0	1	1
1	2	2	0	3	1
1	4	0	0	2	1
1	5	1	0	2	2
2	1	0	1	2	1
2	1	1	1	3	1
2	1	1	2	5	1
2	1	2	1	2	2
2	3	0	1	1	2
2	4	1	1	1	3
3	2	1	0	3	1
3	2	5	0	3	3
3	3	2	0	2	2
4	1	2	0	2	2
5	1	1	0	1	3
5	8	0	0	2	3
6	1	3	1	3	3
6	7	5	1	1	7
10	4	5	2	1	7

Proof. Let (a, b, c, d, e, f) be another solution. From computer data, $e > 10$ or $f > 7$.

Lemma 3.F. (a) $a \geq 4$.

(b) (i) *If* $c \geq 1$, *then* $e \equiv f \pmod 2$.

(ii) *If* $c \geq 2$, *then* $e \equiv f \pmod 6$.

(iii) *If* $c \geq 3$, *then* $e + 5f \equiv 0 \pmod{18}$.

(iv) *If* $c \geq 4$, *then* $e + 23f \equiv 0 \pmod{54}$.

(c) *If* $b, e \geq 2$, *then* $d = 0$.

(d) $\text{Min}\,\{b, e\} \leq 2$.

(e) $\text{Min}\,\{d, f\} \leq 2$.

Proof of Lemma . Part (a) follows from Lemma 1.2. (For, in each of the cases $a = 0, 1, 2$ or 3, after all possible cancellations we obtain an equation of the form $x + y = 2^g 5^h$ where either $g > 8$ or $h > 6$.) Part (b) then follows from the moduli 3, 9, 27 and 81. Part (c) follows from mod 4 and mod 5, and part (d) from mod 8. To prove (e), assume the contrary, so that, from mod 125, $a \equiv 50 \pmod{100}$. Define $s = b - e$, $t = d - f$. Then $2^s 3^c 5^t \equiv 1 \pmod{p}$, where $p = 101$, 1811 or 394201. Using primitive roots 2, 7 and 7 for the moduli 101, 1811, 394201, respectively, we have the system: $s + 69c + 24t \equiv 0 \pmod{100}$, $835s + 177c + 914t \equiv 0 \pmod{1180}$ and $173174s + 287766c + 126172t \equiv 0 \pmod{394200}$. Thus, we have the system: $s + 9c + 4t \equiv 0, 5s + 3c + 6t \equiv 0$, $7s + 3c + 6t \equiv 0 \pmod{20}$. We easily conclude that $s, c, t \equiv 0 \pmod{10}$, so that $2 + 2^b 5^d \equiv 2^b 5^d \pmod{11}$. $\nmid\!\!\!\mid$. □

From Program BFJ, the lemma, and the moduli in N, we determine six cases, which are listed in Table 3.F.2.

TABLE 3.F.2. $(a, b, c, d, e, f) \pmod{(10800, 5400, 10800, 10800, 5400, 10800)}$.

	a	b	c	d	e	f
1.	3*	2*	5	0*	3*	3
2.	5	1*	1*	0*	1*	3
3.	5	8	0*	0*	2*	3
4.	6	1*	3*	1*	3*	3
5.	6	7	5	1*	1*	7
6.	10	4*	5	2*	1*	7

Cases 1 *and* 2. $7 + 3^x = 2 \cdot 5^f$, $(x, f) \equiv (5, 3) \pmod{(10800, 5400)}$, $x = c$ or a. Here Lemma 1.1 produces a contradiction.

Case 3. $1 + 3^a + 2^b = 4 \cdot 5^f$, $(a, b, f) \equiv (5, 8, 3)$ (mod (10800, 5400, 10800)). From mod 251, mod 751 and 625, $f = 3$. $\nparallel$.

Case 4. $271 + 3^a = 8 \cdot 5^f$, $(a, f) \equiv (6, 3) \pmod{10800}$. Without calculation, from mod 251 and mod 625, $f = 3$. $\nparallel$.

Case 5. $1 + 3^a + 5 \cdot 2^b 3^c = 2 \cdot 5^f$, $(a, b, c, f) \equiv (6, 7, 5, 7)$ (mod (10800, 5400, 10800, 10800)) $(f > 7)$. Immediately from mod 243 and mod 1621, $(b, f) \equiv (7, 7) \pmod{162}$, so that, from mod 163, $(a, c) \equiv (6, 5) \pmod{162}$. If $c > 5$, then, from mod 729, $f \equiv 169 \pmod{486}$, a contradiction mod 9721. Hence, $1 + 3^a + 1215 \cdot 2^b = 2 \cdot 5^f$. Without calculation, from the sequence of moduli 729, 9721, 4861, 1459, 2917 and 2187, $a = 6$, $73 + 243 \cdot 2^s = 5^t$, where $s = b - 1$, $t = f - 1$. Thus, we have a contradiction mod 128 and mod 97.

Case 6. $1 + 3^a + 400 \cdot 3^c = 2 \cdot 5^f$, $(a, c, f) \equiv (10, 5, 7) \pmod{10800}$. From mod 243 and mod 163, $(a, c, f) \equiv (10, 5, 7) \pmod{162}$. If $c \neq 5$, we thus have a contradiction mod 729 and mod 9721. Hence, $97201 + 3^a = 2 \cdot 5^f$ $(a > 10)$. We conclude from mod 177147 that $f \equiv 157471 \pmod{472392}$. Thus we have a contradiction mod 472393. Thus, Theorem 3.F is proven. □

Theorem 3.G. *The solutions to* (3.G) *are given in* Table 3.G.1.

Proof. Let (a, b, c, d, e, f) be another solution. From computer data, $e > 11$ or $f > 7$.

