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# THE ONE-QUARTER CLASS OF ORTHOGONAL POLYNOMIALS

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ABSTRACT. An important class of orthogonal polynomials consists of those which satisfy a three-term recurrence relation with unbounded coefficients which have certain convergence properties. This paper reviews some of the known spectral properties of this class of polynomials. In order to put the discussion in the proper perspective, it includes an expository survey of the current knowledge of spectral properties of orthogonal polynomials in general as predictable on the basis of the behavior of the coefficients in the three-term recurrence relation.

**1. Introduction.** An important class of orthogonal polynomials consists of those whose three-term recurrence relation,

$$\begin{aligned} P_n(x) &= (x-c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \\ P_{-1}(x) = 0, \quad P_0(x) &= 1, \quad c_n \text{ real}, \ \lambda_n > 0, \end{aligned} \tag{1.1}$$

have coefficients which satisfy the conditions

$$\lim_{n\to\infty} c_n = \infty, \quad \lim_{n\to\infty} \frac{\lambda_{n+1}}{c_n c_{n+1}} = \frac{1}{4} \tag{1.2}$$

Here we have assumed without loss of generality that our polynomials are monic. The classical prototype of this class is, of course, the sequence of Laguerre polynomials. There are a large number of natural questions concerning the spectral properties of the orthogonal polynomials of this class. For, under the hypotheses (1.2), it is possible for the zeros of the corresponding orthogonal polynomials to (i) form a dense subset of the interval $(0,\infty)$, (Laguerre polynomials, Wilson's continuous dual Hahn polynomials), (ii) have a derived set that forms a sequence converging to $\infty$ (certain Al-Salam and Carlitz polynomials, Askey-Ismail polynomials), (iii) spread from $-\infty$ to $+\infty$. The existence

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of orthogonal polynomials with the last property is easy to establish, but no explicit examples of such seem to be known (which should pose an interesting challenge to the special function experts).

In this article, we plan to discuss this class of OP which, lacking a better name, we call the "one-quarter class" of orthogonal polynomials. In particular, we will point out criteria for each of the three cases mentioned above to occur, for various other spectral properties to be possessed and also give criteria for determinacy or indeterminacy of the associated moment problems. In order to put this discussion into the proper perspective, we will include a survey of the relation of the behavior of the coefficients in (1.1) to the spectral properties of the orthogonal polynomials in the general case.

**2. Preliminaries and notation.** We will be concerned with real, monic orthogonal polynomial sequences (OPS). That is, we consider sequences $\{P_n(x)\}$ such that (a) $P_n(x)$ is a monic polynomial of degree $n$ and (b) there exists a distribution function $\psi$ such that

$$\int_{-\infty}^{\infty} P_m(x)P_n(x)\,d\psi(x) = K_n\delta_{mn}, \quad K_n > 0. \tag{2.1}$$

Here, by *distribution function* we will mean a bounded, nondecreasing function whose moments

$$\mu_n = \int_{-\infty}^{\infty} t^n\,d\psi(t), \quad n = 0, 1, 2, \dots, \tag{2.2}$$

are all finite and whose spectrum (= support of $d\psi$)

$$S(\psi) = \{t : \psi(t+\varepsilon) - \psi(t-\varepsilon) > 0 \text{ for all } \varepsilon > 0\} \tag{2.3}$$

is an infinite set.

It is a classical result that every (monic) OPS satisfies a three term recurrence relation of the form (1.1). Conversely, by the so-called "theorem of Favard," any polynomial sequence that satisfies a recurrence of the form (1.1) is an OPS (e.g., see [**14**]). This result is actually contained in the work of Stieltjes [**42**] and of Hamburger [**29**] with continued fractions and the moment problems that bear their names. As such, it has been at least implicitly long well known to mathematicians who work with continued fractions. However, the latter were only rarely interested in orthogonal polynomials per se and, although the result was rediscovered or rederived in other contexts, it was the announcement of the result by Favard [**26**] that called the fact to the attention of those interested in orthogonal polynomials. This eventually led to the current activity involving investigations into discovering properties of orthogonal polynomials hidden in the coefficients of (1.1). In some cases, this led investigators to consider continued fraction theory, thereby bringing us full circle. Thus, there is some justification in continuing to associate this result with Favard's name. In any case, to do so should cause no greater harm than is done by references to Schwarz's inequality, Taylor's series and other such technically misnamed objects.

