ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 4, Fall 1991

CONVOLUTION OF SET FUNCTIONS

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ABSTRACT. In this paper we study a class of functional operators under which the convexity of convex set functions is preserved. In particular, convolution of convex set functions is defined and its convexity verified.

1. Introduction. The study of set functions has been motivated by recent theoretical results (e.g., [**3**–**8**]) and many applications in different fields (e.g., [**1, 2, 10**]). Since the definition of the convexity for set functions is given in a more general form than that of the ordinary one, it is expected that not all functional operators defined in [**9**; Part I, Section 5] will preserve the convexity of set functions. However, one of the most significant operators, convolution, does preserve the convexity of set functions. The main purpose of this paper is to prove that fact. Some other basic facts concerning the algebra of convex set functions are also explored.

Throughout this paper, it is assumed that $(X, \mathcal{A}, m)$ is an atomless finite measure space with $L_1(X, \mathcal{A}, m)$ separable. For $\Omega \in \mathcal{A}$, χ_Ω denotes the characteristic function of Ω, I the interval $[0,1]$, and $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$. We adopt the convention rules as in [**9**; Part I, Section 4] for arithmetic calculations involving $+\infty$ and $-\infty$. In [**8**] Morris showed that for any given Ω, $\Lambda \in \mathcal{A}$ and $\lambda \in I$, there exists a sequence $\{\Gamma_n\} \subset \mathcal{A}$ such that

$$\chi_{\Gamma_n} \xrightarrow{w^*} \lambda\chi_\Omega + (1-\lambda)\chi_\Lambda,$$

where $\xrightarrow{w^*}$ denotes weak* convergence of elements in L_∞. We shall call such a sequence a Morris-sequence associated with $\langle \lambda, \Omega, \Lambda\rangle$. Using Morris-sequence instead of usual convex combinations, a subfamily $\mathcal{S} \subset \mathcal{A}$ is said to be convex if, for every $\langle \lambda, \Omega, \Lambda\rangle \in I \times \mathcal{S} \times \mathcal{S}$ and every Morris-sequence $\{\Gamma_n\}$ associated with $\langle \lambda, \Omega, \Lambda\rangle$ in $\mathcal{S}$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ in $\mathcal{S}$.

AMS *Mathematics Subject Classification.* Primary 90C25.
Received by the editors on October 4, 1988, and in revised form on April 21, 1989.

Let $\mathcal{S}$ be a convex subfamily of $\mathcal{A}$ and $F : \mathcal{S} \to \overline{\mathbf{R}}$ be a set function defined on $\mathcal{S}$. Let $\overline{\mathcal{S}} = \{\Omega \in \mathcal{S} \mid F(\Omega) < \infty\}$ be the effective domain of F. F is said to be convex on $\mathcal{S}$ if, given any $\langle \lambda, \Omega, \Lambda \rangle \in I \times \overline{\mathcal{S}} \times \overline{\mathcal{S}}$ and any Morris-sequence $\{\Gamma_n\} \subset \overline{\mathcal{S}}$ associated with $\langle \lambda, \Omega, \Lambda \rangle$, we have

$$\limsup_{n\to\infty} F(\Gamma_n) \leq \lambda F(\Omega) + (1-\lambda)F(\Lambda).$$

A convex set function F on $\mathcal{S}$ is proper if $F(\Omega) < +\infty$ for at least one $\Omega \in \mathcal{S}$ and $F(\Omega) > -\infty$ for every $\Omega \in \mathcal{S}$. The epigraph of F over $\mathcal{S}$, $[F : \mathcal{S}]$, is defined as the set $\{(r, \Omega) \in \mathbf{R} \times \mathcal{S} \mid \Omega \in \mathcal{S},\ F(\Omega) \leq r\}$. A subset $C \subset \mathbf{R} \times \mathcal{A}$ is said to be convex if, given $(r, \Omega), (s, \Lambda) \in C$ and $\lambda \in I$, then, for every Morris-sequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ and a sequence $\{t_k\}$ such that $t_k \to \lambda r + (1-\lambda)s$ and $\{(t_k, \Gamma_{n_k})\} \subset C$. It was shown in [**4**, Theorem 3.3] that $F : \mathcal{S} \to \mathbf{R}$ is convex if and only if its epigraph is a convex subset of $\mathbf{R} \times \mathcal{A}$, and this result can be easily extended to the extended real valued set functions. We shall conclude this section with a theorem which demonstrates that a convex set function can be generated by a convex subset of $\mathbf{R} \times \mathcal{A}$ in a natural way.

