ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 1, Winter 1991

THE RATIONAL INTERPOLATION PROBLEM REVISITED

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Dedicated to Wolfgang Thron on his 70th birthday

ABSTRACT. We present a general, homogeneous treatment of the rational interpolation problem in the extended complex domain. Interpolation conditions at the point ∞ and prescribed poles are allowed.

1. Introduction. Rational functions are functions meromorphic in the extended plane $\overline{\mathbf{C}}$. In $\overline{\mathbf{C}}$ the number ω of poles of a rational function $r \neq 0$ is equal to the number of its zeros; ω is called the *order* of r. We denote the set of rational functions of order at most ω by $\mathcal{R}_\omega$ and define the set $\overline{\mathcal{R}}_\omega := \mathcal{R}_\omega \cup \{\infty\}$ by adding the constant function ∞ to the set. (See Section 2 for more details.) Then, if t_1, t_2 are two Möbius transforms (i.e., meromorphic one-to-one maps of $\overline{\mathbf{C}}$ onto itself), the composition $t_1 \circ r \circ t_2$ is a rational function of the same order. Hence, in the theory of rational functions there is nothing special about the point ∞, neither in the domain nor in the range. However, in the usual treatment of the rational interpolation problem (or multipoint Padé or Newton-Padé approximation problem—as it is also called) one assumes that the prescribed function values are finite, and one treats the cases where data are given at the point ∞, or poles are prescribed, completely independently. As a consequence, the resulting theory lacks symmetry. For example, the result that r is the interpolant of f if and only if $1/r$ is the interpolant of $1/f$ does not always hold, since $1/f$ may take the value ∞. Also, while the rational interpolant is often constructed as a terminating Thiele fraction, some Thiele fractions correspond to

1980 *Mathematics Subject Classification.* 41A21, 41A05, 65D05
Keywords and phrases. Rational interpolation, Newton-Padé approximation, multipoint Padé approximation.
Received by the editors on August 1, 1988, and in revised form on October 28, 1988.

unbounded data: For example,

$$1+\cfrac{z-1}{1-\cfrac{z-2}{1+\cfrac{z-3}{3}}}$$

corresponds to the data $(1,1),(2,2),(3,\infty)$. This difficulty has nothing to do with the well-known fact that a rational interpolation problem need not have a solution, which can occur even if, as we always assume, the number of free parameters, $2\omega+1$, equals the number of data.

As usual, we define in general the interpolant via the linearized interpolation problem, which leads always to a unique result. We should also point out that we never assume that the interpolation points are distinct, i.e., we allow that at any interpolation point the first few derivatives are prescribed in addition to the function value. (Often the names *osculatory* and *Hermite* rational interpolation problem have been used for this case.) Since we allow that one of the interpolation points is ∞ and that in some of the interpolation points poles of an arbitrary order are prescribed, our treatment indeed covers problems that usually go under a variety of names: Padé approximation (all data are given at 0), two-point Padé approximation (half of the data are given at each of the points 0 and ∞), Newton-Padé or multipoint Padé approximation (data are given at several finite points), Padé-type approximation (all poles of the approximants are prescribed, the other data are given at 0). But here only some poles may be prescribed, other data may be given at any point of the extended plane. In the same generality, the problem has recently been studied by Stahl [**10**], but our treatment and our results are quite different from his; see Section 7 for a brief discussion of the connection between his and our approach.

We do not at all attempt to give a full discussion of all aspects of the rational interpolation problem. Essentially, we restrict ourselves to three basic theorems: the existence and uniqueness of the rational function solving the linearized problem (we call this function the *multipoint Padé approximant* (MPA), even in our general situation); the description of the general form of the pairs of polynomials solving the linearized problem (we call these pairs *multipoint Padé forms* (MPFs)); and the characterization of the MPA. Some simple conclusions are then drawn from these results, e.g., one can give several criteria for the MPA to be a solution of the original nonlinear rational interpolation problem.

After having treated the general case in Sections 2–5, we will exclude the point ∞ as interpolation point in Section 6 except for assuming a certain number of zeros or poles there. In other words, we will then prescribe the *type* (m, n) of the rational interpolant (i.e., its maximum numerator degree m and denominator degree n) rather than its order. It is no surprise that this problem can be treated as a special case of the general one, although some details need to be discussed and verified. In contrast to the standard treatment, we still allow that given function values are ∞, i.e., that poles are prescribed.

Our treatment picks up ideas from the well-known bigradient formulation in Padé approximation [**2, 4**], which in effect has been generalized by Warner [**14**] to the rational interpolation problem. But since Warner required the second data function (called g below) to be nonvanishing, he missed the generality we are heading for, even in the case where all interpolation points are finite.