A further classical consequence of orthogonality is the fact that $P_n(x)$ has only real, simple zeros $x_{ni} : x_{n1} < x_{n2} < \cdots < x_{nn}$. We then also have the familiar separation theorem for the zeros of two consecutive OPs:

$$x_{n+1,i} < x_{ni} < x_{n+1,i+1}, \quad i = 1, 2, \ldots, n.$$

From this follows the existence of the limits (in the extended real number system)

$$\begin{gathered} \xi_i = \lim_{n\to\infty} x_{ni}, \quad \eta_j = \lim_{n\to\infty} x_{n,n-j+1}, \\ -\infty \le \xi_1 \le \xi_2 \le \cdots \le \cdots \le \eta_2 \le \eta_1 \le \infty. \end{gathered} \tag{2.2}$$

$[\xi_1, \eta_1]$ is what Shohat called the "true" interval of orthogonality. It is the smallest closed interval that contains all of the zeros of the orthogonal polynomials, and there always exists a solution of the moment problem (2.2) whose spectrum is contained in $[\xi_1, \eta_1]$; it is therefore also referred to as the "spectral interval." Finally, we note the limits

$$\sigma = \lim_{i\to\infty} \xi_i, \quad \tau = \lim_{j\to\infty} \eta_j. \tag{2.3}$$

When the Hamburger moment problem given by (2.2) is determined, $\sigma$ and $\tau$ are the smallest and largest limit points of the spectrum. When the moment problem is indeterminate, either $\sigma = \tau = \infty$ (or, essentially equivalently, $\sigma = \tau = -\infty$) or $\sigma = -\tau = -\infty$. In the former case, there is a unique extremal solution of the moment problem whose spectrum consists of the distinct points $\xi_i$ $(\eta_j)$ and thus is a denumerable set whose only limit point is $\infty$ $(-\infty)$ [**11**]. In the latter case, rather little can be said other than that the zeros spread from $-\infty$ to $+\infty$.

## 3. The spectral interval is bounded.

We first review results known when the interval $[\xi_1, \eta_1]$ is bounded. In this case, the corresponding Hamburger moment problem is determined so the distribution function $\psi$ is unique up to an obvious equivalence class. The following is essentially contained in Stieltjes' work [**42**].

THEOREM 3.1. *$[\xi_1, \eta_1]$ is bounded if and only if both coefficient sequences $\{c_n\}$ and $\{\lambda_n\}$ are bounded.*

The first spectral theorem of the type we are most concerned with is also due to Stieltjes [**42**]. We denote by $E'$ the derived set of $E$.

THEOREM 3.2. *Let $c_n = 0$ for all $n$, and let $\lambda_n \to 0$ (as $n \to \infty$). Then $S(\psi)' = \{0\}$.*

Thus, Stieltjes gives conditions for the spectrum to be a symmetric, denumerable set with 0 as its only accumulation point. Stieltjes proved his result using convergence properties of the corresponding $S$-fraction. Using the theory of completely continuous operators applied to the appropriate J-matrix, Krein in 1938 proved a remarkable generalization of Stieltjes' theorem.

THEOREM 3.3. *Let $E$ be any finite set of real numbers. Krein states necessary and sufficient conditions on $\{c_n\}$ and $\{\lambda_n\}$ so that $S(\psi)' = E$.*

The conditions given by Krein would require the introduction of a little too much notation to be reasonably summarized in this survey, so we refer to [**1**] (see also [**14**]) for specifics in the general case. In the special case, $E = \{a\}$, Krein's conditions become the natural generalization of Stieltjes' result:

$$\lim_{n\to\infty} c_n = a, \quad \lim_{n\to\infty} \lambda_n = 0,$$

while, for $E = \{a, b\}$, his conditions are

$$\lim_{n\to\infty} [\lambda_n + \lambda_{n+1} + (c_n - a)(c_{n+1} - b)] = 0,$$
$$\lim_{n\to\infty} \lambda_{n+1}(c_n + c_{n+1} - a - b) = 0,$$
$$\lim_{n\to\infty} \lambda_n \lambda_{n+1} = 0.$$

Somewhat more recently, Maki [**32**] proved the following result. Let

$$\mathcal{L} = \{t : t \text{ is a subsequential limit point of } \{c_n\}\}. \tag{3.1}$$

THEOREM 3.4. *Let* $\lim_{n\to\infty} \lambda_n = 0$. *Then* $\mathcal{L} \subseteq S(\psi)$.

Maki also used operator methods to prove his result which, it should be pointed out, does not require that $\{c_n\}$ be bounded. He also noted that, by choosing the $c_n$ to be, for example, an enumeration of the rationals, one would have an example whose spectrum is the real line. Maki also conjectured that, still with the hypothesis $\lim_{n\to\infty} \lambda_n = 0$, a point is a limit point of $\{c_n\}$ if and only if it is an *accumulation* point of the spectrum. I was able to verify Maki's conjecture [**12**]:

THEOREM 3.5. *Let* $\lim_{n\to\infty} \lambda_n = 0$. *Then* $\mathcal{L} = S(\psi)'$.