TABLE 3.G.1. The solutions (a, b, c, d, e, f) to (3.G).

a	b	c	d	e	f
1	3	0	2	4	1
1	3	2	1	2	2
1	7	0	4	3	4
1	7	0	6	5	4
1	7	1	4	1	5
1	7	2	4	4	4
1	7	4	2	8	2
1	7	5	4	1	7
2	0	1	1	2	1
2	0	1	2	4	1
2	0	2	1	1	2
2	4	1	2	4	2
2	4	3	2	3	3
2	4	5	2	8	2
3	1	0	2	1	2
3	1	1	1	3	1
3	1	1	2	2	2
3	1	1	3	4	2
3	1	2	2	1	3
3	1	3	1	5	1
4	2	1	1	5	1
5	3	1	5	11	1
5	3	3	1	3	3
6	0	1	1	4	1
6	0	3	1	3	2
6	0	5	1	8	1
6	4	5	1	8	2
7	1	1	1	4	2
7	1	5	1	6	2
10	0	2	2	1	4

Lemma 3.G. (a) *If* $a, e \geq 2$, *then* c *is odd*; *if* $a, e \geq 3$, *then* c *and* d *are odd*; *if* $a, e \geq 4$, *then* $c \equiv d \equiv 1$ *or* $3 \pmod 4$; *if* $a, e \geq 5$, *then* $(c, d) \equiv (1, 5)$, $(3, 7)$, $(5, 1)$ *or* $(7, 3) \pmod 8$; *if* $a, e \geq 6$, *then* $(c, d) \equiv (1, 13)$, $(3, 7)$, $(5, 1)$, $(7, 11)$, $(9, 5)$, $(11, 15)$, $(13, 9)$ *or* $(15, 3) \pmod{16}$.

(b) $b, c \geq 1$ *if and only if* $e+f$ *is even*; *if* $b, c \geq 2$, *then* $e \equiv f \pmod 6$; *if* $b, c \geq 3$, *then* $e + 5f \equiv 0 \pmod{18}$; *if* $b, c \geq 4$, *then* $e + 23f \equiv 0 \pmod{54}$.

(c) $a - b \equiv 2 \pmod 4$; *if* $d, f \geq 2$, *then* $a + 7b \equiv 10 \pmod{20}$; *if* $d, f \geq 3$, *then* $a + 7b \equiv 50 \pmod{100}$.

Proof of Lemma . These results follow routinely from the moduli 4, 8, 16, 32, 64, 3, 9, 27, 81, 5, 25 and 125. □

From Program EGI, the lemma, and the moduli in N, we have the 13 cases listed in Table 3.G.2, which we now consider.

TABLE 3.G.2. $(a, b, c, d, e, f) \pmod{(5400, 10800, 10800, 10800, 5400, 10800)}$.

	a	b	c	d	e	f
1.	1*	7	0*	4	3*	4
2.	1*	7	0*	6	5*	4
3.	1*	7	1*	4	1*	5
4.	1*	7	2*	4	4*	4
5.	1*	7	4	2	8	2
6.	1*	7	5	4	1*	7
7.	2*	4	3*	2*	3*	3
8.	2*	4	5	2*	8	2*
9.	5*	3*	1*	5	11	1*
10.	6	0*	5	1*	8	1*
11.	6	4	5	1*	8	2*
12.	7	1*	5	1*	6	2*
13.	10	0*	2*	2*	1*	4

Case 1. $1+2\cdot3^b+5^d=8\cdot5^f$, $(b,d,f)\equiv(7,4,4) \pmod{10800}$ $(f>4)$. Immediately, from mod 625, $b\equiv 7 \pmod{500}$, $\pmod{5400}$. Hence, using mod 751, we deduce that $(d,f)\equiv(4,4) \pmod{5400}$. From mod 3125, we conclude that $b\equiv 2257$ or $9007 \pmod{11250}$, according to whether $d>4$ or $d=4$. In either case, we have a contradiction mod 22501.

Case 2. $1+2\cdot3^b+5^d=32\cdot5^f$, $(b,d,f)\equiv(7,6,4) \pmod{10800}$ $(f>4)$. As in the previous case, from mod 625 and mod 751, $(b,d,f)\equiv(7,6,4)$ $\pmod{54000}$. Also, from mod 3125, $b\equiv 2257 \pmod{11250}$, yielding an absurdity mod 22501.

Case 3. $1+2\cdot3^b+3\cdot5^d=2\cdot5^f$, $(b,d,f)\equiv(7,4,5) \pmod{10800}$ $(f>5)$. Again, $b\equiv 7 \pmod{54000}$ and, from mod 751, $(d,f)\equiv(4,5)$ $\pmod{54000}$. As above, if $d\neq 4$, then $b\equiv 2257 \pmod{11250}$, which produces an absurdity mod 22501. Hence, $938+3^b=5^f$. From mod 15625, we conclude that $b\equiv 5007 \pmod{12500}$, an impossibility mod 37501.

Case 4. $1+2\cdot3^b+9\cdot5^d=16\cdot5^f$, $(b,d,f)\equiv(7,4,4) \pmod{10800}$ $(f>4)$. As in the previous cases, $(b,d,f)\equiv(7,4,4) \pmod{54000}$. From mod 3125, $b\equiv 2257$ or $6757 \pmod{11250}$, according as $d>4$ or $d=4$. In either case, mod 22501 yields a contradiction.

Case 5. $1+2\cdot3^b+3^c5^d=2^e5^f$, $(b,c,d,e,f)\equiv(7,4,2,8,2)$ $(\text{mod } (10800,10800,10800,5400,10800))$. Immediately, from mod 125, either $d=f=2$ or $d,f\neq 2$. In either case, from mod 625, $b\equiv 7 \pmod{500}$, $\pmod{54000}$. Thus, from mod 251, $c\equiv 4$ $\pmod{54000}$. Since $2^{146}\equiv 5 \pmod{751}$, it follows that $d\equiv 2$ $\pmod{375}$, $\pmod{54000}$, and $e+146f\equiv 300 \pmod{375}$. Thus, since $2^{798}\equiv 5 \pmod{3001}$, it follows immediately that $e+798f\equiv$ $104 \pmod{1500}$, $\pmod{375}$, so that $(e,f)\equiv(8,2) \pmod{375}$, $\pmod{54000}$. If $d,f\neq 2$, then, as above, $b\equiv 2257 \pmod{11250}$, an absurdity mod 22501. Hence, $d=f=2$, $1+2\cdot3^b+25\cdot3^c=25\cdot2^e$. Immediately from mod 1621 and mod 243, $c=4$, $1013+3^b=25\cdot2^g$, $g=e-1\gg 7$, so that, from mod 1024 and mod 257, we are finished.