Theorem 1.1. *Let C be a convex subset of $\mathbf{R} \times \mathcal{A}$. Let $\mathcal{S} = \{\Omega \in \mathcal{A} \mid (u, \Omega) \in C$, for some $u \in \mathbf{R}\}$. Define a set function F on $\mathcal{S}$ by $F(\Omega) = \inf\{u : (u, \Omega) \in A\}$; then $\mathcal{S}$ is a convex subfamily of $\mathcal{A}$ and F is a convex set function on $\mathcal{S}$.*

Proof. The convexity of $\mathcal{S}$ follows directly from the definitions. Let $\Omega, \Lambda \in \mathcal{S}, \lambda \in I$, and $\{\Gamma_n\}$ be a Morris-sequence associated with $\langle \lambda, \Omega, \Lambda \rangle$ in $\mathcal{S}$.

Case (i). Assume $F(\Omega) = -\infty$ and $F(\Lambda)$ is finite. Then for all small enough $\varepsilon < 0$, we may assume $(\varepsilon, \Omega) \in C$. Let η be a number such that $(\eta, \Lambda) \in C$. The convexity of C implies that there exists a sequence $\{t_k\}$ and a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ such that $t_k \to \lambda\varepsilon + (1-\lambda)\eta$ and $(t_k, \Gamma_{n_k}) \in C$. The latter implies that $F(\Gamma_{n_k}) \leq t_k$ for all k. Therefore, $\limsup F(\Gamma_{n_k}) \leq \limsup_{k\to\infty} t_k = \lambda\varepsilon + (1-\lambda)\eta$. If $\lambda > 0$, then $\limsup_{k\to\infty} F(\Gamma_{n_k}) = -\infty$ since $\varepsilon < 0$ is arbitrarily small. If $\lambda = 0$, then $\limsup_{k\to\infty} F(\Gamma_{n_k}) \leq \eta$ and, therefore, $\limsup_{k\to\infty} F(\Gamma_{n_k}) \leq F(\Lambda) = 0F(\Omega) + (1-0)F(\Lambda)$.

Case (ii). If both $F(\Omega) = -\infty$ and $F(\Lambda) = -\infty$, following the similar arguments given in Case (i), we can prove that $\limsup F(\Gamma_{n_k}) = -\infty$.

Case (iii). Assume that both $F(\Omega) = \alpha$ and $F(\Lambda) = \beta$ are finite. Then, there exist sufficiently small $\varepsilon, \eta > 0$, such that $(\alpha + \varepsilon, \Omega) \in C$ and $(\beta + \eta, \Lambda) \in C$. Since C is convex, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ and sequence $\{t_k\}$ such that

$$\begin{aligned} t_k &\longrightarrow \lambda(\alpha + \varepsilon) + (1 - \lambda)(\beta + \eta) \\ &= \lambda F(\Omega) + (1 - \lambda)F(\Lambda) + \lambda\varepsilon + (1 - \lambda)\eta \end{aligned}$$

and

$$\{(t_k, \Gamma_{n_k})\} \in C, \quad \text{i.e., } F(\Gamma_{n_k}) \leq t_k.$$

Therefore,

$$\begin{aligned} \limsup_{k\to\infty} F(\Gamma_{n_k}) &\leq \limsup_{k\to\infty} \{t_k\} \\ &= \lambda F(\Omega) + (1 - \lambda)F(\Lambda) + \lambda\varepsilon + (1 - \lambda)\eta. \end{aligned}$$

