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EXTENDED DOMAINS OF SOME INTEGRAL OPERATORS

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ABSTRACT. The extended domain is determined for a class of integral operators with rapidly oscillating kernels of modulus one.

1. Introduction. For a σ-finite measure space S (or respectively T), $L^0(S)$ (or respectively $L^0(T)$) denotes the space of all measurable finite a.e. complex valued functions on S (or T) with the metric topology of convergence in measure on all subsets of finite measure.

$K : D_K \subset L^0(S) \to L^0(T)$ is an integral operator with kernel $k(t,s)$ and with the (proper) domain $D_K = \{u \in L^0(S) : \int_S |k(t,s)||u(s)|\,ds < \infty \text{ a.e.}\}$:

$$Ku(t) = \int_S k(t,s)u(s)\,ds, \qquad u \in D_K. \tag{1.1}$$

We assume that K is nonsingular, i.e., $\exists\, g \in D_K$, $g > 0$ a.e.

The extended domain $\tilde{D}_K$ of K was first introduced in Aronszajn-Szeptycki [**1**]. It is the maximal solid topological vector subspace of $L^0(S)$ to which K can be extended by continuity.

For more information about these notions we refer to Labuda-Szeptycki [**3**, **4**] and to bibliographies in these papers.

In [**3**] the extended domain $\tilde{D}_K$ was found for kernels k of the form

$$k(t,s) = \exp\, iP(t-s), \qquad t,s \in \mathbf{R}, \tag{1.2}$$

where P is a real polynomial in one variable. The spaces which occur in this context are of independent interest and are referred to as compressed amalgams (see Fournier-Stewart [**2**]).

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The aim of the present note is to show that the approach used in [**3**] applies to more general kernels of the form

$$k(x,y) = \exp\, i\phi(t,s), \qquad s,t \in \mathbf{R}, \tag{1.3}$$

where $\phi : \mathbf{R}^2 \to \mathbf{R}$ is a continuous function satisfying suitable regularity conditions and suitable growth conditions with respect to the variable s. Obviously in this case $D_K = L^1(\mathbf{R})$. $\tilde{D}_K$ turns out to be a suitable compressed $\ell^2(L^1)$ amalgam determined by the rate of growth of ϕ with respect to s.

The approach applies to the case where $\phi(t,s)$ is an arbitrary real polynomial with nonzero mixed derivative ϕ_{ts} or when $\phi(t,s) = |t-s|^\alpha$ where $\alpha > 1$. (Note that when $\phi_{ts} = 0$, k is a one dimensional operator and the questions we are addressing become trivial.) This complements the result in Labuda-Szeptycki [**3**] and gives the positive answer to a question stated in that paper.

We note that some special cases of k of the form (1.3) can be reduced by a suitable change of variables to the Fourier transform in which case the extended domain is the (ordinary) amalgam $\ell^2(L^1)$. The results below could be viewed as perturbations of this approach, even though we were unable to derive them directly from those corresponding to the Fourier transform.

2. Some definitions and notations. We recall the concept of the extended domain of an integral operator defined by (1.1).

Denote by $\tilde{\mathcal{T}}$ the weakest (locally) solid topology in $L^0(S)$ which makes the operator $K : D_K \subset L^0(S) \to L^0(T)$ continuous. It turns out that $\tilde{\mathcal{T}}$ is a complete metric group topology whose restriction to D_K is a vector topology.

The closure $\tilde{D}_K$ of D_K in $L^0(S)$ equipped with $\tilde{\mathcal{T}}$ is referred to as the extended domain of K.

The same construction can be carried out with $L^0(T)$ replaced by a smaller image space $L \subset L^0(T)$. This gives rise to the extended domain relative to L, $\tilde{D}_{K,L}$.

We do not describe the explicit construction of the distance function giving rise to the topology $\tilde{\mathcal{T}}$, as it will not be needed in this paper. We will use, however, the following characterization of $\tilde{D}_K$.