Case 6. $1+2\cdot3^b+3^c5^d=2\cdot5^f$, $(b,c,d,f)\equiv(7,5,4,7) \pmod{10800}$. Easily, from mod 243 and mod 1621, we conclude that $(d,f)\equiv$ $(4,7) \pmod{32400}$. Thus, from mod 163, $(b,c)\equiv(7,5) \pmod{162}$, $\pmod{32400}$. Also, from mod 487 and mod 9721, $f\equiv 7 \pmod{4860}$ so that, from mod 729, $c=5$, $1+2\cdot3^b+243\cdot5^d=2\cdot5^f$. Clearly, from mod 625, $b \equiv 7 \pmod{54000}$, so that, from mod 751, $(d, f) \equiv (4, 7)$ $\pmod{54000}$. If $d \neq 4$, then, as above, $b \equiv 2257 \pmod{11250}$, an absurdity mod 22501. Hence, $75938 + 3^b = 5^f$. Without calculation, using the moduli 2187, 58321, 17497, we have $f \equiv 7 \pmod{1458}$, (mod 3645), $b \equiv 7 \pmod{729}$, $f \equiv 7 \pmod{17496}$, $b = 7$. $\nparallel$.

Case 7. $169 + 3^b = 2 \cdot 5^f$, $(b, f) = (4, 3) \pmod{10800}$. Immediately from the moduli 487, 811 and 243, we have $b = 4$. $\nparallel$.

Case 8. $1+4\cdot 3^b+25\cdot 3^c = 25\cdot 2^e$, $(b, c, e) \equiv (4, 5, 8) \pmod{(10800, 10800, 5400)}$. Without calculation, from mod 1621 and mod 243, $e \equiv 8$ $\pmod{1620}$, $b = 4$, $1 + 2^2 3 + 3^c = 2^e$, so that, by Theorem 2.A.2, $e = 8$. $\nparallel$.

Case 9. $173 + 3 \cdot 5^g = 2^e$, $g = d - 1 > 4$, $(g, e) \equiv (4, 11)$ $\pmod{(10800, 5400)}$. Clearly, from mod 625, $e \equiv 11 \pmod{500}$, (mod 27000), and, from mod 751, $g \equiv 4 \pmod{375}$, (mod 54000). Hence, from mod 22501, $(g, e) \equiv (4, 11) \pmod{(5625, 22500)}$, an obvious contradiction mod 3125.

Case 10. $1 + 2^a + 5 \cdot 3^c = 5 \cdot 2^e$, $(a, c, e) \equiv (6, 5, 8) \pmod{(5400, 10800, 5400)}$. Suppose that $a > 6$. Then, from mod 1024, $c \equiv 245$ $\pmod{256}$, which produces a contradiction mod 257. Hence, $a = 6$, $1 + 2^2 3 + 3^c = 2^e$, $e = 8$. $\nparallel$.

Case 11. $1 + 2^a 3^b + 5 \cdot 3^c = 25 \cdot 2^e$, $(a, b, c, e) \equiv (6, 4, 5, 8)$ $\pmod{(5400, 10800, 10800, 5400)}$ $(e \neq 8)$. Using mod 97, $(a, e) \equiv (6, 8)$ $\pmod{48}$. From mod 1024, we conclude that $c \equiv 245$ or $69 \pmod{256}$, according as $a > 6$ or $a = 6$. In either case we have a contradiction mod 257.

Case 12. $1+3\cdot 2^a+5\cdot 3^c = 25\cdot 2^e$, $(a, c, e) \equiv (7, 5, 6) \pmod{(5400, 10800, 5400)}$ $(e \neq 6)$. From mod 97, $(a, e) \equiv (7, 6) \pmod{48}$, so that, immediately from mod 257, $c \equiv 5 \pmod{256}$, an absurdity mod 1024.

Case 13. $113 + 2^g = 5^f$, $g = a - 1 > 9$, $(g, f) \equiv (9, 4)$ $\pmod{(5400, 10800)}$. Here mod 1024 and mod 257 produce a contradiction. Thus, Theorem 3.G is proven. □

Theorem 3.H. *The solutions to* (3.H) *are given in* Table 3.H.

TABLE 3.H. The solutions (a, b, c, d, e, f) to (3.H).

a	b	c	d	e	f
1	2	1	0	1	1
1	3	2	0	1	2
2	1	0	1	1	1
2	3	0	1	2	1
2	4	0	2	4	1
3	1	1	0	1	1
3	2	2	0	2	1
3	3	3	0	2	2
5	2	1	0	2	1
5	6	1	0	2	2
6	1	0	1	1	2
6	5	0	1	2	2
7	1	1	0	3	1
7	5	1	0	2	2
9	1	4	0	3	2
10	2	0	2	2	3
10	3	0	3	4	2
15	2	2	0	8	1

Proof. Let (a, b, c, d, e, f) be another solution. From computer data, either $e > 9$ or $f > 5$.

Lemma 3.H. (a) $a \geq 4$.

(b) *Either* (i) $c = 0$, $d \neq 0$, $a \equiv 2 \pmod 4$ *and* $b \equiv d \pmod 2$, *or*

(ii) $d = 0$, $c \neq 0$, $(a, b - c) \equiv (1, 1)$ *or* $(3, 0) \pmod 4$.

(c) *If* $b \geq 2$, *then* e *is even*; *if* $b \geq 3$, *then* e *and* f *are even*; *if* $b \geq 4$, *then* $e \equiv f \equiv 0$ *or* $2 \pmod 4$; *if* $a, b \geq 5$, *then* $e \equiv f \equiv 0, 2, 4$ *or* $6 \pmod 8$; *if* $a, b \geq 6$, *then* $(e, f) \equiv (0, 0)$, $(2, 10)$, $(4, 4)$, $(6, 14)$, $(8, 8)$, $(10, 2)$, $(12, 12)$ *or* $(14, 6) \pmod{16}$.

(d) Min $\{d, f\} \leq 2$.

(e) Min $\{c, e\} \leq 3$.

Proof of Lemma . Part (a) follows easily from Lemma 1.2 while part (b) follows easily from mod 15. Part (c) follows from the moduli 4, 8, 16, 32 and 64. Define $u = c - e$, $v = d - f$. Assume that (d) is false so that $c = 0$ and, from mod 125, $a \equiv 50 \pmod{100}$. Thus $2^b 3^u 5^v \equiv 1 \pmod{p}$, where $p = 41$, 101 or 8101. Using primitive roots 7, 2 and 6 modulo the primes 41, 101 and 8101, respectively, we have the system

$$14b + 25u + 18v \equiv 0 \pmod{40}, \tag{3.H.1}$$

$$b + 69u + 24v \equiv 0 \pmod{100}, \tag{3.H.2}$$

$$4131b + 3970u + 2104v \equiv 0 \pmod{8100}. \tag{3.H.3}$$

Considering the system (3.H.1)–(3.H.3) modulo 10 we conclude that $(b, u, v) \equiv (0, 0, 0) \pmod{(10, 10, 5)}$. Thus, we have an easy contradiction mod 11. To prove (e), assume the contrary. Then alternative (ii) of (b) is true. We suppose first that $c, e \geq 5$, so that, from mod 243, $a \equiv 81 \pmod{162}$. Thus, $2^b 3^u 5^v \equiv 1 \pmod{q}$, where $q = 19, 163, 135433$ or 87211. Using the primitive roots 2, 2, 5 and 13 for these primes we have