Since $\lambda\varepsilon + (1 - \lambda)\eta$ can be made arbitrarily small, we have

$$\limsup F(\Gamma_{n_k}) \leq \lambda F(\Omega) + (1 - \lambda)F(\Lambda).$$

This completes the proof. $\square$

2. Some basic functional operators on convex set functions. In this section we shall explore some basic functional operations that preserve the convexity of set functions.

Theorem 2.1. *Let $\mathcal{C}$ be a convex subfamily of $\mathcal{A}$.*

(i) *If F is a convex set function on $\mathcal{C}$ and $c > 0$ is any scalar, then cF is convex on $\mathcal{C}$.*

(ii) *If both F, G are proper convex set functions on $\mathcal{C}$, then $F + G$ is convex on $\mathcal{C}$.*

(iii) *If F is a convex set function on $\mathcal{C}$, then $F + c$ is convex on $\mathcal{C}$ for any constant c.*

The proof of this theorem is trivial.

Remark . The properness in the hypothesis of (ii) is for the sake of avoiding $\infty - \infty$ when $F + G$ is formed.

Theorem 2.2. *Let F be a proper convex set function on $\mathcal{A}$, and let ϕ be an upper semi-continuous, nondecreasing convex function from* $\mathbf{R}$ *to* $(-\infty, +\infty]$. *Then* $H(\Omega) = \phi(F(\Omega))$ *is convex on* $\mathcal{A}$ (where one defines $\phi(+\infty) = +\infty$).

Proof. Given $\Omega, \Lambda \in \mathcal{A}$ and $\lambda \in I$, let $\{\Gamma_n\}$ be a Morris-sequence associated with $\langle \lambda, \Omega, \Lambda \rangle$. We need to show that

$$\limsup_{n\to\infty} \phi(F(\Gamma_n)) \leq \lambda\phi(F(\Omega)) + (1-\lambda)\phi(F(\Lambda)). \tag{1}$$

Case (i). $\limsup_{n\to\infty} F(\Gamma_n) = -\infty$. Then $\lim_{n\to\infty} F(\Gamma_n) = -\infty$ and, for sufficiently large n, $F(\Gamma_n) \leq \lambda F(\Omega) + (1-\lambda)F(\Lambda)$. Since ϕ is nondecreasing and convex, we have for sufficiently large n, $\phi(F(\Gamma_n)) \leq \lambda\phi(F(\Omega)) + (1-\lambda)\phi(F(\Lambda))$, and thus (1) is satisfied.

Case (ii). $\limsup_{n\to\infty} F(\Gamma_n) > -\infty$. Invoking the convexity of F, we have

$$\limsup_{n\to\infty} F(\Gamma_n) \leq \lambda F(\Omega) + (1-\lambda)F(\Lambda). \tag{2}$$

Applying the nondecreasing and convex function ϕ to both sides of (2), we have

$$\phi(\limsup_{n\to\infty} F(\Gamma_n)) \leq \lambda\phi(F(\Omega)) + (1-\lambda)\phi(F(\Lambda)). \tag{3}$$

Now, since ϕ is upper semi-continuous,

$$\limsup_{n\to\infty} \phi(F(\Gamma_n)) \leq \phi(\limsup_{n\to\infty} F(\Gamma_n)). \tag{4}$$

Combining (3) and (4), (1) follows. □

Example. If u is a nondecreasing, upper semi-continuous convex function, then

$$F(\Omega) = u\bigg(\int f\, dm\bigg), \qquad f \in L_1(X, \mathcal{A}, m)$$

is convex.