For a function $u \in L^0(S)$ we denote by $\mathcal{F}_u$ the collection of all sequences $\{g_n\} \subset D_K$ with supports of g_n for distinct n's intersecting along null sets and such that $|g_n| \leq |u|$ a.e. for every n.

Theorem 1.1. $u \in \tilde{D}_K \Leftrightarrow \Sigma|Kg_n(t)|^2 < \infty$ *a.e. on* T $\forall\{g_n\} \in \mathcal{F}_u$.

There are some variants of this theorem; we state two of them.

Let $S = T = \mathbf{R}$.

For $u \in L^0(\mathbf{R})$ let $\mathcal{F}'_u$ be the collection of all sequences $\{g_n\}$ in $\mathcal{F}_u$ with supports of g_n contained in nonoverlapping intervals.

Theorem 1.2. *If* k *is continuous and* $\neq 0$ *everywhere on* $\mathbf{R}^2$*, then* $\tilde{D}_K \subset L^1_{\text{loc}}$ *and* $u \in \tilde{D}_K \Leftrightarrow \Sigma|Kg_l(t)|^2 < \infty$ $\forall\{g_l\} \in \mathcal{F}_u$ *and a.e.* $t \in \mathbf{R}$.

We next consider the extended domain relative to L^2_{loc}. Let k be as in Theorem 1.2 and let this time T be any subset of $\mathbf{R}$.

Theorem 1.3. $u \in \tilde{D}_{K,L^2_{\text{loc}}(t)} \Leftrightarrow \Sigma \int_C |Kg_l(t)|^2\, dt < \infty$ $\forall\{g\} \in \mathcal{F}'_l$, $\forall u$ *compact* $C \subset T$.

The proof of Theorem 1.3 is quite similar to that of Theorem 1.2 in [**3**] and uses a known property of series whose partial sums with all changes of signs form a bounded set in L^2 (if $\{\sum_{n=1}^{N} \pm f_n;\ N = 1, 2, \dots\}$ is a bounded set in L^2 then $\Sigma||f_n||^2 < \infty$).

Remark . We don't know a useful characterization of the extended domain relative to L^1_{loc}, similar to those in Theorems 1.2 and 1.3.

We recall next the notion of compressed amalgam of ℓ^q with L^p (see [**2**]).

For an increasing sequence $\{\beta_n\}_{n=-\infty}^{\infty}$ such that $\beta_n \uparrow \infty$ as $n \to \infty$ and $\beta_n \downarrow -\infty$ as $n \to -\infty$, $\ell^q(\beta_n, L^p) = \{u \in L^1_{\text{loc}} : \Sigma||u||^q_{L^p(\beta_n,\beta_{n+1})} < \mu\}$. With an obvious norm $\ell^q(\beta_n, L^p)$ is a Banach space referred to as a compressed, or stretched, amalgam of ℓ^q with L^p, depending on the behavior of the sequence $\beta_{n+1} - \beta_n$.

When $\beta_n = n$ one gets the "ordinary" amalgam denoted by $\ell^p(L^q)$ and for $p = q$, $\ell^p(\beta_n, L^p) = L^p(\mathbf{R})$ for every $\{\beta_n\}$.

Note the obvious isomorphism of $\ell^q(\beta_n, L^p)$ onto $\ell^q(\mathbf{Z}, L^p)$ of all ℓ^q sequences with values in $L^p(0,1)$. This isomorphism is topological only when $\{\beta_{n+1} - \beta_n\}$ is bounded and bounded away from 0. In this case $\ell^q(\beta_n, L^p)$ coincides with $\ell^q(L^p)$.

3. Statement of results. Let $\phi(t,s)$ be a real valued polynomial of two real variables. We write ϕ in the form

$$\phi(t,s) = \phi(0,s) + \phi_0(t)s^m + \sum_{j=1}^{m} \phi_j(t)s^{m-j}, \tag{3.1}$$

where $\phi_j(t)$ are polynomials in $t, j = 0, \dots, m$.