$$b + 13u - 16f \equiv 0 \pmod{18}, \tag{3.H.4}$$

$$b + 101u - 15f \equiv 0 \pmod{162}, \tag{3.H.5}$$

$$91124b + 132322u - f \equiv 0 \pmod{135432}, \tag{3.H.6}$$

$$79135b + 28155u - 53732f \equiv 0 \pmod{87210}. \tag{3.H.7}$$

Considering the system (3.H.4)–(3.H.6) mod 18, we have $(b, u, f) \equiv (0, 0, 0)$ or $(9, 9, 0) \pmod{18}$. In the first case, there is a contradiction mod 7. In the second, from mod 7, $(c, e) \equiv (3, 0) \pmod 6$, and, hence, $(a, b-c) \equiv (3, 0) \pmod 4$, so that $a \equiv 27 \pmod{36}$. From (3.H.6), $f \equiv 2 \pmod 4$, so that $f \equiv 18 \pmod{36}$. Thus, we have a contradiction mod 37. Hence, either $c = 4$ or $e = 4$. Also, $a \equiv 27 \pmod{54}$ so that (3.H.4) and (3.H.7) hold.

Case 1. $c = 4$, $1 + 2^a + 81 \cdot 2^b = 3^e 5^f$. From (3.H.4) and (3.H.7), $b - 13e - 16f \equiv 2$, $7b - 3e - 2f \equiv 6 \pmod{18}$, so that $e \equiv 4 - b \pmod 6$, $f \equiv -b \pmod 3$. Also, from (b), $(a, b) \equiv (1, 1)$ or $(3, 0) \pmod 4$ and either $b = 1$ or $b > 3$. In the former case, $(e, f) \equiv (0, 5) \pmod 9$, $2^a \equiv 3^e 5^f \pmod{163}$, and (using the primitive root 2), $a - 101e - 15f \equiv 0 \pmod{162}$ so that $7e + 3f \equiv 0 \pmod 9$. $\nparallel$. In the latter case, from mod 8, e, f and, hence b, are even, so that $f \equiv -b \pmod 6$ and we have an easy contradiction mod 7 and mod 13.

Case 2. $e = 4$, $1 + 2^a + 2^b 3^c = 81 \cdot 5^f$. From (3.H.4) and (3.H.7), $b + c + 2f \equiv -2$, $b + 3c - 2f \equiv 0 \pmod 6$, so that $c \equiv b + 4 \pmod 6$. Further, $(a, b-c) \equiv (3, 0) \pmod 4$ so that $a \equiv 3 \pmod{12}$. If $b = 2$, then $c \equiv 0 \pmod 3$ and we have an easy contradiction modulo 13. Thus, from mod 8, $b \geq 3$ and f is even, so that we have another easy contradiction from mod 7, and the lemma is proven. □

From Program AHL, the lemma, and the moduli in N, we determine three possibilities: $(a, b, c, d, e, f) \equiv (10, 2^*, 0^*, 2^*, 2^*, 3)$, $(10, 3^*, 0^*, 3, 4, 2^*)$ or $(15, 2^*, 2^*, 0^*, 8, 1^*)$ (mod (5400, 5400, 10800, 10800, 10800, 10800)). In the first case, $101 + 2^a = 9 \cdot 5^f$ $(f > 3)$, and we have a contradiction mod 625 and mod 751. In the second case, $1 + 2^a + 8 \cdot 5^d = 25 \cdot 3^e$ and, if $d > 3$, we have a contradiction mod 625 and mod 751. Thus, $1001 + 2^a = 25 \cdot 3^e$ in this case, and immediately, from the moduli 1621 and 243, we have $e = 4$. $\nparallel$. Finally, in the third case, $37 + 2^a = 5 \cdot 3^e$ $(e > 8)$. Here $2^a \equiv 19646 \pmod{19683}$ so that $a \equiv 8763 \pmod{13122}$, a contradiction from mod 52489. Thus, Theorem 3.H is proven. □

Theorem 3.I. *The solutions to* (3.I) *are given in* Table 3.I.1.

TABLE 3.I. The solutions (a, b, c, d, e, f) to (3.I).

a	b	c	d	e	f
1	3	2	1	1	2
1	3	4	1	3	1
1	3	6	1	1	3
1	7	1	4	2	4
1	7	2	5	3	4
1	7	3	4	1	5
1	7	7	4	3	5
2	4	1	2	1	3
2	4	4	1	4	1
2	4	5	2	2	3
3	1	1	2	1	2
3	1	2	1	2	1
3	1	3	2	2	2
3	1	4	3	4	2
3	5	4	1	4	2
4	2	4	1	2	2
7	1	2	1	4	1

Proof. Let (a, b, c, d, e, f) be another solution. From computer data, $e > 4$ or $f > 5$.

Lemma 3.I. (a) *If* $a, c \geq 2$, *then* e *is even*; *if* $a, c \geq 3$, *then* e *and* f *are even*; *if* $a, c \geq 4$, *then* $e \equiv f \equiv 0$ *or* $2 \pmod 4$; *if* $a, c \geq 5$, *then* $e \equiv f \equiv 0,\ 2,\ 4$ *or* $6 \pmod 8$; *if* $a, c \geq 6$, *then* $(e, f) \equiv (0, 0),\ (2, 10),\ (4, 4),\ (6, 14),\ (8, 8),\ (10, 2),\ (12, 12)$ *or* $(14, 6) \pmod{16}$.

(b) $c + d$ *is odd*; *if* $b, e \geq 2$, *then* $c - d \equiv 3 \pmod 6$; *if* $b, e \geq 3$, *then* $c + 5d \equiv 9 \pmod{18}$; *if* $b, e \geq 4$, *then* $c + 23d \equiv 27 \pmod{54}$.

(c) $a - b \equiv 2 \pmod 4$; *if* $d, f \geq 2$, *then* $a + 7b \equiv 10 \pmod{20}$; *if* $d, f \geq 3$, *then* $a + 7b \equiv 50 \pmod{100}$.