At the opposite extreme from the preceding results, there is the following theorem due to Blumenthal [**7**].

THEOREM 3.6. *Let*

$$\lim_{n\to\infty} c_n = c, \quad \lim_{n\to\infty} \lambda_n = \lambda \quad (\textit{both finite}). \tag{3.2}$$

*Then*

$$\sigma = c - 2\sqrt{\lambda}, \ \tau = c + 2\sqrt{\lambda},$$

*and the zeros of the* $P_n(x)$ *are dense in the interval* $[\sigma, \tau]$.

The density of the zeros would suggest (but not imply) that the interval $[\sigma, \tau]$ belongs to the spectrum. This is, however, in fact true and follows from an argument due to Nevai (oral communication in 1983) based on another theorem of Nevai himself [ **35**, Theorem 4.2.14]:

THEOREM 3.6. *Under the conditions* (3.2), $[\sigma, \tau] \subseteq S(\psi)$.

The conclusion in Theorem 3.6 implies Blumenthal's conclusion that the zeros are dense in $[\sigma, \tau]$. With additional conditions on the rate of convergence in (3.2), more specific properties of the distribution function have been deduced. For example, Nevai [**35**] has proved

THEOREM 3.8. *Let*

$$\sum(|c_n - c| + |\lambda_n - \lambda|) < \infty.$$

*Then* $\psi$ *is absolutely continuous on* $(\sigma, \tau)$, *and* $\psi'$ *is positive and continuous on* $(\sigma, \tau)$.

For additional results along these lines, see papers by Nevai and his collaborators (e.g., [**25, 28, 33, 35, 36, 37**]).

Finally, we note the following result of Geronimo and Case [**27**] which has significance for applications to scattering theory in physics.

THEOREM 3.9. *Let*

$$\sum n(|c_n - c| + |\lambda_n - \lambda|) < \infty.$$

*Then* $S(\psi)$ *has at most finitely many points on the complement of* $(\sigma, \tau)$, *and* $\psi$ *is continuous at* $\sigma$ *and* $\tau$.

For alternate and more direct proofs of Theorem 3.9, see [**21**].

Relative to the important conclusions in Theorem 3.9, the conditions

$$c_n - c = O(n^{-2}), \quad \lambda_n - \lambda = O(n^{-2})$$

describe a borderline case. For amplification of this last comment and for related results, see [**15**]. For other studies of (1.1) with bounded coefficients directed at determining various other properties of the polynomials (asymptotics, bounds for zeros, etc.), see, for example, [**9, 13, 35, 37, 38, 39, 44, 45**].

**4.** The spectral interval is unbounded. We now take up the case where $(\sigma, \tau)$ is unbounded, which is equivalent to the condition that at least one of the sequences $\{c_n\}$ and $\{\lambda_n\}$ is unbounded.

*Case* A. $\sigma = \infty$. The important special case where $\sigma = \infty$ is represented by many specific examples in the literature and include the classic examples of the polynomials associated with the names of Charlier, Stieltjes-Wigert, Meixner, Hahn, as well as more recent examples discovered by Al-Salam and Carlitz [**3**], Al-Salam and Chihara [**4**] (see also [**5**]), and Askey and Ismail (see below). This case is equivalent to the condition that

$$\xi_1 < \xi_2 < \cdots < \xi_n, \quad \xi_n \to \infty.$$

When the Hamburger moment problem is determined, $S(\psi) = \{\xi_i : i \geq 1\}$ so that $S(\psi)' = \{\infty\}$, while if the moment problem is indeterminate, there is always a unique (extremal) solution of the moment problem with the above spectrum (see [**11**]).

The following was initially proved under the assumption that the Hamburger moment problem is determined [**9**] and was then rediscovered as an equivalent statement about the convergence of continued fractions without assumptions about the moment problem [**31**] (see also [**14**]).

THEOREM 4.1. *Let*

$$\lim_{n\to\infty} c_n = \infty, \quad \limsup_{n\to\infty} \frac{\lambda_{n+1}}{c_n c_{n+1}} < \frac{1}{4}.$$

*Then* $\sigma = \infty$.

This result is sufficiently general to cover the majority of specific examples found in the literature, but certain special cases of the Al-Salam and Carlitz polynomials have the property that the ratio $\lambda_{n+1}/(c_n c_n)$ converges to $1/4$ (that is, we have polynomials of the one-quarter class). The following result [**17**] covers the latter.

THEOREM 4.2. *Let*

$$\sum \frac{n}{c_n} < \infty, \quad \sum n \left| \frac{\lambda_{n+1}}{c_n c_{n+1}} - \frac{1}{4} \right| < \infty.$$

*Then* $\sigma = \infty$.