Theorem 2.3. *Let* $F_i : \mathcal{A} \to \overline{\mathbf{R}}$ *be convex set functions for* $i = 1, \ldots, m$. *Then* $F(\Omega) = \max\{F_i(\Omega) : i = 1, \ldots, m\}$ *is convex.*

Proof. The theorem follows from the fact that $[F : \mathcal{A}] = \cap_{i=1}^{m} [F_i : \mathcal{A}]$ which again is convex in $\mathbf{R} \times \mathcal{A}$. □

3. Convolution of set functions. In this section we shall introduce a functional operational which corresponds to the addition of epigraphs.

Definition 3.1. Let F_1 and F_2 be two set functions. The convolution of F_1 and F_2, $F_1 \,\square\, F_2$, is defined by

$$(F_1 \,\square\, F_2)(\Omega) = \inf\{F_1(\Omega_1) + F_2(\Omega_2) : \Omega_1 \dot{\bigcup} \Omega_2 = \Omega, \Omega_1, \Omega_2 \in \mathcal{A}\},$$

where $\dot{\bigcup}$ denotes the disjoint union.

Lemma 3.1. [**8**, Lemma 3.3]. *Let* $(X, \mathcal{A}, m)$ *be a finite atomless measure space with* $L_1(X, \mathcal{A}, m)$ *separable. Then for* $\Omega \in \mathcal{A}$ *and* $\lambda \in I$, *it follows that* $\lambda \chi_\Omega$ *is in the weak* closure of* $\chi = \{\chi_\Lambda : \Lambda \in \mathcal{A}\} \subset L_\infty(X, \mathcal{A}, m)$.

Theorem 3.2. *Let* C_1, C_2 *be two sets in* $\mathbf{R} \times \mathcal{A}$. *We define* $C = C_1 \dot{+} C_2 = \{(u, \Omega) \in \mathbf{R} \times \mathcal{A} : u_1 + u_2 = u, \Omega_1 \dot{\bigcup} \Omega_2 = \Omega, (u_i, \Omega_i) \in C_i, i = 1, 2\}$.

If both C_1 *and* C_2 *are convex in* $R \times \mathcal{A}$, *then* $C_1 \dot{+} C_2$ *is convex in* $\mathbf{R} \times \mathcal{A}$.

Proof. Let $(r, \Omega), (s, \Lambda) \in C$, $\lambda \in I$, and let $\{\Gamma_n\}$ be any Morris-sequence associated with $\langle \lambda, \Omega, \Lambda \rangle$. Since $(r, \Omega) \in C$, there exist r_1, r_2, Ω^1 and Ω^2 such that $r_1 + r_2 = r$, $\Omega^1 \dot{\bigcup} \Omega^2 = \Omega$ and $(r_i, \Omega_i) \in C_i$ for $i = 1, 2$. Also, $(s, \Lambda) \in C$ implies the existence of s_1, s_2, Λ^1 and Λ^2 such that $s_1 + s_2 = s$, $\Lambda^1 \dot{\bigcup} \Lambda^2 = \Lambda$, and $(s_i, \Lambda^i) \in C_i$, $i = 1, 2$.

Let $\{\Delta_n\}$ be a sequence in $\Omega^1 \cap \Lambda^2$ with

$$\chi_{\Delta_n} \xrightarrow{w^*} \lambda \chi_{\Omega^1 \cap \Lambda^2},$$

and let $\{\Delta'_n\}$ be a sequence in $\Omega^2 \cap \Lambda^1$ with

$$\chi'_{\Delta_n} \xrightarrow{w^*} (1-\lambda)\chi_{\Lambda^1 \cap \Omega^2}.$$