Proof of Lemma . These assertions follow routinely from the moduli 4, 8, 16, 32, 64, 3, 9, 27, 81, 5, 25 and 125. □

From Program EGI, the lemma, and the moduli in N, we identify 13 cases. The following six cases are immediately eliminated by Lemma 1.2 (since e or $f > 10800$): $(a,b,c,d,e,f) \equiv (1^*,3,4^*,1^*,3,1^*)$, $(2^*,4,4^*,1^*,4,1^*)$, $(3^*,1^*,4^*,3,4,2^*)$, $(3^*,5,4^*,1^*,4,2^*)$, $(4^*,2,4^*,1^*,2,2^*)$ and $(7,1^*,2^*,1^*,4,1^*)$. The remaining seven cases are listed in Table 3.I.2. We consider these separately.

TABLE 3.I.2. $(a,b,c,d,e,f) \pmod{(5400, 10800, 5400, 10800, 10800, 10800)}$.

	a	b	c	d	e	f
1.	1*	3*	6	1*	1*	3
2.	1*	7	1*	4	2*	4
3.	1*	7	2*	5	3*	4
4.	1*	7	3*	4	1*	5
5.	1*	7	7	4	3*	5
6.	2*	4	1*	2*	1*	3
7.	2*	4	5*	2*	2*	3

Case 1. $11 + 2^c = 3 \cdot 5^g$, $(c, g) \equiv (6, 2) \pmod{(5400, 10800)}$, $g = f - 1$. Immediately from mod 125, $g = 2$. $\nparallel$.

Case 2. $1 + 2 \cdot 3^b + 2 \cdot 5^d = 9 \cdot 5^f$, $(b, d, f) \equiv (7, 4, 4) \pmod{10800}$ $(f > 4)$. Immediately, from mod 625, $b \equiv 7 \pmod{500}$. Thus, from mod 751, $(d, f) \equiv (4, 4) \pmod{375}$, $\pmod{1125}$. If $d \neq 4$, then, from mod 3125, $b \equiv 2257 \pmod{11250}$ and we have a contradiction mod 22501. Hence, $139 + 2 \cdot 3^g = 5^f$, where $g = b - 2 > 5$. Thus, we have a contradiction mod 729 and mod 1459. $\nparallel$.

Case 3. $1+2\cdot 3^b+4\cdot 5^d=27\cdot 5^f$, $(b,d,f)\equiv(7,5,4) \pmod{10800}$, $(f>4)$. From mod 625 and mod 751 we conclude that $(b,d,f)\equiv(7,5,4) \pmod{54000}$. Thus, from mod 3125, mod 22501, we have an impossibility. $\nparallel$.

Case 4. $1+2\cdot 3^b+8\cdot 5^d=3\cdot 5^f$, $(b,d,f)\equiv(7,4,5) \pmod{10800}$ $(f>5)$. From mod 625 and mod 751, we conclude that $(b,d,f)\equiv(7,4,5) \pmod{54000}$. If $d\neq 4$, we have a contradiction mod 3125 and mod 22501. Hence, $1667+2\cdot 3^g=5^f$, $g=b-1$. Without calculation, from the moduli 729, 1459, 58321 and 2187, $g=6$. $\nparallel$.

Case 5. $1+2\cdot 3^b+2^c 5^d=27\cdot 5^f$, $(b,c,d,f)\equiv(7,7,4,5)$ (mod $(10800,5400,10800,10800)$). From mod 625, $b\equiv 7 \pmod{500}$ so that, from mod 751 (since $2^{146}\equiv 5$), we have

$$c+146d\equiv 216 \pmod{375}, \tag{3.I.1}$$

$f\equiv 5 \pmod{375}$. Thus, immediately from mod 3001 (since $2^{798}\equiv 5$), we have

$$c+798d\equiv 199 \pmod{1500}, \quad \pmod{375}. \tag{3.I.2}$$

Easily, from (3.I.1), (3.I.2), we have $(c,d)\equiv(7,4) \pmod{375}$. If $d>4$, we have a contradiction mod 3125 and 22501. Hence, $1+2\cdot 3^b+625\cdot 2^c=27\cdot 5^f$. Without calculation, from the moduli 128, 97, 577, 1601 and 256, we conclude that $c=7$, $2963+2\cdot 3^g=5^f$, $g=b-3$. Thus, immediately from mod 487, mod 811 and mod 243, we have $g=4$. $\nparallel$.

Case 6. $17+4\cdot 3^g=5^f$, $(g,f)\equiv(3,3) \pmod{10800}$, $g=b-1$. Immediately from mod 81, $g=3$. $\nparallel$.

Case 7. $89+4\cdot 3^g=5^f$, $(g,f)\equiv(2,3) \pmod{10800}$, $g=b-2$. Immediately, from mod 27, $g=2$. $\nparallel$. Thus, Theorem 3.I is proven. $\square$

Theorem 3.J. *The solutions to* (3.J) *are given in* Table 3.J.

TABLE 3.J. The solutions (a, b, c, d, e, f) to (3.J).

a	b	c	d	e	f
2	1	2	2	1	1
2	2	1	1	1	1
2	4	1	1	2	1
2	5	2	1	4	1
3	1	0	1	1	1
3	5	0	2	1	1
3	9	0	2	3	1
4	3	0	1	2	1
6	2	1	1	1	3
6	4	1	1	4	1
7	9	0	2	3	2

Proof. Let (a, b, c, d, e, f) be another solution. From computer data, either $d > 10$, $e > 4$ or $f > 5$.

Lemma 3.J. (a) $a \geq 4$. *It trivially follows from* mod 5 *and* mod 15 *that in fact* $a \neq 0, 1$ *for* all *solutions of* (3.J).

(b) $b + c \equiv 1 \pmod 2$; *if* $e \geq 2$, *then* $b - c \equiv 3 \pmod 6$; *if* $e \geq 3$, *then* $b + 5c \equiv 9 \pmod{18}$; *if* $e \geq 4$, *then* $b + 23c \equiv 27 \pmod{54}$.

(c) *If* $b, d \geq 2$, *then* $c = 0$.

(d) $\text{Min}\,\{b, d\} \leq 2$.

(e) $\text{Min}\,\{c, f\} \leq 1$.