A very important class of orthogonal polynomials has been discovered recently by Askey and Ismail [**5**]. These polynomials are denoted by $v_n(x;q;a,b,c)$ $(|q| < 1)$. If we consider the equivalent monic form

$$P_n(x) = (-1)^n q^{-n(n-1)/2} (q;q)_n v_n(x),$$

then, in the corresponding recurrence relation (1.1), we have

$$c_{n+1} = aq^{-n}, \quad \lambda_{n+1} = q^{-2n+1}(1-q^n)(b - cq^{n-1}),$$

with $a$ real and $b$ and $c$ restricted so that $\lambda_{n+1} > 0$. Askey and Ismail prove that the corresponding Hamburger moment problem is determined if and only if

$$a^2 > 4b \text{ and } |q| \geq \frac{|a| - \sqrt{a^2 - 4b}}{|a| + \sqrt{a^2 - 4b}}.$$

Thus, in particular, the moment problem is indeterminate for the polynomials of the one-quarter class. When the moment problem *is* determined, they obtain the distribution function $\psi$ explicitly. In particular, they show that the spectrum consists of the points $x_n$ where

$$x_n = A(Bq^n) + (Bq^n)^{-1}, \quad n = 0, 1, \dots .$$

Here $A$ and $B$ are certain, explicitly given, positive constants. In particular, this means that, for $q > 0$ (and $a > 0$), $c_n \to \infty$ and $\lambda_{n+1}/(c_n c_{n+1}) \to b/a^2 < 1/4$ (which is consistent with Theorem 4.1). However, if $q < 0$, then the specific interval is $(-\infty, \infty)$ (as is also implied by the fact that $\inf_n c_n = -\sup_n c_n = -\infty$). We therefore will next look briefly at this case. However, before leaving the present case, we mention that criteria that can predict determinacy or indeterminacy of the moment problems in many cases can be given in terms of the coefficients in (1.1) [**20**]. For example, if the conditions of

Theorem 4.1 hold, the Hamburger moment problem will be determined if $c_n = O(n^p)$. On the other hand, if $c_n = f_n q^{-n}$, where $0 < q < 1$ and the $f_n$ are bounded and bounded away from 0, let $L$ denote the limit superior that appears in Theorem 4.1. Then the moment problem is determined if $L < q(q+1)^{-2}$, while it is indeterminate if the opposite (strict) inequality holds.

*Case* B. The spectral interval is $(-\infty, \infty)$. When $(\xi_1, \eta_1) = (-\infty, \infty)$, we also have $(\sigma, \tau) = (-\infty, \infty)$ (see Sherman [**38**]). Sufficient conditions for this case are, for example (see [**9, 17**]),

$$\begin{aligned} &\text{(i)} \quad \inf_n c_n = -\infty, \quad \sup c_n = \infty; \\ &\text{(ii)} \quad \lim_{n\to\infty} c_n = \infty, \quad \liminf_{n\to\infty} \frac{\lambda_{n+1}}{c_n c_{n+1}} > \frac{1}{4}; \\ &\text{(iii)} \quad \lim_{n\to\infty} \frac{c_n}{n^2} = \infty, \quad \frac{\lambda_{n+1}}{c_n c_{n+1}} > \frac{1}{4} + \frac{t}{16n^2}, \ t > 1, \end{aligned} \tag{4.1}$$

The Askey-Ismail polynomials with $q < 0$ are examples with a discrete spectrum satisfying (i). The Meixner polynomials of the second kind (Meixner-Pollaczek polynomials) (see [**14**]) provide an example whose spectrum is the entire real line satisfying (ii). Note that, for the latter polynomials, the Hamburger moment problem is determined.

General spectral theorems dealing with case B are rather rare. In the symmetric case ($c_n = 0$), one can consider the related polynomials on $[0, \infty)$ (see (5.3)) and translate results from theorems dealing with the case $(\xi_1, \eta_1) \subseteq (0, \infty)$ to obtain conclusions for the symmetric case. For the nonsymmetric case, one theorem which predicts a discrete spectrum is ([**18**])

THEOREM 4.3. *Let the Hamburger moment problem be determined and suppose*

(i) $\lim_{n\to\infty} |c_n| = \infty$;

(ii) $\inf_n c_n = -\infty$, $\sup_n c_n = \infty$;

(iii) $\limsup_{n\to\infty} |\lambda_{n+1}/(c_n c_{n+1})| < \frac{1}{4}$.

*Then* $\sigma = -\infty$, $\tau = \infty$ *and* $S(\psi)$ *has no finite points of accumulation.*

A second general result [**18**] should be compared with Theorem 3.5. Recall (3.1) that $\mathcal{L}$ denotes the set of subsequential limit points of $\{c_n\}$.