2. The set of rational functions of order ω and a representation of the data of the corresponding interpolation problems. Let $\mathcal{P}_\omega$ denote the space of polynomials in z of degree at most ω. The exact degree of a polynomial p is denoted by ∂p. Let $\mathcal{R}_\omega$ be the set of rational functions of order at most ω, i.e., the set of functions r which can be written as $r = p/q$ with $p, q \in \mathcal{P}_\omega$, $q \neq 0$. (Of course, this representation is never unique unless some normalization is agreed upon. A first requirement would be to use relatively prime polynomials p and q. However, here we have no need for a special normalization.) We enlarge here this standard set of rational functions by the constant function ∞, which has the representation $1/0$, and we set $\overline{\mathcal{R}}_\omega := \mathcal{R}_\omega \cup \infty$. Note that $r \in \overline{\mathcal{R}}_\omega$ if and only if $1/r \in \overline{\mathcal{R}}_\omega$. More generally, if t_1, t_2 are any two Möbius transforms, then $r \in \overline{\mathcal{R}}_\omega$ if and only if $t_1 \circ r \circ t_2 \in \overline{\mathcal{R}}_\omega$. The *defect* δ of $r \in \overline{\mathcal{R}}_\omega$ is the difference of ω and the actual order of r. For example, all the constant functions have defect ω in $\overline{\mathcal{R}}_\omega$. This also holds for the constants 0 and ∞.

Let $Z' := \{z'_1, \dots, z'_J\} \subset \mathbf{C}$ be a set of distinct interpolation points, and let $K := \{\kappa_1, \dots, \kappa_J\}$ be the associated set of multiplicities, $\kappa_j \geq 1$ $(\forall j)$. Furthermore, let $z'_0 := \infty$, let $\kappa_0 := \kappa \geq 0$ be an associated multiplicity, and assume that

$$\sum_{j=0}^{J} \kappa_j = 2\omega + 1. \tag{1}$$

The rational interpolation problem consists in finding $r \in \overline{\mathcal{R}}_\omega$ with the property that at each point z'_j the first κ_j terms of the power series of either r or $1/r$ (depending on whether f is bounded or unbounded at z'_j) coincide with prescribed data. Although in principle only the prescribed function values and derivatives need to be known, we may assume here for ease of notation and without restricting the generality that the data are given in terms of an everywhere meromorphic, hence, rational function $h = f/g$. Clearly, there is abundance of freedom in choosing h, and even more so in choosing f and g. In particular, we might demand that

$$\begin{aligned} |h(z)| > 1 &\Rightarrow f(z) = 1, \quad z \in Z', \\ |h(z)| \le 1 &\Rightarrow g(z) = 1, \quad z \in Z', \end{aligned} \tag{2a}$$

and, if $\kappa > 0$,

$$\begin{aligned} |h(\infty)| > 1 &\Rightarrow f(z) = z^{2\omega} + O(z^{2\omega-1}), \quad z \to \infty, \\ |h(\infty)| \le 1 &\Rightarrow g(z) = z^{2\omega} + O(z^{2\omega-1}), \quad z \to \infty. \end{aligned} \tag{2b}$$

To be more specific, we may choose

$$\begin{aligned} f(z) &:= f_{2\omega-\kappa}(z) + t(z)\phi(z), \\ g(z) &:= g_{2\omega-\kappa}(z) + t(z)\psi(z), \end{aligned} \tag{3}$$

where

$$t(z) := \prod_{j=1}^{J} (z - z'_j)^{\kappa_j} \tag{4}$$

is the polynomial of degree $2\omega - \kappa + 1$ whose zeros are the finite interpolation points with the appropriate multiplicity, $f_{2\omega-\kappa}$ and $g_{2\omega-\kappa}$ are interpolation polynomials of degree at most $2\omega - \kappa$ chosen in accordance with (2a) for the data at these points, and ϕ and ψ are polynomials of degree at most $\kappa - 1$ chosen in accordance with (2b) for prescribing the behavior at ∞. Note that ϕ or ψ has exact degree $\kappa - 1$ if $\kappa > 0$, and $\phi = \psi = 0$ if $\kappa = 0$. Hence, $f, g \in \mathcal{P}_{2\omega}$ and $\max\{\partial f, \partial g\} = 2\omega$.

If we let $\hat{\phi}(1/z) := z^{-\kappa+1}\phi(z)$ and $\hat{\psi}(1/z) := z^{-\kappa+1}\psi(z)$, and if we assume for the moment that $\phi \neq 0$ and $\psi \neq 0$, the quotient f/g behaves at $z = \infty$ asymptotically as

$$\begin{aligned}\frac{f(z)}{g(z)} &= \frac{\hat{\phi}(1/z) + O(z^{-\kappa})}{\hat{\psi}(1/z) + O(z^{-\kappa})} = \frac{\hat{\phi}(1/z)[1 + O(z^{-\partial\phi-1})]}{\hat{\psi}(1/z)[1 + O(z^{-\partial\psi-1})]} \\ &= \frac{\hat{\phi}(1/z)}{\hat{\psi}(1/z)}[1 + O(z^{-\min\{\partial\phi+1, \partial\psi+1\}})] = O\left(\left(\frac{1}{z}\right)^{\partial\psi-\partial\phi}\right),\end{aligned} \tag{5}$$

from which it is clear that ϕ and ψ determine the first κ terms of the power series in $1/z$ of f/g at ∞ if $\partial\phi \leq \partial\psi = \kappa - 1$, and the first κ terms of the one of g/f if $\partial\psi \leq \partial\phi = \kappa - 1$.