Proof of Lemma . Part (a) follows easily from Lemma 1.2. Part (b) follows from part (a) and the moduli 3, 9, 27 and 81. Part (c) follows from mod 4 and mod 5, while part (d) follows from mod 8. Define $s = d - b$, $t = f - c$. To prove (e), let us first note that $\min\{c, f\} \leq 2$. For, if not, then, as in the proof of Lemma 3.F, we have a system of congruences: $s + 9e + 4t \equiv 0$, $5s + 3e + 6t \equiv 0$, $7s + 3e + 6t \equiv 0 \pmod{20}$, which implies that $(s, e, t) \equiv (0, 0, 0) \pmod{10}$, which yields a contradiction mod 11. Suppose now that $\min\{c, f\} > 1$, so that, either $c = 2$, $f \geq 2$ or $f = 2$, $c \geq 2$, and, from mod 25 and mod 1181, we have $a \equiv 10$, $5s + 3e + 6t \equiv 0 \pmod{20}$, and, in particular,

$$c \equiv 3e + f \pmod{5}. \tag{3.J.1}$$

Suppose that $c = 2$. From part (b) and mod 16, either $(b, d, e) = (1, 2, 1)$ or $b \equiv 5 \pmod 6$, $d = 1$, e is even, and f is odd. In the former case, from (3.J.1) and mod 11, we have an easy contradiction. In the latter case, $2 + 3 \cdot 2^b \equiv 6 \cdot 9^e \pmod{11}$, and we conclude that $(b, e, f) \equiv (11, 4, 5)$ or $(5, 6, 9) \pmod{(30, 10, 10)}$, which produces a contradiction mod 61. Hence, $f = 2$, $c \equiv 3e + 2 \pmod 5$. From mod 16, $(b, d) = (1, 2)$ or (3,1). Thus, from (3.J.1), $2 + 6 \cdot 4^e \equiv 3^e$ or $2 + 2 \cdot 4^e \equiv 6 \cdot 3^e \pmod{11}$, again a contradiction. □

From Program BFJ, the lemma, and the moduli in N, we conclude that $(a, b, c, d, e, f) \equiv (2, 4^*, 1^*, 1^*, 2, 1^*)$, $(6, 2^*, 1^*, 1^*, 1^*, 3)$ or $(6, 4^*, 1^*, 1^*, 4, 1^*) \pmod{(10800, 5400, 10800, 5400, 10800, 10800)}$. In the first and last cases, $81 + 3^a = 10 \cdot 3^e$, $e \gg 4$. $\nparallel$. In the second case, $7 + 3^{a-1} = 2 \cdot 5^f$, a contradiction by Lemma 1.1. Thus, Theorem 3.J is proven. □

Theorem 3.K. *The solutions to* (3.K) *are given in* Table 3.K.

Proof. Let (a, b, c, d, e, f) be another solution. From computer data, either $d > 9$, $e > 6$ or $f > 4$.

Lemma 3.K. (a) $a \neq 0, 1$ *or* 3.

(b) $b - c \equiv 2 \pmod 4$; *if* $a, f \geq 2$, *then* $b + 7c \equiv 10 \pmod{20}$; *if* $a, f \geq 3$, *then* $b + 7c \equiv 50 \pmod{100}$.

TABLE 3.K. The solutions (a, b, c, d, e, f) to (3.K).

a	b	c	d	e	f
1	1	3	2	1	1
1	3	1	1	1	1
1	4	2	1	1	2
2	2	0	1	1	1
2	6	0	1	2	1
3	1	3	2	2	1
3	1	7	2	2	3
3	2	4	1	2	2
3	3	1	1	1	2
3	4	2	1	3	1
3	8	2	1	5	1
5	1	7	2	1	4

(c) *$b = 1$ or $d = 1$; if $b = 1$, then $d = 2$, $c \equiv 3 \pmod 4$ and a is odd.*

(d) *$\mathrm{Min}\,\{c, e\} \leq 2$. It easily follows from* mod 3, mod 15, *that $a \neq 0$ for all solutions of* (3.K). *Thus* (b)–(d) *are also true for such solutions.*

Proof of Lemma . Part (a) follows from Lemma 1.2. Part (b) follows from part (a) and the moduli 5, 25 and 125. To prove (c), note that, from mod 4, b or d is one. If $b = 1$, from (b), $c \equiv 3 \pmod 4$, so that from mod 3 and mod 8, a is odd and $d = 2$. To prove (d), define $s = d - b$, $t = e - c$, and assume the contrary. First, let $c, e \geq 4$, so that, from mod 81, $a \equiv 27 \pmod{54}$, $2^s 3^t 5^f \equiv 1 \pmod p$, $p = 7$, 163, 487 or 5167. As in the proof of Lemma 3.D (with f replacing d in (3.D.1)–(3.D.4)), we have $s \equiv 0$, $t \equiv f \equiv 0$ or $9 \pmod{18}$. From mod 19 (since $b \equiv d \equiv 1 \pmod{18}$ and c is odd, from parts (b) and (c), in either case we have a contradiction. Thus, either c or e is equal to 3, $a \equiv 9 \pmod{18}$, and, again as in the proof of Lemma 3.D, we have

$$\text{(3.K.1)} \qquad t + f \equiv 0 \pmod 6, \qquad s + t \equiv 0 \pmod 3.$$

There are four cases: $(d, e) = (1, 3)$, $(d, c) = (1, 3)$, $(b, c) = (1, 3)$ or $(b, e) = (1, 3)$. In the first case, $1 + 5^a + 2^b 3^c = 54 \cdot 5^f$, and, from (3.K.1), $b + 4c \equiv b + c \equiv 1 \pmod 3$, so that, from mod 13, part (c) and mod 16, we have a contradiction. In the second case, $1+5^a+27\cdot 2^b = 2\cdot 3^e 5^f$, $b \equiv 1 \pmod 4$, and, from (3.K.1), $e \equiv b-1 \pmod 3$, $e \not\equiv f \pmod 2$, which yields a contradiction from mod 16 and mod 13. In the third case, easily, $11+5^g = 4\cdot 3^e$, $g = a-1 \equiv 8 \pmod{18}$, and, from (3.K.1), $e \equiv 2 \pmod 6$, so that we have an absurdity mod 19. In the last case, $1+5^a+2\cdot 3^c = 108\cdot 5^f$, an absurdity mod 13. □

From Program CDK, the lemma and the moduli in N, we identify three cases, which we now consider.

Case 1. $1+5^a+2\cdot 3^c = 36\cdot 5^f$, $(a,c,f) \equiv (3,7,3) \pmod{10800}$ $(f>3)$. Here we have a contradiction from part (a) of the lemma and the moduli 251, 751, 3125 and 22501.