We assume that $\phi_0 \not\equiv 0$ and that $\phi_0(0) = 0$.

Let $k(t,s) = \exp[i\phi(t,s)]$ and let K be the corresponding integral operator (1.1). Obviously $D_K = L^1$.

Theorem 3.1. *With K as above, $\tilde{D}_K = \ell^2(\beta_n, L^1)$ where $\beta_n = (\text{sign } n)|n|^{1/m}$. Moreover, in this case $\tilde{D}_K = \tilde{D}_{KL^2_{\text{loc}}(T)}$ where $T = \mathbf{R}\backslash\{\text{roots of } \phi_0'\}$ and $K : \ell^2(\beta_n, L^1) \to L^2_{\text{loc}}(T)$ is continuous.*

In the case of convolution operator with $\phi(t,s) = P(t-s)$, where P is the real polynomial of one variable, the result is given in [**3**]; in that case $T = \mathbf{R}$. The next theorem confirms a conjecture made in that paper.

Theorem 3.2. *Let $\alpha > 1$, let $k(t,s) = \exp[i|t-s|^\alpha]$, and let K be the corresponding integral operator. Then $\tilde{D}_K = \ell^2(\beta_n, L^1)$ where $\beta_n = (\text{sign}\, n)|n|^{1/(\alpha-1)}$. In this case $\tilde{D}_K = \tilde{D}_{KL^1_{\text{loc}}(\mathbf{R})} = \tilde{D}_{KL^2_{\text{loc}}(\mathbf{R})}$ and $K : \ell^2(\beta_n, L^1) \to L^2_{\text{loc}}(\mathbf{R})$ is continuous.*

Recall that, for $\alpha \in (0,1]$ and K as above, $\tilde{D}_K = D_K = L^1$.

Theorems 3.1 and 3.2 can be obtained as special cases of a single result which may be of independent interest.

Let $\omega(s)$ be a strictly increasing function, $\omega(s) \uparrow \infty$ as $s \to \infty$ and $\omega(s) \downarrow -\infty$ as $s \to -\infty$. Consider the following condition on a function $\phi(t,s)$.

R can be written as the union of an at most countable collection Γ of closed intervals $\{I\}$, bounded or unbounded, such that for every $I \in \Gamma$, there is an $M \geq 0$ such that, for $t \in I$ and $s \geq M$ and, respectively, for $t \in I$, $s \leq -M$, $\phi(t,s)$ can be represented in the form similar to (3.1):

$$\phi(t,s) = \phi_{-1}(s) + \phi_0(t)\omega(s) + \sum_{j\rho<1} \phi_j(t)|\omega(s)|^{1-j\rho} + \psi(t,s) \tag{3.2}$$

where $\phi_{-1}(s)$ is measurable, $\rho \in (0,1)$, $\phi_j(t)$ are $m+1$ times continuously differentiable on I where m is the least integer $> 1/\rho$ and ϕ_0 is strictly monotone on I.

The function $\psi(t,s)$ is continuous and bounded together with partial derivatives with respect to t up to order $m+1$ and satisfies the condition (3.3)

$$\max\{|\psi(t,s) - \psi(t,\omega^{-1}(n))| : s \in [\omega^{-1}(n), \omega^{-1}(n+1)], t \in I\} \to 0$$
$$\text{as } |n| \to \infty.$$

Note that the representation (3.2) is allowed to have different coefficients ϕ_j and ψ for $s > M$ and for $s < -M$.