The existence of such sequences is given by Lemma 3.1. Therefore,

$$\begin{aligned}\chi_{(\Omega^1\cap\Lambda^2)\setminus\Delta_n} &= \chi_{\Omega^1\cap\Lambda^2} - \chi_{\Delta_n} \xrightarrow{w^*} \chi_{\Omega^1\cap\Lambda^2} - \lambda\chi_{\Omega^1\cap\Lambda^2} \\ &= (1-\lambda)\chi_{\Omega^1\cap\Lambda^2}\end{aligned}$$

and

$$\begin{aligned}\chi_{(\Omega^1\cap\Lambda^2)\setminus\Delta'_n} &= \chi_{\Omega^2\cap\Lambda^1} - \chi_{\Delta'_n} \xrightarrow{w^*} \chi_{\Omega^2\cap\Lambda^1} - (1-\lambda)\chi_{\Omega^2\cap\Lambda^1} \\ &= \lambda\chi_{\Omega^2\cap\Lambda^1}.\end{aligned}$$

Now, define sequences $\{\Omega_n^1\}$, $\{\Lambda_n^1\}$, $\{\Omega_n^2\}$ and $\{\Lambda_n^2\}$ by

$$\Omega_n^1 = (\Omega_n \cap \Omega^1) \cup \Delta_n, \qquad \Lambda_n^1 = (\Lambda_n \cap \Lambda^1) \cup \Delta'_n,$$

$$\begin{aligned}\Omega_n^2 &= (\Omega_n \cap \Omega^2) \cup \{(\Omega^1 \cap \Lambda^2) - \Delta_n\}, \\ \Lambda_n^2 &= (\Lambda_n \cap \Lambda^2) \cup \{(\Omega^2 \cap \Lambda^1) - \Delta'_n\}.\end{aligned}$$

Then

$$\begin{aligned}\chi_{\Omega_n^1} &= \chi_{\Omega_n\cap\Omega^1} + \chi_{\Delta_n} - \chi_{(\Omega_n\cap\Omega^1)\cap\Delta_n}, \\ \chi_{\Lambda_n^1} &= \chi_{\Lambda_n\cap\Lambda^1} + \chi_{\Delta'_n} - \chi_{(\Lambda_n\cap\Lambda^1)\cap\Delta'_n}.\end{aligned}$$

Note that, since $\Omega^1 \cap \Delta_n$ is a subset of $\mathcal{A}$, for any $f \in L_1(\chi, \mathcal{A}, m)$,

$$\langle f, \chi_{\Omega_n\cap\Omega^1}\rangle = \langle f\chi_{\Omega^1}, \chi_{\Omega_n}\rangle \to \lambda \left\langle f\chi_{\Omega^1}, \chi_{\Omega\setminus\Lambda}\right\rangle = \lambda \left\langle f, \chi_{\Omega^1\setminus\Lambda}\right\rangle$$

and

$$\left\langle f, \chi_{\Omega_n\cap\Omega'\cap\Delta_n}\right\rangle = \langle f\chi_{\Omega^1\cap\Delta_n}, \chi_{\Omega_n}\rangle \to \left\langle f\chi_{\Omega^1\cap\Delta_n}, \chi_{\Omega\setminus\Lambda}\right\rangle = 0,$$

so

$$\chi_{\Omega_n^1} \xrightarrow{w^*} \lambda\chi_{\Omega^1\setminus\Lambda} + \lambda\chi_{\Omega^1\cap\Lambda^2} = \lambda(\chi_{\Omega^1\setminus\Lambda} + \chi_{\Omega^1\setminus\Lambda^2}) = \lambda\chi_{\Omega^1\setminus\Lambda^1}.$$

Similarly,

$$\chi_{\Lambda_n^1} \xrightarrow{w^*} (1-\lambda)\chi_{\Lambda^1\setminus\Omega}.$$