Case 2. $1+5^a+9\cdot 2^b = 10\cdot 3^e$, $(a,b,e) \equiv (3,8,5) \pmod{(10800, 5400, 10800)}$. Immediately from mod 243, mod 1621 and mod 163, we conclude that $(a,b,e) \equiv (3,8,5) \pmod{1620}$. From mod 9721, $(a,e) \equiv (3,5) \pmod{4860}$, which forces $e=5$ from mod 729. $\nparallel$.

Case 3. $1+5^a+2\cdot 3^c = 12\cdot 5^f$, $(a,c,f) \equiv (5,7,4) \pmod{10800}$ $(f>4)$. From the lemma (part (a)) and the moduli 625, 751, 3125 and 22105, we again have a contradiction. Thus, Theorem 3.K is proven. □

Theorem 3.L. *There are no solutions to* (3.L).

Proof. This follows trivially from mod 15. □

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Appendix A.

The solutions to equation (1.1). Here we list all solutions to (1.1) in order of increasing z, with $x \leq y$.

x	y	z	x	y	z	x	y	z
1	1	3	9	20	30	8	81	90
1	2	4	1	30	32	9	80	90
1	3	5	4	27	32	25	64	90
2	2	5	6	25	32	5	90	96
1	4	6	15	16	32	15	80	96
2	3	6	3	32	36	20	75	96
1	6	8	5	30	36	45	50	96
2	5	8	8	27	36	3	96	100
3	4	8	10	25	36	9	90	100
2	6	9	15	20	36	18	81	100
3	5	9	3	36	40	24	75	100
4	4	9	9	30	40	27	72	100
1	8	10	12	27	40	45	54	100
3	6	10	15	24	40	27	80	108
4	5	10	4	40	45	32	75	108
1	10	12	8	36	45	4	120	125
2	9	12	12	32	45	16	108	125
3	8	12	20	24	45	24	100	125
5	6	12	2	45	48	60	64	125
2	12	15	15	32	48	2	125	128
4	10	15	20	27	48	27	100	128
5	9	15	1	48	50	6	128	135
6	8	15	4	45	50	9	125	135
3	12	16	9	40	50	54	80	135
5	10	16	24	25	50	8	135	144
6	9	16	3	50	54	15	128	144

Appendix A. (Continued)

x	y	z	x	y	z	x	y	z
1	16	18	5	48	54	18	125	144
2	15	18	8	45	54	5	144	150
5	12	18	5	54	60	24	125	150
8	9	18	9	50	60	9	150	160
1	18	20	27	32	60	15	144	160
3	16	20	3	60	64	24	135	160
4	15	20	9	54	64	1	160	162
9	10	20	15	48	64	36	125	162
3	20	24	18	45	64	80	81	162
5	18	24	27	36	64	54	125	180
8	15	24	2	72	75	64	135	200
4	20	25	10	64	75	15	200	216
6	18	25	20	54	75	80	135	216
8	16	25	24	50	75	90	125	216
9	15	25	4	75	80	8	216	225
12	12	25	15	64	80	24	200	225
1	25	27	25	54	80	32	192	225
2	24	27	5	75	81	64	160	225
6	20	27	8	72	81	80	144	225
8	18	27	16	64	81	96	128	225
10	16	27	20	60	81	2	240	243
2	27	30	30	50	81	50	192	243
4	25	30	32	48	81	80	162	243
5	24	30	40	40	81	6	243	250

Appendix A. (Continued)

x	y	z	x	y	z
9	240	250	375	648	1024
24	225	250	100	1024	1125
5	250	256	324	800	1125
12	243	256	225	1024	1250
15	240	256	64	1215	1280
30	225	256	15	1280	1296
75	180	256	45	1250	1296
120	135	256	80	1215	1296
125	144	270	320	1215	1536
125	162	288	384	1215	1600
3	320	324	512	1215	1728
80	243	324	24	2000	2025
50	324	375	80	1944	2025
54	320	375	1000	1024	2025
8	375	384	125	2304	2430
15	384	400	512	2187	2700
24	375	400	80	3375	3456
75	324	400	512	3375	3888
4	400	405	45	4050	4096
20	384	405	450	3645	4096
80	324	405	720	3375	4096
125	324	450	1215	2880	4096
5	480	486	125	4374	4500
80	405	486	625	4374	5000
125	360	486	1250	4374	5625
243	256	500	1875	4374	6250
25	486	512	324	6075	6400
27	512	540	1215	5184	6400
75	500	576	2025	4374	6400

Appendix A. (Continued)

x	y	z	x	y	z
125	450	576	80	6480	6561
200	375	576	160	6400	6561
24	600	625	800	5760	6561
48	576	625	1440	5120	6561
144	480	625	2560	4000	6561
192	432	625	3125	4374	7500
240	384	625	1215	8000	9216
300	324	625	4374	5000	9375
135	512	648	4374	5625	10000
162	512	675	864	9375	10240
8	720	729	72	15552	15625
80	648	729	1800	13824	15625
128	600	729	4374	11250	15625
216	512	729	4374	12500	16875
20	729	750	1250	18432	19683
9	800	810	4374	15625	20000
80	729	810	36	32768	32805
27	972	1000	4374	80000	84375
135	864	1000	729	155520	156250
270	729	1000	4374	151875	156250
324	675	1000	59049	97200	156250
			19440	512000	531441

APPENDIX B.

Orders and least positive primitive roots mod m. In this paper we have considered equation 1.1 modulo m, where m, $\text{ord}_m n$, $n = 2, 3$ and 5, and the least positive primitive root (LPPR) mod m, for $m > 10$, are listed below. An (i) to the left of m in the table indicates that either m is a power of i or is especially useful in conjunction with powers of i. For the sake of clarity and convenience, some larger numbers appearing in the table are factored.

	m	$\text{ord}_m 2$	$\text{ord}_m 3$	$\text{ord}_m 5$	LPPR
(5)	11	10	5	5	2
	13	12	3	4	2
	15	4	–	–	–
(2)	16	–	4	4	–
(2)	17	8	16	16	3
(3)	19	18	18	9	2
(5)	25	20	20	–	2
(3)	27	18	–	18	2
(3),(5)	31	5	30	3	3
(2)	32	–	8	8	–
(3)	37	36	18	36	2
(2)	41	20	8	20	6
(7)	43	14	42	42	3
(7)	49	21	42	42	3
(2),(3),(5)	61	60	10	30	2
(2)	64	–	16	16	–
(2),(3)	73	9	12	72	5
(3)	81	54	–	54	2
	91	12	6	12	–
(2)	97	48	48	96	5
(5)	101	100	100	25	2
(3)	109	36	27	27	6
(5)	125	100	100	–	2
(2)	128	–	32	32	–
(5)	151	15	50	75	6