THEOREM 4.4. *Let the Hamburger moment problem be determined, and let*

$$\lim_{n\to\infty} \frac{\lambda_{n+1}}{c_n c_{n+1}} = 0.$$

*Then* $\sigma = \inf \mathcal{L}$, $\tau = \sup \mathcal{L}$ *and* $S(\psi)' \subseteq \mathcal{L}$.

Note that this theorem applies more generally than just to cases where $(\sigma, \tau) = (-\infty, \infty)$. Comparing this result with Theorem 3.5, we see that there are some natural questions raised by this theorem. In particular, under what additional conditions, *if any*, will it be true that $S(\psi)' = \mathcal{L}$?

Criteria for deciding determinacy of the Hamburger moment problem in this case are rare. The simplest is Carleman's criterion (see [**41**]) which says that the moment problem is determined if $\sum \lambda_n^{-1/2} = \infty$. However, this is rarely applicable when $\sigma = -\infty$. For another which sometimes applies, see Dennis and Wall [ **22**].

**5.** $\sigma$ **is finite.** Before listing some specific results involving the conclusion that $|\sigma| < \infty$, we review the concept of "chain sequences" which is involved in the derivation of many of the previous results as well as those to come. In particular, this concept will help explain why the number 1/4 seems to be so central to the determination of $\sigma$.

DEFINITION. $\{a_n\}_{n=1}^{\infty}$ is a *chain sequence* if there exists a sequence $\{g_n\}_{n=0}^{\infty}$ such that

(i) $0 \le g_0 < 1$, $0 < g_n < 1$ for $n \ge 1$;

(ii) $a_n = (1 - g_{n-1})g_n$, $n \ge 1$.

The spectral interval can now be related to the coefficients in (1.1) via this concept of chain sequences (see [**9, 14**]). In order to relate the coefficients in (1.1) with $\sigma$ (and $\tau$), we need a modification of this concept.

DEFINITION. $\{a_n\}_{n=1}^{\infty}$ is an *eventual chain sequence* if there exists an index $N$ such that $\{a_{N+n}\}_{n=1}^{\infty}$ is a chain sequence.

NOTATION. Let $\mathcal{E}$ denote the class of all eventual chain sequences, and let

$$\alpha_n(x) = \frac{\lambda_{n+1}}{(c_n - x)(c_{n+1} - x)}. \tag{5.1}$$

We can then relate $\sigma$ to the coefficients in (1.1) by ([**17**])

THEOREM 5.1. *Let* $c_n \to \infty$ $(n \to \infty)$.

(i) *If* $\{\alpha_n(x)\} \in \mathcal{E}$*, then* $\sigma \geq x$.

(ii) *If* $\{\alpha_n(x)\} \notin \mathcal{E}$*, then* $\sigma \leq x$.

Theorem 5.1 can be used to arrive easily at, for example, Theorem 4.1. For the constant sequence, $\{1/4\}$ is a chain sequence and, by Wall's comparison theorem, any sequence (weakly) dominated by a chain sequence is itself a chain sequence [**14**]. Under the hypotheses (i), for every real $x$, $\alpha_n(x) < 1/4$ for all sufficiently large $n$; this says that $\{a_n(x)\} \in \mathcal{E}$ for every $x$, hence $\sigma = \infty$. Similarly, the criterion (4.1 ii) follows since the inequality given there implies that $\{a_n(x)\} \notin \mathcal{E}$ for any real $x$. The borderline case thus occurs when the limit is exactly 1/4. For example, let

$$a_n = \frac{1}{4} + \frac{1 + e_n}{16n(n+1)}. \tag{5.2}$$

Then it can be shown that [**17**]

(i) If $e_n = O(1/n)$ or if $\sum e_n$ converges, then $\{a_n\} \in \mathcal{E}$.

(ii) If $e_n \geq 0$ and $\sum e_n/n = \infty$, then $\{a_n\} \notin \mathcal{E}$.

Recently, Jacobson and Masson [**30**] have closed the gap between (i) and (ii) above, so this will permit more precise determination of the nature of $S(\psi)$ or calculation of $\sigma$ in many situations. (See, for example, [**18**, p. 668] for an example where this could be useful.)

Thus, with Theorem 5.1 available to decide whether or not $\sigma$ is finite, we then note the following analog [**10**] of Blumenthal's theorem (Theorem 3.6)).