If only the order of a pole or a zero at ∞ is prescribed, then $\psi = 0$ or $\phi = 0$, respectively. If $\kappa = 0$, then $\phi = \psi = 0$. These cases are not excluded here (except in Formula (5)) and will be especially addressed in Section 6.

We will always refer to the data as f/g, although we really mean that the pair (f, g) is available, and not just the values of the quotient. The representation of the data by f and g as defined in (3) is suitable for the theoretical treatment of a single rational interpolation problem. We would turn to another representation, if we discussed sequences of interpolation problems, as, e.g., for investigations of the multipoint Padé table. In any case, the results obtained do at most superficially depend on the representation of the data, and the proofs become simpler. (With respect to homogeneous notation, Stahl's treatment [**10**] is more consequent, at the cost of introducing auxiliary functions, see Section 7.)

Whenever we will consider the special case where $g(z) \equiv 1$ and $\kappa = 0$, we will refer to it as the *standard situation*. Actually, most of the *classical* work up to 1973 is only concerned with nonosculatory interpolation, or, as we will say here, interpolation *at distinct points* (see Meinguet [**7**] for a survey). The few exceptions include [**5, 9**]; the real change started with [**14, 16**].

3. The rational interpolation problem and its linearized version. Given Z', K, κ, and $h = f/g$, where f and g are defined by (3), the (*true*) *rational interpolation problem* is to find $r \in \overline{\mathcal{R}}_\omega$ such that

(6a)
$$\frac{f(z)}{g(z)} - r(z) = \begin{cases} O(t(z)) & \text{as } z \to z' \in Z', \quad \text{if } g(z') \neq 0, \\ O(z^{-\kappa}) & \text{as } z \to \infty, \qquad \text{if } \partial g = 2\omega,\ \kappa > 0, \end{cases}$$

(6b)
$$\frac{g(z)}{f(z)} - \frac{1}{r(z)} = \begin{cases} O(t(z)) & \text{as } z \to z' \in Z', \quad \text{if } f(z') \neq 0, \\ O(z^{-\kappa}) & \text{as } z \to \infty, \qquad \text{if } \partial f = 2\omega,\ \kappa > 0, \end{cases}$$

where the notation

$$O(t(z)) \quad \text{as } z \to z' \in Z' \quad \text{if } g(z') \neq 0$$

is an equivalent for

$$O((z - z_j')^{\kappa_j}) \quad \text{as } z \to z_j' \quad \text{if } 1 \leq j \leq J \text{ and } g(z_j') \neq 0.$$

Note that, for those $z' \in Z'$ with $f(z') \neq 0$, $g(z') \neq 0$, the conditions (6a) and (6b) are equivalent. Actually, one could always replace (6b) by a condition about the vanishing of certain coefficients of the Laurent series of $f/g - r$ at z' and at ∞.

It is well known from the standard situation that this problem may have no solution in general, but that, whenever a solution exists, it can be found by solving a linearized problem. (Often the latter has been called the *modified* rational interpolation problem.) In our general situation this *linearized rational interpolation problem* becomes the following: Given Z', K, κ, f, and g as above, find a pair of polynomials $(p, q) \in \mathcal{P}_\omega \times \mathcal{P}_\omega$ such that

$$(fq - gp)(z) = \begin{cases} O(t(z)) & \text{as } z \to z' \in Z', \\ O(z^{3\omega-\kappa}) & \text{as } z \to \infty; \end{cases} \tag{7}$$

then let $r := p/q$. (There may be common factors that cancel.) As in the standard situation, there holds

Theorem 1. *For every data f/g, there exists a nontrivial subspace of $\mathcal{P}_\omega \times \mathcal{P}_\omega$ consisting of the solutions (p, q) of (7), and all these* *pairs* (p, q) *determine the same rational function* $r := p/q \in \overline{\mathcal{R}}_\omega$, *which is the unique solution of the linearized rational interpolation problem. The function* r *is also the solution of the nonlinear true rational interpolation problem* (6) *if the latter has a solution.*