Theorem 3.3. *If ϕ satisfies the above conditions and if $k(t,s) = \exp(i\phi(t,s))$, then $\tilde{D}_K = \ell^2(\beta_n, L^1)$ where $\beta_n = \omega^{-1}(n)$. Moreover, $\tilde{D}_K = \tilde{D}_{KL^2_{\text{loc}}(T)}$ where $T = \cup_\Gamma I^{\text{int}}$ and $\tilde{K} : \ell^2(\beta_n, L^1) \to L^2_{\text{loc}}(T)$ is continuous.*

We notice that in the special case when $\phi_j = \psi = 0$, $j > 0$, and ω is locally absolutely continuous, Theorem 3.3 can be obtained by a change of variables from the known characterization of the extended domain of the Fourier transform as the amalgam $\ell^2(L^1)$.

We next explain how Theorem 3.3 implies Theorems 3.1 and 3.2.

Since $\phi_0(t)$ in (3.1) is a nonconstant polynomial, $\phi_0(t)$ is monotone on each connected component of $\mathbf{R}\backslash\{\text{roots of } \phi_0'\}$. In this case $\omega(s) = s^m$, $\rho = 1/m$ and $\psi(t,s) = 0$. For each of the components I of T, M may be taken to be 0; $\phi_j(t)$ has to be replaced by $-\phi_j(t)$ when $s^{m-j} < 0$ to reconcile (3.1) with (3.2).

To accommodate Theorem 3.2 write $\Gamma = \{[-M,M]; M = 1,2,\dots\}$ and for each M and $|t| \le M < |s|$,

$$|s-t|^\alpha = |s|^\alpha \left(1-\frac{t}{s}\right)^\alpha = |s|^\alpha \sum_{\ell=0}^{\infty} \binom{\alpha}{\ell} \left(\frac{t}{s}\right)^\ell . \tag{3.4}$$

We get the representation (3.2) with $\phi_{-1}(s) = |s|^\alpha$, $\phi_0(t) = t$, $\omega(s) = \operatorname{sign} s|s|^{\alpha-1}$, $\phi_j = \binom{\alpha}{j+1}t^{j+1}$ when $s > M$ and $(-1)^{j+1}\binom{\alpha}{j+1}t^{j+1}$ when $1 < M$, $\rho = 1/(\alpha-1)$ (if $1 < \alpha < 2$ then $\phi_j = 0$ for $j > 0$) and $\psi(t,s)$ is the part of the series 3.4 for $l > [\alpha]$—the integer part of α.

4. Outline of proofs. We outline the proof of Theorem 3.1 with some indications of changes needed to obtain Theorem 3.2. The idea is quite similar to the corresponding result in [**3**].

To show that $\tilde{D}_K \subset \ell^2(\beta_n, L^1)$ we take any $u \in \tilde{D}_K$. Since $\tilde{D}_K$ is solid we may assume that $\exp(i\phi(0,s))u(s) \ge 0$ and hence that $\phi(0,s) = 0$. Theorem 1.2 implies that $u \in L^1_{\text{loc}}$ and it follows that the sequence $\{\chi_{[\beta_n,\beta_{n+1}]}u\} \in \mathcal{F}_u$, where χ stands for the characteristic function. It follows now from Theorem 1.1 that

$$S(t) = \sum \left| \int_{\beta_n}^{\beta_{n+1}} k(t,s)u(s)\,ds \right|^2 \le \infty \text{ a.e.} \tag{4.1}$$

For $s \in [\beta_n, \beta_{n+1}]$ we write $\phi(t,s)$ in the form

$$\phi(t,s) = \phi_0(t)(s^m - \beta_n^m) + \Sigma\phi_j(t)(s^j - \beta_n^j) + \phi(t,\beta_n)$$

and we factor $k(t,s) = \exp[i\phi(t,s)]$ accordingly.