THEOREM 5.2. *Let $c_n \to \infty$ (as $n \to \infty$), and let*

$$\lim_{n\to\infty} \frac{\lambda_{n+1}}{c_n c_{n+1}} = \frac{1}{4}.$$

*If, in addition, $\sigma$ is finite, then the zeros of the $P_n(x)$ are dense in $(\sigma, \infty)$.*

The additional assumption that $\sigma$ is finite is essential. The Al-Salam and Carlitz polynomials (with $a = 1$) were already mentioned in Section 4 as examples satisfying the conditions of Theorem 5.2 but having $\sigma = \infty$ (and Theorem 4.2 provides an extensive subclass of the one-quarter class having $\sigma = \infty$). The Hamburger moment problem is indeterminate for this case of the Al-Salam and Carlitz polynomials (in fact, the distribution function given by Al-Salam and Carlitz is an extremal solution (see [**16**]) which is, in itself, an interesting fact). However, the example

$$c_n = n^2 f_n, \quad \lambda_{n+1} = \frac{1}{4} n^2 (n+1)^2 f_n f_{n+1},$$

where $f_n \to \infty$, $\sum f_n^{-1} = \infty$, would provide an example with $\sigma = \infty$, and the moment problem would be determined [**20**, Theorem 4]. One can also construct examples with $\sigma = -\infty$. For example, if

$$c_n = 2n, \quad \lambda_{n+1} = n^2 + n^\gamma, \ 1 < \gamma < 2,$$

then $\sigma = -\infty$ [**14**, p. 125] and the corresponding moment problem is determined (by Carleman's criterion).

Some examples of systems in the one-quarter class with $|\sigma| < \infty$ are the classical Laguerre polynomials, two classes of orthogonal polynomials related to Meixner polynomials of the second kind studied by Al-Salam [**2**] (see also [**14**, p. 180]), and various hypergeometric orthogonal polynomials (such as the continuous dual Hahn polynomials) studied by Askey and Wilson [**6, 7, 48**].

Carleman's criterion for determinacy of the Hamburger moment problem is frequently applicable for the one-quarter class when $|\sigma| < \infty$, as is the following criterion for indeterminacy [**20**]: the moment problem is indeterminate if

$$\liminf_{n\to\infty} c_n^{1/n} > 1.$$

The obvious question that arises upon seeing Theorem 5.2 is, "where (what) is the analog of Nevai's 'half' of the Blumenthal-Nevai theorem (Theorem 3.6)?" An analogue of Theorem 3.7 would also be highly desirable. A step in this direction has been taken by Dombrowski [**23**] who has considered the symmetric case. Let us write the recurrence relation for a symmetric OPS as

$$S_n(x) = xS_{n-1}(x) - \gamma_n S_{n-2}(x), \tag{5.3}$$

and denote the symmetric distribution with respect to which the $S_n(x)$ are orthogonal as $d\varphi(x)$. Dombrowski assumes that the coefficients in (5.1) satisfy:

(i) $\{\gamma_n\}$ increases monotonically to $\infty$;

(ii) $\sum \gamma_n^{-1/2} = \infty$.

Condition (ii) ensures that the Hamburger moment problem is determined (again by Carleman's criterion). Dombrowski then uses operator theory to prove

THEOREM 5.3. *Let* $D_n = \sqrt{\gamma_{n+1}} - \sqrt{\gamma_n}$. *If*

(iii) $\{D_n\}$ *is bounded*

*and*

(iv) *either*

(a) $D_{n+1} - D_n \le D_n - D_{n-1}$

*or*

(b) $D_{n+1}^2 \le D_n D_{n+2}$

*then* $\varphi$ *is absolutely continuous on* $(-\infty, \infty)$.

In order to relate Dombrowski's theorem to the one-quarter class, we set

$$P_n(x) = S_{2n}(\sqrt{x}).$$

Then [**14**] $\{P_n(x)\}$ is orthogonal over $[0, \infty)$ with respect to $d\psi(x) = d\varphi(\sqrt{x})$ and satisfies (1.1) with

$$c_1 = \gamma_2, \quad c_{n+1} = \gamma_{2n+1} + \gamma_{2n+2}, \quad \lambda_{n+1} = \gamma_{2n}\gamma_{2n+1}. \tag{5.4}$$

The conditions (i) and (iii) above imply that $\{P_n(x)\}$ belongs to the one-quarter class of OP. However, it is not immediately clear how to state conditions in terms of $\{c_n\}$ and $\{\lambda_{n+1}\}$ which will be sufficient to guarantee that the conditions of Dombrowski's theorem are satisfied.

For a related result dealing with (5.2) under conditions (i), (ii), see Dombrowski [**25**].