Proof. The conditions (7) yield a total of $2\omega+1$ linear restrictions for $(p, q) \in \mathcal{P}_\omega \times \mathcal{P}_\omega$. Since this space has dimension $2\omega + 2$, there always exist nontrivial solutions. To prove the uniqueness of $r = p/q$, assume that (p, q) and $(\tilde{p}, \tilde{q})$ are two nontrivial solutions of (7). Then

$$[(\tilde{q}p - q\tilde{p})g](z) = [\tilde{q}(fq + O(\cdot)) - q(f\tilde{q} + O(\cdot))](z) = \begin{cases} O(t(z)) & \text{as } z \to z' \in Z', \\ O(z^{4\omega-\kappa}) & \text{as } z \to \infty, \end{cases}$$

and the same asymptotic behavior is obtained for $(\tilde{q}p - q\tilde{p})f$. Consequently,

$$(\tilde{q}p - q\tilde{p})(z) = \begin{cases} O(t(z)) & \text{as } z \to z' \in Z', \\ O(z^{2\omega-\kappa}) & \text{as } z \to \infty, \end{cases}$$

which means that $\tilde{q}p - q\tilde{p} \in \mathcal{P}_{2\omega-\kappa}$ and that it has $\partial t = 2\omega - \kappa + 1$ zeros. Therefore, $\tilde{q}p - q\tilde{p} = 0$, i.e., $p/q = \tilde{p}/\tilde{q}$ after canceling common factors.

If the nonlinear interpolation problem (6) has a solution $r = p/q$, then multiplication of (6a) by $g(z)q(z)$ and of (6b) by $f(z)p(z)$ yields (7), which means that the pair (p, q) is a solution of (7). □

We call the unique solution $r = p/q$ obtained from (7) the *multipoint Padé approximant* (MPA) *of order* ω for the given data f/g, even if r does not solve (6). If it does, we call it a *true rational interpolant.* Moreover, in analogy to the notion of a Padé form [**2**] we call a nontrivial solution $(p, q) \in \mathcal{P}_\omega \times \mathcal{P}_\omega$ of (7) a *multipoint Padé form* (MPF) *of order* ω. (At first sight, possible confusion may arise from the fact that, even in an MPF (p, q) with degrees of p and q as small as possible, these two polynomials need not be mutually prime; but it will become clear that, when r is not a true interpolant, both contain necessarily a common polynomial factor s.)

4. The general form of multipoint Padé forms and the characterization of multipoint Padé approximants. The next theorem describes the general solution of (7), i.e., the set of MPFs belonging to some given data. The corresponding result for the standard situation can be found in Maehly and Witzgall [**6**] for interpolation at distinct points and in Claessens [**1**] for osculatory interpolation.

Theorem 2. *The set of MPFs of order* ω *for the data* f/g *consists of the pairs*

$$(p, q) = (\hat{p}sw, \hat{q}sw) \in \mathcal{P}_\omega \times \mathcal{P}_\omega, \tag{8}$$

where $(\hat{p}, \hat{q}) \in \mathcal{P}_\omega \times \mathcal{P}_\omega$ *is a pair of fixed relatively prime polynomials,* s *is a fixed monic polynomial divisor of* t *(defined in (4)) of degree* $\partial s \leq \min\{\delta, \eta\}$*,* δ *being the defect of* $r := \hat{p}/\hat{q}$ *in* $\overline{\mathcal{R}}_\omega$*, and* η *being such that the degree of* $f\hat{q} - g\hat{p}$ *equals*

$$\partial(f\hat{q} - g\hat{p})(z) = 3\omega - \kappa - \eta. \tag{9}$$

Finally, $w \neq 0$ *is an arbitrary polynomial of degree*

$$\partial w \leq \min\{\delta, \eta\} - \partial s. \tag{10}$$

The polynomial s *can be characterized as the one of minimum degree satisfying*

$$(f\hat{q} - g\hat{p})(z) = O(t(z)/s(z)) \text{ as } z \to z' \in Z'. \tag{11}$$

If $\kappa = 0$*, (10) simplifies to*

$$\partial w \leq \delta - \partial s. \tag{12}$$

Proof. Let $(p, q) \in \mathcal{P}_\omega \times \mathcal{P}_\omega$ be any MPF. First, we can write $(p, q) = (\hat{p}d, \hat{q}d)$ with $\hat{p}, \hat{q}$ relatively prime and $d \in \mathcal{P}_\delta$. Second, d can be split as $d = sw$, where s is a monic divisor of minimum degree with the property that $(\hat{p}s, \hat{q}s)$ is still an MPF. Clearly, s is a divisor of t, since one can divide through (7) by any common linear factor of p and q but not t without changing the $O(t(z))$-term; the $O(z^{3\omega-\kappa})$-term is even improved, and thus is not violated. Hence, (p, q) is of the form (8), and $(\hat{p}s, \hat{q}s)$ is a solution of (7); dividing through by s then gives (11), and, for η defined as the largest integer satisfying (9), one obtains $\eta \geq \partial s$. Moreover, since s has been chosen as the divisor of minimum degree for which $(\hat{p}s, \hat{q}s)$ is still an MPF, (11) would not hold for any proper divisor of s. Therefore, at those points z'_j which are zeros of s, formula (11) gives the exact order, and at the other points the order is at least the same as the one of $t(z)$. Now we know from Theorem 1 that $\hat{p}$ and $\hat{q}$ are uniquely determined by the data up to a common scalar factor, and, of course, the same holds then for the left-hand side of (11). Consequently, s is also uniquely determined by the data.