The term $\exp[i\phi(t,\beta_n)]$ can be taken out of the integral sign. In the remaining terms we use the estimates

$$s^m - \beta_n^m \le (n+1) - n = 1,\ \ s_n^j - \beta_n^j \le (n+1)^{j/m} - n^{j/m} \le n^{j/m-1}.$$

We also recall that $\phi_0(0) = 0$. It follows that it is possible to choose $\delta > 0$ and n_0 such that for $|t| \le \delta$ and for $|n| \ge n_0$,

$$\left| \phi_0(t)(s^m - \beta_n^m) + \sum_{j=1}^{m-1} \phi_j(t)(s^j - \beta_n^j) \right| < \frac{\pi}{3} \tag{4.2}$$

and, consequently,

$$\left|\int_{\beta_n}^{\beta_{n+1}} k(t,s)u(s)\right| \geq \int_{\beta_n}^{\beta_{n+1}} \operatorname{Re} k(t,s)u(s)\,ds \geq \frac{1}{2}\int_{\beta_n}^{\beta_{n+1}} u(s)\,ds.$$

It follows that for every $t \in [-\delta, \delta]$,

$$\sum_{|n|\geq n_0} \left(\int_{\beta_n}^{\beta_{n+1}} u(s)\,ds\right)^2 \leq 4S(t);$$

(4.1) implies that the last sum is finite for some $t \in [-\delta, \delta]$, and it follows that $u \in \ell^2(\beta_n, L^1)$.

In the proof of Theorem 3.3 the assumption (3.3) is used to arrive at the estimate (4.2).

The reverse inclusion will be established in a stronger form $\ell^2(\beta_n, L^1) \subset \tilde{D}_{K,L^2_{\mathrm{loc}}} \subset \tilde{D}_K$, where L^2_{loc} stands for $L^2_{\mathrm{loc}}(T)$ as explained in the remarks after Theorem 3.3.

Let $u \in \ell^2(\beta_n, L^1)$ and let $\sigma = \{I\}$ be any sequence of closed nonoverlapping intervals. Let

$$S(t,\sigma) = \sum_{I\in\sigma} |K(\chi_I u)(t)|^2. \tag{4.3}$$

Let $g \geq 0$ be a C_0^∞ function with a connected support in T (i.e., in one of the intervals on which $\phi_0(t)$ is monotone). We will show now that

$$\int g(t)S(t,\sigma)\,dt < \infty, \tag{4.4}$$

which according to Theorem 1.3 suffices to establish the desired inclusion.

To get (4.4) we denote by J_n the intervals $[\beta_n, \beta_{n+1}]$ occurring in the definition of $\ell^2(\beta_n, L^1)$. Clearly then

$$S(t, \{J_n\}) \leq ||u||_{\ell^2(\beta_n, L^1)}. \tag{4.5}$$

Next let $\sigma = \sigma' \cup \sigma''$ where $\sigma' = \{I \in \sigma : I \subset J_n \cup J_{n+1} \text{ for some } n\}$. Then

$$S(t,\sigma') \le \sum_n \sum \{|K\chi_I u(t)|^2 : I \subset J_n \cup J_{n+1}\}$$
$$\le \sum_n (|K|\chi_{J_n \cup J_{n+1}}|u|(t))^2 \le 4S(\{J_n\}, t)$$

which by (4.5) is bounded.

For each $I \in \sigma'' = \sigma \backslash \sigma'$, we let $\tilde{I} = \cup\{J_n : J_n \cap I^{\text{int}} \neq \varnothing\}$, $\tilde{\sigma} = \{\tilde{I}\}_{I \in \sigma''}$. Note that no more than two intervals of $\tilde{\sigma}$ may overlap at a time and then the intersection is one of the intervals J_n. Also $I = \cup\{J_n : J_n \subset I\}$ for every $I \in \tilde{\sigma}$.

It is easy to check the inequality

$$S(\tilde{\sigma}, t) \le 2S(\sigma'', t) + 4||u||^2_{\ell^2(\beta_n, L^1)}$$

and also the same inequality with $\tilde{\sigma}$ and σ'' reversed.

In conclusion, to check (4.4) for a function in $\ell^2(\beta_n, L^1)$ it suffices to do it with σ replaced by $\tilde{\sigma}$.