We also mention here the following analogue of the Geronimo and Case result, Theorem 3.6 (see [**17**]):

THEOREM 5.3. *Let*

$$\sum \frac{1}{c_n} = \infty \text{ and } \sum n \left| \frac{\lambda_{n+1}}{c_n c_{n+1}} - \frac{1}{4} \right| < \infty.$$

*Then* $\sigma = 0$, $S(\psi) \cap (-\infty, 0)$ *is a finite set, and* $\psi$ *is continuous at 0.*

Other conditions that lead to the type of conclusions given in Theorem 5.3 can also be found in [**17**].

To emphasize the point that the additional hypothesis that $\sigma$ is finite must be made in Theorem 5.2, we mention ([**17**])

THEOREM 5.4. *Given any sequence* $\{c_n\}$ *with* $c_n \to \infty$, *there exists a sequence* $\{\lambda_{n+1}\}$ *such that* $\lambda_{n+1}/(c_n c_{n+1}) \to 1/4$ *and* $\sigma = \infty$ *and there exists* $\{\lambda_{n+1}\}$ *such that the above limit holds and* $\sigma = -\infty$.

In this connection, it may also be informative to consider the important special cases where $c_n$ and $\lambda_n$ are polynomials in $n$ (see [**17**]). In this case, $\lim_{n\to\infty} c_n = \infty$ and $\lim_{n\to\infty} \lambda_{n+1}/(c_n c_{n+1}) = L$ always exist (and, of course, $|\sigma| = \infty$ if $L \neq 1/4$). In the critical case $L = 1/4$, one can give conditions in terms of the coefficients $c_n$ and $\lambda_{n+1}$ for $|\sigma| = \infty$ or for $\sigma$ to be finite. In the latter case, one can determine $\sigma$ precisely in terms of the coefficients in $c_n$ and $\lambda_{n+1}$. In particular, it can be shown that when the degree of $c_n$ is at least three, $\sigma$ is always infinite.

We conclude by mentioning a few additional studies of (1.1) applicable when the coefficient sequences are unbounded which deal with other properties of the orthogonal polynomials: [**13, 34, 39, 43, 46, 47**].

## REFERENCES

**1.** N.I. Ahiezer and M.G. Krein, *Some Questions in the Theory of Moments*, Transl. Math. Monographs, vol. 2, Amer. Math. Soc., 1962.

**2.** W.A. Al-Salam, *Characterization of a certain class of orthogonal polynomials related to elliptic functions*, Ann. Mat. Pura Appl. **67** (1965), 75–94.

**3.** ——— and L. Carlitz, *Some orthogonal q-polynomials*, Math. Nachr. **30** (1965), 47–61.

**4.** ——— and T.S. Chihara, *Convolutions of orthogonal polynomials*, SIAM J. Math. Anal. **7** (1976), 16 –28.

**5.** R.A. Askey and M.E.H. Ismail, *Recurrence relations, continued fractions and orthogonal polynomials*, Mem. Amer. Math. Soc., No. 300, 1984.

**6.** ——— and J.A. Wilson, *A set of hypergeometric polynomials*, SIAM J. Math. Anal. **13** (1982), 651–655.

**7.** ——— and ———, *some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc., No. 319, 1985.

**8.** O. Blumenthal, *Über die Entwicklung einer willkürlichen Funktion nach den Nennern des Kettenbruches fűr* $\int_{-\infty}^{0}[\phi(\xi)/(z-\xi)]d\xi$, Dissertation, Göttingen, 1898.

**9.** T.S. Chihara, *Chain sequences and orthogonal polynomials*, Trans. Amer. Math. Soc. **104** (1962), 1–16.

**10.** ———, *Orthogonal polynomials whose zeros are dense in intervals*, J. Math. Anal. Appl. **24** (1968), 362–371.

**11.** ———, *On indeterminate Hamburger moment problems*, Pacific J. Math. **27** (1968), 475–484.

**12.** ———, *The derived set of the spectrum of a distribution function*, Pacific J. Math. **35** (1970), 571–574.

**13.** ———, *Convergent sequences of orthogonal polynomials*, J. Math. Anal. Appl. **38** (1972), 335–347.

**14.** ———, *An Introduction to Orthogonal Polynomials*, Gordon and Breach Science Publishers, New York, 1978.

**15.** ———, *Orthogonal polynomials whose distribution functions have finite point spectra*, SIAM J. Math. Anal. **11** (1980), 358–364.

**16.** ———, *Indeterminate symmetric moment problems*, J. Math. Anal. Appl. **85** (1982), 331–346.

**17.** ———, *Spectral properties of orthogonal polynomials on unbounded sets*, Trans. Amer. Math. Soc. **270** (1982), 623–639.