From (9) and the second line in (7) one gets $\partial w + \partial s \leq \eta$, and from (8) one has clearly $\partial w + \partial s \leq \delta$. Therefore, w satisfies (10). Moreover, since $f, g \in \mathcal{P}_{2\omega}$, (9) gives

$$\kappa + \eta \geq \delta. \tag{13}$$

Consequently, if $\kappa = 0$, (10) becomes (12).

On the other hand, given this particular MPF $(\hat{p}s, \hat{q}s)$, any pair of the form in (8), with $w \neq 0$ satisfying the degree restriction (10), is also a nontrivial solution of (7), i.e., an MPF. □

We call s the *deficiency polynomial* and

$$\sigma := \begin{cases} 0 & \text{if } \eta \geq \delta, \\ \delta - \eta & \text{if } \eta < \delta, \end{cases} \tag{14}$$

the *deficiency at* ∞. Note that, in view of (13) and $\eta \geq \partial s$,

$$0 \leq \sigma \leq \min\{\kappa, \delta - \partial s\}. \tag{15}$$

As one can expect from (9) and (11), and as will indeed be shown in the following theorem, the two quantities s and σ exhibit the shortcoming of the MPA if the latter is tried as solution of the original interpolation problem (6). The next theorem not only states this property of the MPA but characterizes the MPA among the elements of $\overline{\mathcal{R}}_\omega$. For the standard situation this characterization theorem can be found in Maehly and Witzgall [**6**, p. 296] and Wuytack [**15**, Lemma 2] for the case of distinct interpolation points, and in Warner [**14**, Theorem 7] and

Wuytack [**16**, Lemma 2] for Hermite data. For Padé approximation, it goes back to Padé [**8**], but its usefulness has been overlooked for decades and only recently been pointed out by Trefethen [**12, 13**].

Theorem 3 (Characterization Theorem). *The function $r \in \overline{\mathcal{R}}_\omega$ with defect δ is the MPA of order ω of $h = f/g$ if and only if there exists a polynomial divisor s of t (defined by (4)) of degree at most δ and an integer σ satisfying (15) such that*

(16a)
$$\frac{f(z)}{g(z)} - f(z) = \begin{cases} O(t(z)/s(z)) & \text{as } z \to z' \in Z', \text{ if } g(z') \neq 0, \\ O(z^{-\kappa+\sigma}) & \text{as } z \to \infty, \text{ if } \partial g = 2\omega,\ \kappa > 0, \end{cases}$$

(16b)
$$\frac{g(z)}{f(z)} - \frac{1}{r(z)} = \begin{cases} O(t(z)/s(z)) & \text{as } z \to z' \in Z', \text{ if } f(z') \neq 0, \\ O(z^{-\kappa+\sigma}) & \text{as } z \to \infty, \text{ if } \partial f = 2\omega,\ \kappa > 0, \end{cases}$$

i.e., if and only if at least $2\omega-\delta+1$ of the $2\omega+1$ interpolation conditions in (6) are fulfilled.

If r is the MPA of $h = f/g$, (16) holds in particular with s and σ equal to the deficiency polynomial and the deficiency at ∞, respectively. The minimum number of interpolation conditions fulfilled, $2\omega - \delta + 1$, is then attained whenever $\sigma + \partial s = \delta$. Any other pair $(\tilde{s}, \tilde{\sigma})$ satisfying (16) has the properties that s is a divisor of $\tilde{s}$ and $\sigma \leq \tilde{\sigma}$.

Proof. Assume r is the MPA of h. Then, by Theorem 2 and (15), s and σ with the quoted restrictions exist. For any $z' \in Z'$ with $g(z') \neq 0$, there follows from (11) that z' is not a common zero of $\hat{q}$ and t/s, since otherwise it would also be a zero of $\hat{p}$. Hence, division of (11) by $g\hat{q}$ yields the first line of (16a). If $\kappa > 0$ and $\partial f \leq \partial g = 2\omega$, it follows from (7) that $\partial\hat{p} \leq \partial\hat{q} = \omega - \delta$. Then (9) yields the second line of (16a):

$$\begin{aligned}\frac{f(z)}{g(z)} - r(z) = \frac{f\hat{q} - g\hat{p}}{g\hat{q}}(z) &= O(z^{3\omega-\kappa-\eta-(3\omega-\delta)}) \\ &= O(z^{-\kappa-\eta+\delta}) = O(z^{-\kappa+\sigma}) \quad \text{as } z \to \infty.\end{aligned}$$

(In the last step we replace $\delta - \eta$ by its positive part σ since an error of order $z^{-\kappa}$ at ∞ means that already all prescribed data are matched there.) The derivation of (16b) is completely analogous.

If s and σ still denote the quantities of Theorem 2 and (15), the total number of interpolation conditions fulfilled is exactly $\partial t - \partial s + \kappa - \sigma$, which by (15) satisfies

$$\partial t - \partial s + \kappa - \sigma \geq 2\omega - \delta + 1. \tag{17}$$

The lower bound on the right-hand side is attained whenever $\sigma + \partial s = \delta$. From the minimum degree property of s and the maximality of η (i.e., the minimality of σ), there finally follows the statement of the last sentence of the theorem, cf. the proof of Theorem 2.