$S(\tilde{\sigma}, t)$ can be written in the form

$$S(\tilde{\sigma}, t) \le \sum_{I \in \tilde{\sigma}} \Big| \sum_{J_n \subset I} \int k(t,s)u(s)\,ds \Big|^2$$

and

$$\int g(t)S(\tilde{\sigma}, t) = \sum_{I \in \tilde{\sigma}} \int g(t) \Big| \sum_{J_n \subset I} \int k(t,s)u(s)\,ds \Big|^2 dt$$
$$= \sum_{I \in \tilde{\sigma}} \sum \bigg\{ \int_{J_n} \int_{J_l} \int g(t)k(t,s)\overline{k(t,r)}\,dt\,u(s)\overline{u(r)}\,ds\,dr \bigg\}$$
$$\le \sum_{I \in \tilde{\sigma}} \sum \bigg\{ a_{ln} \int_{J_l} |u| \int_{J_n} |u| : J_l, J_n \subset I \bigg\}$$

where $a_{ln} = \max\{|\int g(t)k(t,s)\overline{k(t,r)}\,dt| : s \in J_l,\ r \in J_n\}$.

The proof is now concluded by checking that the (symmetric) matrix a_{ln} defines a bounded operator in $\ell^2(\mathbf{Z})$.

This is accomplished by showing that

$$\sum_n a_{ln} \le C \tag{4.6}$$

with C independent of l.

The first estimate of a_{ln} is obtained by repeated use of integration by parts. We change the variable of integration $\tau = \phi_0(t)$ and denote $g(t(\tau))dt/dt$ by $g(\tau)$ and $\phi_j(t(\tau))$ by $\phi_j(\tau)$. We can then write, integrating by parts,

$$\begin{aligned}
&\int g(t)k(t,s)\overline{k(t,r)}\,dt \\
&\quad = i(s^m - r^m)^{-1}\int g'(\tau)k(\tau,s)\overline{k(\tau,r)}\,d\tau \\
&\qquad + \sum_{j=1}^{m-1}\left[(s^j - r^j)\int g(\tau)\phi_j'(\tau)k(\tau,s)\overline{k(\tau,r)}\,d\tau\right] \\
&\quad = (s^m - r^m)^{-1}\int\left(g(\tau) + \sum_{j=1}^{m-1}(s^j - r^j)g_j(\tau)\right)k(\tau,s)\overline{k(\tau,r)}\,d\tau,
\end{aligned}$$

where $g_j = -g\phi_j'$, $g_0 = ig'$ are all in C_0^∞ and have supports in T.

The integration by parts is now performed again resulting in a sum of terms of the form

$$-(s^m - r^m)^{-2}\int g''(\tau)k(\tau,s)\overline{k(\tau,r)}\,d\tau \tag{4.7}$$

and

(4.8)

$$(s^m - r^m)^{-j_0-\mu}(s^{j_1} - r^{j_1})\cdots(s^{j_m} - r^{j_m})\int g_{j_1}\ldots_{j_\mu}(\tau)k(\tau,s)\overline{k(\tau,r)}\,d\tau$$

where $1 \le j_1,\ldots,j_\mu \le m-1$, $j_0 = 0$ or 1, and $g_{j_1}\ldots g_{j_n} \in C_0^\infty$ are supported in T.

The integration by parts is repeated in all terms where $-m(j_0 + \mu) + j_1 + \cdots + j_\mu \ge -m$ and the procedure ends after at most $m+1$ repetitions, resulting in a sum of terms as in (4.7) or (4.8) with

$\mu \leq m+1$ and degree of homogeneity of the factors in front of integrals at most $-m-1$.

Clearly all the integrals are bounded functions of s, r.

We now estimate a_{ln} by a sum of terms denoted by $b_{ln}^{(j)}$, each obtained from an estimate of (4.7) or (4.8).