Appendix B. (Continued)

	m	$\mathrm{ord}_m 2$	$\mathrm{ord}_m 3$	$\mathrm{ord}_m 5$	LPPR
(3)	163	$162(2 \cdot 3^4)$	$162(2 \cdot 3^4)$	54	2
(3)	181	180	45	15	2
(2)	193	96	16	192	5
(3),(5)	217	15	30	6	–
(2)	241	24	120	40	7
(3)	243	$162(2 \cdot 3^4)$	–	$162(2 \cdot 3^4)$	2
(5)	251	50	125	25	6
(2)	256	–	64	64	–
(2)	257	16	256	256	3
(3)	271	$135(3^3 5)$	30	27	6
	313	156	39	8	10
(2),(5)	401	$200(2^3 5^2)$	$400(2^4 5^2)$	25	3
(2)	433	72	27	$432(2^4 3^3)$	5
(3)	487	$243(3^5)$	$486(2 \cdot 3^5)$	54	3
(2)	$512(2^9)$	–	$128(2^7)$	$128(2^7)$	–
(3)	541	$540(2^2 3^3 5)$	$135(3^3 5)$	$135(3^3 5)$	2
(2)	577	$144(2^4 3^2)$	48	$576(2^6 3^2)$	5
(5)	601	25	75	12	7
(5)	$625(5^4)$	$500(2^2 5^3)$	$500(2^2 5^3)$	–	2
(2)	641	64	$640(2^7 5)$	64	3
(5)	671	60	10	30	–
(3)	703	36	18	36	–
(3)	$729(3^6)$	$486(2 \cdot 3^5)$	–	$486(2 \cdot 3^5)$	2
(5)	751	$375(3 \cdot 5^3)$	$750(2 \cdot 3 \cdot 5^3)$	$375(3 \cdot 5^3)$	3
(2)	769	$384(2^7 3)$	48	$128(2^7)$	11
(3)	811	$270(2 \cdot 3^3 5)$	$810(2 \cdot 3^4 5)$	$405(3^4 5)$	3
	829	828	207	9	2
(2)	$1024(2^{10})$	–	$256(2^8)$	$256(2^8)$	–
(2)	1153	$288(2^5 3^2)$	$576(2^6 3^2)$	$1152(2^7 3^y 2)$	5
	1181	236	20	590	7
(3)	1459	$486(2 \cdot 3^5)$	$1458(2 \cdot 3^6)$	$243(3^5)$	3
(2),(5)	1601	$400(2^4 5^2)$	$1600(2^6 5^2)$	$400(2^4 5^2)$	3

APPENDIX B. (Continued)

	m	$\text{ord}_m 2$	$\text{ord}_m 3$	$\text{ord}_m 5$	LPPR
(3)	1621	$1620(2^2 3^4 5)$	45	$405(3^4 5)$	2
(2)	$2048(2^{11})$	–	$512(2^9)$	$512(2^9)$	–
(3)	$2187(3^7)$	$1458(2\cdot 3^6)$	–	$1458(2\cdot 3^6)$	2
(5)	2251	$750(2\cdot 3\cdot 5^3)$	$250(2\cdot 5^3)$	$1125(3^2 5^3)$	7
(3)	2917	$972(2^2 3^5)$	$1458(2\cdot 3^6)$	$2916(2^2 3^6)$	5
(5)	3001	$1500(2^2 3\cdot 5^3)$	$500(2^2 5^3)$	$250(2\cdot 5^3)$	14
(5)	$3125(5^5)$	$2500(2^2 5^4)$	$2500(2^2 5^4)$	–	2
(2),(3)	3457	$576(2^6 3^2)$	$1728(2^6 3^3)$	$1152(2^7 3^2)$	7
(2),(5)	4001	$1000(2^3 5^3)$	$4000(2^5 5^3)$	$200(2^3 5^2)$	3
(2)	$4096(2^{12})$	–	$1024(2^{10})$	$1024(2^{10})$	–
(3)	4861	$972(2^2 3^5)$	$1215(3^5 5)$	81	11
	5167	861	738	18	6
(3)	$6561(3^8)$	$4374(2\cdot 3^7)$	–	$4374(2\cdot 3^7)$	2
(3),(5)	8101	100	$810(2\cdot 3^4 5)$	$2025(3^4 5^2)$	6
(5)	9001	$2250(2\cdot 3^2 5^3)$	$1500(2^2 3\cdot 5^3)$	$750(2\cdot 3\cdot 5^3)$	7
(3)	9721	$810(2\cdot 3^4 5)$	$4860(2^2 3^5 5)$	$4860(2^2 3^5 5)$	7
(5)	11251	$2250(2\cdot 3^2 5^3)$	$2250(2\cdot 3^2 5^3)$	$1125(3^2 5^3)$	13
(5)	$15625(5^6)$	$12500(2^2 5^5)$	$12500(2^2 5^5)$	–	2
(3)	17497	$4374(2\cdot 3^7)$	$729(3^6)$	$17496(2^3 3^7)$	5
(3)	$19683(3^9)$	$13122(2\cdot 3^8)$	–	$13122(2\cdot 3^8)$	2
(5)	22501	$22500(2^2 3^2 5^4)$	$11250(2\cdot 3^2 5^4)$	$5625(3^2 5^4)$	2
(2)	$32768(2^{15})$	–	$8192(2^{13})$	$8192(2^{13})$	–
(5)	37501	$37500(2^2 3\cdot 5^5)$	$6250(2\cdot 5^5)$	$18750(2\cdot 3\cdot 5^5)$	2
(2)	40961	$10240(2^{11} 5)$	$40960(2^{13} 5)$	$10240(2^{11} 5)$	3
(3)	52489	$13122(2\cdot 3^8)$	$2187(3^7)$	$8748(2^2 3^7)$	7
(3)	58321	$29160(2^3 3^6 5)$	$729(3^6)$	$3645(3^6 5)$	11
	87211	54	5814	2295	13
	135433	162	67716	135432	5
(3)	$177147(3^{11})$	$118098(2\cdot 3^{10})$	–	$118098(2\cdot 3^{10})$	2
	394201	197100	100	98550	7
(3)	472393	$59049(3^{10})$	$78732(2^2 3^9)$	$472392(2^3 3^{10})$	5

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