Conversely, assume that $r = \hat{p}/\hat{q} \in \overline{\mathcal{R}}_\omega$ (with $\hat{p}$ and $\hat{q}$ relatively prime) satisfies at least $2\omega - \delta + 1$ of the $2\omega + 1$ conditions in (6). Define s as the monic polynomial whose zeros are, with the appropriate multiplicity, the finite interpolation points where (6) is not fulfilled, and let σ likewise be the number of conditions neglected at ∞. Then r satisfies (16), $s \in \mathcal{P}_\delta$ is a divisor of t, and σ is an integer satisfying (15). Multiplication of the first line of (16a) and (16b) by $g\hat{q}s$ and $f\hat{p}s$, respectively, show that the first line of (7) holds for $(p, q) = (\hat{p}s, \hat{q}s)$. Likewise, in view of (15), multiplication of the second line of (16a) and (16b) yields the second line of (7). □

An alternate way to prove Theorem 3 consists in applying both in the domain and the range of h Moebius transforms to the effect that all interpolation points and all corresponding function values become finite. By this simple means, our general situation can be reduced to the standard one. After establishing the result for the standard situation, there only remains to formulate the conclusions for the original general setting.

5. Further conclusions. We restrict ourselves to a few of the many results that now follow easily from Theorems 2 and 3. First of all, these two theorems make the connection between the original and the linearized interpolation problem obvious. There are various ways to characterize those MPAs that are true interpolants, and, indeed, many authors have given such characterizations in the standard situation, see, e.g., [**6, 7, 11, 15**], and, for Hermite data, [**14, 16**]. Here we have the additional difficulty that the behavior at ∞ must be kept under control.

Corollary 4. *Let $r = \hat{p}/\hat{q}$ be the MPA of f/g in reduced form, and let s, δ and η be as in Theorem 2, σ as in (14). Furthermore, among the MPFs of f/g, let (p, q) be one where p and q have minimum degree. The following statements are equivalent*:

(i) *r is a true interpolant.*

(ii) *$s(z) \equiv 1$ and $\sigma = 0$.*

(iii) *$(\hat{p}, \hat{q})$ is an MPF and $\eta \geq \delta$.*

(iv) *$p(z') \neq 0$ or $q(z') \neq 0$ for each $z' \in Z'$ and $\partial(fq - gp) \leq 3\omega - \kappa - \delta$.*

Proof. (i) $\Rightarrow$ (ii). This follows from Theorem 3, in particular (16) and the final sentence. (ii) $\Rightarrow$ (iii). Apply Theorem 2 (choose $w(z) \equiv 1$ in (8)) and (14). (iii) $\Rightarrow$ (iv). Use Theorem 2, where one must have $s(z) \equiv 1$ in (8), in order that $(\hat{p}, \hat{q})$ is an MPF; this MPF then clearly is the one with p and q of minimum degrees; thus, p and q are relatively prime and cannot have a common zero z'. Finally, (9) yields $(fq - gp)(z) = O(z^{3\omega-\kappa-\eta}) = O(z^{3\omega-\kappa-\delta})$ as $z \to \infty$ if $\eta \geq \delta$. (iv) $\Rightarrow$ (i). By (8), one must have $s(z) \equiv 1$ since a zero z' of s would be a zero of both p and q; hence, $(p, q) = (\hat{p}, \hat{q})$. By (9), the order at ∞ is exactly $O(z^{3\omega-\kappa-\eta})$ (since η is chosen as large as possible); therefore, (iv) can only hold if $\eta \geq \delta$, i.e., $\sigma = 0$. The formulas (16) then show that r is a true interpolant. □

As we have mentioned, the conditions (7) impose a total of $2\omega + 1$ linear restrictions for $(p, q) \in \mathcal{P}_\omega \times \mathcal{P}_\omega$. In practice, this means that one obtains a system of $2\omega + 1$ linear equations in the $2\omega + 2$ unknown coefficients of p and q. (These need not be the power series coefficients; one can use as well the Newton interpolation formula representation for the polynomials, i.e., choose the Newton series coefficients as the unknowns.) Often, unlike in our treatment, the derivation of the fundamental results on rational interpolation is based on a discussion of the rank of this linear system. But our Theorem 2, of course, contains this information also:

Corollary 5. *The system of* $2\omega + 1$ *linear equations in the* $2\omega + 2$ *unknown coefficients of* p *and* q *which results from* (7) *has rank*

$$2\omega + 1 + \partial s - \min\{\delta, \eta\}, \tag{18}$$

i.e., the solution subspace has dimension

$$1 + \min\{\delta, \eta\} - \partial s. \tag{19}$$

Proof. The solution is unique up to a constant factor if and only if w is necessarily a constant. Otherwise, the dimension of the solution space is increased by the maximum degree of w, which is given by (10). $\square$

The aim of this paper was to establish the basic results on rational interpolation in such a way that the point ∞ does not play a special role, neither as interpolation point nor as interpolation value. The following result, which exhibits the symmetry attained, is one of the rewards.