Define a Morris sequence $\{\Gamma_n^1\}$, where $\{\Gamma_n^1\} = \Omega_n^1 \cup \Lambda_n^1 \cap (\Omega^1 \cap \Lambda^1)$ is associated with $\langle \lambda, \Omega^1, \Lambda^1\rangle$. In the same manner, we have a Morris-sequence $\{\Gamma_n^2\}$, where $\Gamma_n^2 = \Omega_n^2 \cup \Lambda_n^2 \cap (\Omega^2 \cap \Lambda^2)$ is associated with $\langle \lambda, \Omega^2, \Lambda^2\rangle$. By the definition of Γ_n^1 and Γ_n^2, we also have $\Gamma_n^1 \dot{\bigcup} \Gamma_n^2 = \Gamma_n$. Since (r_1, Ω^1), $(s_1, \Lambda^1) \in C_1$, there exists a subsequence $\{\Gamma_{n_k}^1\}$ of $\{\Gamma_n^1\}$ and a sequence $\{t_k^1\}$ such that $t_k^1 \to r_1 + (1-\lambda)s_1$ with $\{(t_k^1, \Gamma_{n_k}^1)\} \subset C_1$. Similarly, for $(r_2, \Omega^2), (s_2, \Lambda^2) \in C_2$, there exists a subsequence $\{\Gamma_{n_{k_j}}^2\}$ of $\{\Gamma_{n_k}^2\}$ and a sequence $\{t_k^2\}$ such that

$$t_{k_j}^2 \to \lambda r_2 + (1-\lambda)s_2 \quad \text{with } \{(t_{k_j}^2, \Gamma_{n_{k_j}}^2)\} \subset C_2.$$

For convenience, let the index $k_j = m$, and let

$$\Gamma_{n_m} = \Gamma_{n_m}^1 \dot{\bigcup} \Gamma_{n_m}^2 \quad \text{and} \quad t_m = t_m^1 \text{ and } t_m^2.$$

Then $t_m \to \lambda r + (1-\lambda)s$ and $\{(t_m, \Gamma_{n_m})\} \in C$. Hence, C is a convex subset of $\mathbf{R} \times \mathcal{A}$, and the proof is complete. □

We now can establish the main result of this paper.

Theorem 3.3. *Let F_1 and F_2 be two proper convex set functions on A. Then the convolution $F = F_1 \square F_2$ of F_1 and F_2 is convex on $\mathcal{A}$.*

Proof. Note that the epigraph $[F_i : \mathcal{A}]$ is convex in $\mathbf{R} \times \mathcal{A}$ for $i = 1, 2$.

Define $C = [F_i : \mathcal{A}] \dot{+} [F_2 : \mathcal{A}]$, and $G(\Omega) = \inf\{u : (u, \Omega) \in C\}$. Then C is convex in $\mathbf{R} \times \mathcal{A}$ by Theorem 3.2 and G is convex by Theorem 1.1.

We shall show that $F \equiv G$. For given $\Omega \in \mathcal{A}$, let $u^* = G(\Omega)$. Then, for any $\varepsilon > 0$, $(u^* + \varepsilon, \Omega) \in C$. That is, we have u_1, u_2, Ω_1 and Ω_2 such that $u_1 + u_2 = u^* + \varepsilon$, $\Omega_1 \dot{\bigcup} \Omega_2 = \Omega$ and $F_i(\Omega_i) \le u_i$, $i = 1, 2$. Since $\varepsilon > 0$ is arbitrary, $F(\Omega) \le u^* = G(\Omega)$. Now suppose

$G(\Omega) > F(\Omega)$. Then there exists $\Omega_1, \Omega_2 \in A$ such that $\Omega_1 \dot{\bigcup} \Omega_2 = \Omega$ with $F_1(\Omega_1) + F(\Omega_2) < u^*$. Let $u_1 = F_1(\Omega_1)$ and $u_2 = F_2(\Omega_2)$. Then $(u_1 + u_2, \Omega_1 \dot{\bigcup} \Omega_2) \in \mathcal{A}$, which implies $G(\Omega) = \inf\{u : (u, \Omega) \in \mathcal{A}\} \leq u_1 + u_2 < u^*$. This contradicts the fact that $G(\Omega) = u^*$. Hence, $G(\Omega) \leq F(\Omega)$. □