Since $|k(t,s)| = 1$ we have the obvious estimate $a_{ln} \leq ||g||_{L^1}$ and

$$a_{ln} \leq \min(||g||_{L^1}, \sum_j a_{ln}^{(j)}) \leq \sum_j \min(||g||_{L^1}, b_{ln}^{(j)}) = \sum_j a_{ln}^{(j)}.$$

It follows that it suffices to check (4.6) separately for each term of the last sum.

The term $a_{ln}^{(0)}$ corresponding to (4.7) is estimated by

$$\min(||g||_{L^1}, \mathrm{const}\,\mathrm{dist}\,(J_\ell^m, J_n^m)^{-2})$$

where $J^m = \{s^m : s \in J\}$ and dist denotes the minimal distance. This is sufficient to get (4.6) for $a_{ln}^{(0)}$.

In the remaining terms $a_{ln}^{(j)}$ corresponding to (4.8) we estimate each factor $(s^j - r^j)(s^m - r^m)^{-1}$ using the inequalities

$$|(s^j - r^j)(s^m - r^m)^{-1}| \leq (|s|^j + |r|^j)(|s|^m + |r|^m)^{-1} \quad \text{if } s^m r^m < 0$$

and

$$|(s^j - r^j)(s^m - r^m)^{-1}| \leq j \max(|s|, |r|)^{j-1}(|s|^{m-1} + |r|^{m-1})^{-1} \quad \text{if } s^m r^m \geq 0.$$

The factor $(s^m - r^m)^{-1}$ is estimated as in $a_{ln}^{(0)}$ and in each of the above two estimates maximum is taken over $|s|$, $|r|$, $s \in J_l$, $r \in J_n$. This combined with $a_{ln}^{(j)} \leq ||g||_{L^1}$ yields (4.6) for each $a_{ln}^{(j)}$. We omit the straightforward if somewhat tedious details. The last statement concerning continuity of K follows using the closed graph theorem.

The same proof remains valid in the case described in Theorem 3.3—the differentiability assumptions on ϕ_0, ϕ_j and ψ allow for $m+1$

integration by parts when needed. The condition (3.3) is not needed here but boundedness of t derivatives of $\psi(t,s)$ is used.

5. Concluding remarks. The condition in Theorem 3.3, of piecewise monotonicity of ϕ_0, cannot be dispensed with—it suffices for ϕ_0 to be constant on any interval (or a set of positive measure) for $\tilde{D}_K$ to become smaller (e.g., shrink to L^1).

It is not clear to what extent the regularity of ϕ is needed for validity of Theorem 3.3.

It would be interesting to obtain a characterization of $\tilde{D}_K$ similar to that in Theorems 3.1–3.3 in the case when $k(t,s) = b(t,s)\exp(i\phi(t,s))$ where $\phi(t,s)$ is as before and $b(t,s) > 0$. If $b(t,s)$ is independent of t, then $\tilde{D}_K$ is an amalgam of ℓ^2 with L^1 with weight b—the latter space is the domain D_B corresponding to the kernel b.

Another question is that of describing the extended domains corresponding to kernels of the form dealt with above, in the case when $T = \mathbf{R}^{d'}$ and $S = \mathbf{R}^d$ with $d > 1$.

It follows from the general set up (the closed graph theorem) that $K : \tilde{D}_{KL^2_{\mathrm{loc}}(T)} \to L^2_{\mathrm{loc}}(T)$ is continuous. It would be of interest to find a proof of Theorem 3.3 giving an explicit bound of the seminorms of Ku in $L^2_{\mathrm{loc}}(T)$ in terms of the norm of u in $\ell^2(\beta^n, L^1)$. This would avoid the use of Theorem 1.3 in the proof of the inclusion $\ell^2(\beta_n, L^1) \subset \tilde{D}_{K,L^2_{\mathrm{loc}}}$.

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