Corollary 6. *Let* r *be the MPA of order* ω *of* h *with respect to the interpolation points* $z'_0, \dots, z'_J$ *with multiplicities* $\kappa_0, \dots, \kappa_J$*, and let* t_1, t_2 *be two Möbius transforms. Then* $t_1 \circ r \circ t_2$ *is the MPA of order* ω *of* $t_1 \circ h \circ t_2$ *with respect to the interpolation points* $t_2^{-1}(z'_0), \dots, t_2^{-1}(z'_J)$ *with the same multiplicities. In particular,* $1/r$ *is the MPA of order* ω *of* $1/h$.

Proof. $t_1 \circ r \circ t_2$ has the same order as r and, therefore, also the same defect. Moreover, since Möbius transforms are conformal, it interpolates $t_1 \circ h \circ t_2$ in as many points as r interpolates h, even if osculatory interpolation is taken into account. By Theorem 3, an MPA can be characterized alone by the number of interpolation points and the defect. $\square$

6. Rational interpolation in C: The set of rational functions of type $(\mathbf{m}, \mathbf{n})$ and the corresponding Newton-Padé approximation problem. If all interpolation points are finite, the treatment of Sections 2–5 can be simplified considerably. On the other hand, it is then usual to prescribe at least the order of the asymptotic behavior of the interpolant at ∞. (Again, it may happen that no interpolant with this asymptotic behavior exists.) It is no surprise that this interpolation problem is equivalent to a special case of our general one, but this equivalence is not quite so trivial that it can be taken for granted. We have also to point out that the following treatment is more general than the standard one in that it allows us to prescribe poles and the first few Laurent coefficients in these poles.

A rational function $r = p/q \in \overline{\mathcal{R}}_\omega$ is said to be of *type* (m, n) if $\partial p \le m$ and $\partial q \le n$; its *exact type* is $(\partial p, \partial q)$ if p and q are relatively prime. Of course, $\omega \ge \max\{\partial p, \partial q\}$, and the equality sign holds if and only if ω is the exact order. We denote the set of rational functions $r \in \mathcal{R}_\omega$ of type (m, n) by $\mathcal{R}_{m,n}$, and the set of functions $r \in \overline{\mathcal{R}}_\omega$ of type (m, n) by $\overline{\mathcal{R}}_{m,n}$, i.e., $\overline{\mathcal{R}}_{m,n} = \mathcal{R}_{m,n} \cup \{\infty\}$. Clearly, $r \in \overline{\mathcal{R}}_{m,n}$ if and only if $1/r \in \overline{\mathcal{R}}_{n,m}$. The *defect* δ of r in $\overline{R}_{m,n}$ is now defined by $\delta := \min\{m - \partial p, n - \partial q\}$ (again, with p, q relatively prime).

The number of free parameters in $\overline{\mathcal{R}}_{m,n}$ is $m+n+1$, so we can expect that, in general, as many interpolation conditions can be satisfied. As in Section 2, let $Z' := \{z'_1, \ldots, z'_J\} \subset \mathbf{C}$ be a set of distinct interpolation points, and let $K := \{\kappa_1, \ldots, \kappa_J\}$ be the associated set of multiplicities, $\kappa_j \ge 1$ $(\forall j)$, such that

$$\sum_{j=1}^{J} \kappa_j = m + n + 1. \tag{20}$$

We may now assume that the data are given in the form of a quotient f_{m+n}/g_{m+n} of two relatively prime polynomials of degree at most $m+n$, which can but need not be required to satisfy (2a). The function t is again defined by (4).

Given m, n, Z', K, and $h_{m+n} = f_{m+n}/g_{m+n}$, the (*true*) *rational interpolation problem* becomes now to find $r \in \overline{\mathcal{R}}_{m,n}$ such that

(21a)
$$\frac{f_{m+n}(z)}{g_{m+n}(z)} - r(z) = O(t(z)) \quad \text{as } z \to z' \in Z', \text{ if } g_{m+n}(z') \ne 0,$$

(21b)
$$\frac{g_{m+n}(z)}{f_{m+n}(z)} - \frac{1}{r(z)} = O(t(z)) \quad \text{as } z \to z' \in Z', \text{ if } f_{m+n}(z') \ne 0.$$

The corresponding *linearized rational interpolation problem* is: Given m, n, Z', K, f_{m+n}, and g_{m+n} as above, find a pair of polynomials $(p, q) \in \mathcal{P}_m \times \mathcal{P}_n$ such that

$$(f_{m+n}q - g_{m+n}p)(z) = O(t(z)) \quad \text{as } z \to z' \in Z', \tag{22}$$

and then let $r := p/q$.