Remark 3.1. (i) From Definition 3.1, it can be shown that the convolution is associative, i.e.,

$$(F_1 \,\square\, F_2) \,\square\, F_3 = F_1 \,\square\, (F_2 \,\square\, F_3).$$

Therefore,

$$\begin{aligned}(F_1 \,\square\, F_2 \,\square\, \cdots \,\square\, F_n)(\Omega) \\ = \inf\{F_1(\Omega_1) + \cdots + F_n(\Omega_n) : \Omega_1 \dot{\bigcup} \Omega_2 \dot{\bigcup} \cdots \dot{\bigcup} \Omega_n = \Omega\}.\end{aligned}$$

(ii) Let F_1 and F_2 be two set functions defined on $\mathcal{A}$. The convolution of F_1 and F_2 can be expressed by

$$(F_1 \,\square\, F_2)(\Omega) = \inf_{\Lambda \in \mathcal{A}} \{F_1(\Omega \backslash \Lambda) + F_2(\Omega \cap \Lambda)\},$$

since $(\Omega \backslash \Lambda) \dot{\bigcup} (\Omega \cap \Lambda) = \Omega$.

(iii) Let $G = \delta(\cdot \mid \Lambda)$ for a certain $\Lambda \in \mathcal{A}$, where $\delta(\Omega \mid \Lambda) = \infty$ if $\Omega \neq \Lambda$ and $\delta(\Lambda \mid \Lambda) = 0$. Then $(F \,\square\, G)(\Omega) = F(\Omega \backslash \Lambda)$ since $\Omega \backslash \Lambda = \Omega \backslash (\Omega \cap \Lambda)$. It follows that for a certain $\Lambda \in \mathcal{A}$ the set function F_Λ defined by $F_\Lambda(\Omega) = F(\Omega \backslash \Lambda)$ is convex if F is.

(iv) The convolution operation is commutative since $(F_1 \,\square\, F_2)(\Omega) = \inf\{u \mid (\Omega, u) \in [F_1 : \mathcal{A}] \dot{+} [F_2 : \mathcal{A}]\}$ as shown in the proof of Theorem 3.3.

Acknowledgment. The authors greatly appreciate helpful and valuable comments and suggestions received from the referee.

REFERENCES

1. D. Begis and R. Glowinski, *Application de la methode des elements finis a l'approximation d'un probleme de domaine optimal*, *Methodes de resolution des problemes approches*, Appl. Math. Opt. 2, **2** (1975), 130–169.

2. J. Cea, A. Gioan and J. Michel, *Quelques resultats sur l'identification de domaines*, Calcolo **10** (1973), 133–145.

3. J.H. Chou, W.S. Hsia and T.Y. Lee, *On multiple objective programming problems with set functions*, J. Math. Anal. Appl. **105** (1985), 383–394.

4. ———, ——— and ———, *Epigraphs of convex set functions*, J. Math. Anal. Appl. **118** (1986), 247–254.

5. ———, ——— and ———, *Convex programming with set functions*, Rocky Mountain J. Math. **17** (1987), 535–543.

6. W.S. Hsia and T.Y. Lee, *Lagrangian function and duality theory in multiobjective programming with set functions*, JOTA **57** (1988), 239–251.

7. H.B. Lai and S.S. Yang, *Saddle point and duality in the optimization theory of convex set functions*, J. Austral. Math. Soc., Ser. B **24** (1982), 130–137.

8. R.J.T. Morris, *Optimal constrained selection of measurable subset*, J. Math. Anal. Appl. **70** (1979), 546–562.

9. R.T. Rockafellar, *Convex analysis*, Princeton Univ. Press, Princeton, NJ, (1970).

10. P.K.C. Wang, *On a class of optimization problems involving domain variations*, in International Symposium on New Trends in System Analysis, (Versailles, France, Dec., 1976), Lecture Notes in Control and Information Sciences, vol. 2, Springer-Verlag, Berlin, 1977.

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