**18.** ———, *Orthogonal polynomials with discrete spectra on the real line*, J. Approx. Theory **42** (1984), 97–105.

**19.** ———, *On the spectra of certain birth and death processes*, SIAM J. Appl. Math. **47** (1987), 662–669.

**20.** ———, *Hamburger moment problems and orthogonal polynomials*, **315** (1989), 189–203.

**21.** ——— and P. Nevai, *Orthogonal polynomials and measures with finitely many point masses*, J. Approx. Theory **35** (1982), 370–380.

**22.** J.J. Dennis and H.W. Wall, *The limit circle case for a positive-definite J-fraction*, Duke Math. J. **12** (1945), 255–273.

**23.** J. Dombrowski, *Cyclic operators, commutators, and absolutely continuous measures*, Proc. Amer. Math. Soc., **100** (1987), 457–463.

**24.** ———, *Spectral measures, orthogonal polynomials and absolute continuity*, SIAM J. Math. Anal. **19** (1988), 939–943.

**25.** ——— and P. Nevai, *Orthogonal polynomials, measures and recurrence relations*, SIAM J. Math. Anal. **17** (1986), 752–759.

**26.** J. Favard, *Sur les polynomes de Tchebicheff*, C. R. Acad. Sci. Paris **200** (1935), 2052–2053.

**27.** J.S. Geronimo and K.M. Case, *Scattering theory and polynomials orthogonal on the real line*, Trans. Amer. Math. Soc. **258** (1980), 467–494.

**28.** ——— and P.G. Nevai, *Necessary and sufficient conditions relating the coefficients in the recurrence formula to the spectral function for orthogonal polynomials*, SIAM J. Math. Anal. **14** (1983), 622–637.

**29.** H. Hamburger, *Über eine Erweiterung des Stieltjesschen Momentenproblems*, Math. Anal. **81** (1920), 253–319; **82** (1921), 120–164; 168–187.

**30.** L. Jacobsen and D.R. Masson, *A sequence of "best" parabola theorems for continued fractions*, Rocky Mtn. J. Math **21** 1991, 377–385.

**31.** D.P. Maki, *On constructing distribution functions: with applications to Lommel polynomials and Bessel functions*, Trans. Amer. Math. Soc. **130** (1968), 281–297.

**32.** ———, *A note on recursively defined orthogonal polynomials*, Pacific J. Math. **28** (1969), 611–613.

**33.** A. Máté and P.G. Nevai, *Orthogonal polynomials and absolutely continuous measures*, in *Approximation Theory*, IV, C.K. Chui et al., eds., Academic Press, New York, 1983, 611–617.

**34.** ———, ———, and V. Totik, *Asymptotics for the zeros of orthogonal polynomials associated with infinite intervals*, J. London Math. Soc. (2) **33** (1986), 303–310.

**35.** P.G. Nevai, *Orthogonal polynomials*, Mem. Amer. Math. Soc., No. 213, 1979.

**36.** ———, *Orthogonal polynomials defined by a recurrence relation*, Trans. Amer. Math. Soc. **250** (1979), 369–384.

**37.** ———, *On orthogonal polynomials*, J. Approx. Theory **25** (1979), 34–37.

**38.** ———, *Two of my favorite ways of obtaining asymptotics for orthogonal polynomials*, Internat. Ser. Numer. Math. **65** (1984), 417–436.

**39.** ——— and J.S. Dehesa, *On asymptotic average properties of zeros of orthogonal polynomials*, SIAM J. Math. Anal. **10** (1979), 1184–1192.

**40.** J. Sherman, *On the numerators of the convergents of the Stieltjes continued fractions*, Trans. Amer. Math. Soc. **35** (1933), 64–87.

**41.** J.A. Shohat and J.D. Tamarkin, *The Problem of Moments*, Math. Surveys, No. 1, Amer. Math. Soc., 1943.

**42.** T.J. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse Math. **8** (1984), J1-122; **9** (1895), A1-47; Oeuvres, Vol. 2, 398–566.

**43.** E.A. van Doorn, *On orthogonal polynomials with positive zeros and the associated kernel polynomials*, J. Math. Anal. Appl. **113** (1986), 441–450.

**44.** ———, *Representations and bounds for zeros of orthogonal polynomials and eigenvalues of sign-symmetric tri-diagonal matrices*, J. Approx. Theory **51** (1987), 254–2266.

**45.** W. van Assche, *Asymptotic properties of orthogonal polynomials from their recurrence formula, I*, J. Approx. Theory **44** (1985), 258–276.

**46.** ———, *Asymptotic properties of orthogonal polynomials from their recurrence formula, II*, J. Approx. Theory **52** (1988), 322–338.

**47.** ———, *The ratio of q-like orthogonal polynomials*, J. Math. Anal. Appl., to appear.

**48.** J.A. Wilson, *Some hypergeometric orthogonal polynomials*, SIAM J. Math. Anal. **11** (1980), 690–701.

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