Again, if $r = p/q$ is a solution of (21), then (p, q) is also a solution of (22). We call r then a *true* (m, n)-*interpolant*. Any nontrivial solution pair $(p, q) \in \mathcal{P}_m \times \mathcal{P}_n$ of (22) will be called an (m, n)-*Newton-Padé form* (NPF) and the resulting rational function $r = p/q$ the (m, n)-*Newton-Padé approximant* (NPA). We anticipate the well-known fact that the NPA exists and is unique.

The adaptation of our general results to this new situation is based on

Lemma 7. *Given m, n, Z', K, f_{m+n}, and g_{m+n}, let*

$$\omega := \max\{m, n\}, \quad \kappa := |m - n| \tag{23}$$

(so that $2\omega - \kappa = m + n$) and

$$\begin{aligned} &\phi(z) := z^{\kappa-1}, && \psi(z) :\equiv 0, && \text{if } m > n, \\ &\phi(z) :\equiv 0, && \psi(z) :\equiv 0, && \text{if } m = n, \\ &\phi(z) :\equiv 0, && \psi(z) := z^{\kappa-1}, && \text{if } m < n, \end{aligned} \tag{24}$$

and define f and g by (3). Then (p, q) is an MPF of order ω of f/g if and only if (p, q) is an (m, n)-NPF of f_{m+n}/g_{m+n}.

Proof. If $m = n$, the two problems are clearly equivalent since the second line of (7) is always satisfied. Next, assume $m > n$. Then $\partial f = 2\omega$, $\partial g \le 2\omega - \kappa$, and, therefore, the second line of (7) implies that $\partial q \le \omega - \kappa \le n$. Hence, (p, q) satisfying (7) must lie in $\mathcal{P}_m \times \mathcal{P}_n$. On the other hand, if $(p, q) \in \mathcal{P}_m \times \mathcal{P}_n \subset \mathcal{P}_\omega \times \mathcal{P}_\omega$, then the second line of (7) is satisfied. The first line of (7) is equivalent with (22) since f and f_{m+n}, as well as g and g_{m+n}, have, at points z'_j, the same values and the same first $\kappa_j - 1$ derivatives. The case $m < n$ is analogous. □

The above lemma and Theorem 1 imply in particular that, for any pair (m, n) of nonnegative integers and for any given $m + n + 1$ interpolation data, an (m, n)-NPA always exists, is unique, and is equal to the MPA of the associated interpolation problem in $\overline{\mathbf{C}}$. However, we must emphasize that, even if an NPA is a true (m, n)-interpolant, it need not be a true interpolant of the corresponding problem (6) in $\overline{\mathbf{C}}$. In fact, $r \in \overline{\mathcal{R}}_{m,n}$ does not imply that the second line of (6a) or (6b) holds, since r need not be of exact type (m, n). One must also note that the defect of r in $\overline{\mathcal{R}}_{m,n}$ is, in general, different from the defect in $\overline{\mathcal{R}}_{\omega}$.

It would now be an easy matter to formulate Theorems 2 and 3 and Corollaries 4–6 for the (m, n)-NPA problem, see [**3**], where the block structure of the *Newton-Padé table* is discussed on the basis of these results. In Corollary 6 the second Möbius transform t_2, of course, has to be chosen so that $\infty \notin t_2^{-1}(Z')$.

7. Remark on Stahl's approach. Stahl [**10**] investigates in his Section 8 the existence and uniqueness of the solution of the linearized rational interpolation problem in the same generality as in our Section 3. However, his notation and approach is quite different. He avoids the distinction between the first line and the second line of (7) by writing

$$h(z)q_\zeta(z) - p_\zeta(z) = \frac{\prod_{j=1}^{J}[H(z, z_j')]^{\kappa_j}}{[H(z, \zeta)]^{2\omega+1}}, \tag{25}$$

where $\zeta \notin Z'$ can be chosen arbitrarily and where

$$H(z, z') := \begin{cases} z - z', & \text{if } |z'| \leq 1, \\ (z - z')/|z'|, & \text{if } 1 < |z'| < \infty, \\ 1, & \text{if } z' = \infty, \end{cases} \tag{26}$$

and

$$p_\zeta(z) := p\left(\frac{1}{z - \zeta}\right), \quad q_\zeta(z) := q\left(\frac{1}{z - \zeta}\right). \tag{27}$$

Actually, the right-hand side of (25) could be replaced by $t(z)/(z - \zeta)^{2\omega+1}$, but Stahl's choice has the advantage of being continuous in each z_j' even for $z_j' \to \infty$. Stahl allows that the target degrees m and n

for p and q are different, but, of course, this is equivalent to demanding that $r(z) = p_\zeta(z)/q_\zeta(z)$ has at ζ a zero of order $n - m$ if $n > m$, and a pole of order $m - n$ if $m > n$. Although Stahl does not count ζ as one of his interpolation points, his MPA then depends on ζ unless $m = n$. Moreover, if ζ is allowed to coincide with one of the interpolation points, the MPA may be nonunique.

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