ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 1, Winter 1991

SUBORDINATION OF POLYNOMIALS

ANTONIO CÓRDOVA AND STEPHAN RUSCHEWEYH

ABSTRACT. We prove a general sufficient condition for polynomials to be subordinate to certain analytic functions. This generalizes and unifies other known conditions.

1. Introduction. Let $\mathcal{A}$ denote the set of analytic functions in the unit disk $\mathbf{D}$ of the complex plane $\mathbf{C}$, and let $\mathcal{P}_n$ be the set of polynomials of degree $\leq n$. If $f, g \in \mathcal{A}$, then f is called *subordinate* to g (denoted by $f \prec g$) if $f = g \circ \omega$, where $\omega \in \mathcal{A}$ satisfies $|\omega(z)| \leq |z|$ in $\mathbf{D}$. The concept of subordination turned out to be very useful for many applications in function theory, and, therefore, it is of interest to obtain simple, sufficient criteria which guarantee subordination. The most elementary one of such criteria works if g happens to be univalent in $\mathbf{D}$. Then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbf{D}) \subset \mathbf{g}(\mathbf{D})$. Less immediate, but nevertheless easy to prove, is the following known special result.

Theorem A. *Let* $P \in \mathcal{P}_n$, $P(0) = 1$ *and* $0 \notin P(\mathbf{D})$. *Then* $P \prec Q_n := (1-z)^n$.

Note that here already the omission of *one single boundary point* of $Q_n(\mathbf{D})$, namely, $Q_n(1) = 0$, guarantees subordination. Theorem A is fairly surprising, and it seems to represent the only known case, so far, of this kind of subordination criterion. In connection with our work [**1, 2**] on the "maximal range" problem for polynomials, we were led to conjecture that the following might be true.

This work has been supported by the Fondo Nacional Científico y Tecnológico (FONDECYT), Grant 249/87, by the Universidad F. Santa María (Grant 881206), and by the German Academic Exchange Service (DAAD).
Received by the editors on August 18, 1988, and in revised form on December 15, 1989.

Theorem 1. *Let $n \geq 2$ and assume that $Q \in \mathcal{P}_n$ has all of its critical points ζ_j in $\overline{\mathbf{D}}$, $j = 1, \dots, n-1$. Let $P \in \mathcal{P}_n$ satisfy $P(0) = Q(0)$ and*

$$Q(\zeta_j) \notin P(\mathbf{D}), \quad j = 1, \dots, n-1. \tag{1.1}$$

Then $P \prec Q$.

Clearly, Theorem A is a very special case of Theorem 1 (except for the case $n = 1$, where it is trivial anyway), which we are going to establish in the present paper. In a somewhat rough interpretation, it says that the range of P can "leave" the range of Q only through the images of the critical points of Q.

One may ask why the statement in Theorem 1 is restricted to polynomials. This, in fact, is not necessary as far as Q is concerned. Important for Q is only a certain restriction of the curvature of $Q(\partial\mathbf{D})$. And this restriction determines the degree of polynomials P for which the conclusion holds. We shall prove the following much more general (but perhaps less transparent) result.

Theorem 2. *Let F be meromorphic in $\mathbf{C}$, $F' \not\equiv$ const., and assume that, for a certain $n \in \mathbf{N}$,*

$$\operatorname{Re}\left(\frac{zF''(z)}{F'(z)} + 1\right) \geq \frac{n+1}{2}, \quad z \in \partial\mathbf{D}\backslash\mathfrak{C},$$

where $\mathfrak{C}$ is the set of zeros and poles of F' in $\mathbf{C}$. Let $\omega \in \mathcal{A}$ be such that

(i) $\omega(0) = 0$,

(ii) *$P := F \circ \omega$ is the restriction to $\mathbf{D}$ of a polynomial in $\mathcal{P}_n$.*

(iii) $\mathfrak{C} \cap \omega(\mathbf{D}) = \varnothing$.

Then $\omega(\mathbf{D}) \subset \mathbf{D}$, and, in particular, $P \prec F$.

One further remark concerning Theorem 1 is in order. Generally, we will not have that

$$Q(\zeta_j) \notin Q(\mathbf{D}), \quad j = 1, \dots, n-1, \tag{1.2}$$

so that, in these cases, Theorem 1 does not give the trivial result $Q \prec Q$. If, however, Q is univalent in $\mathbf{D}$, then $Q' = 0$ only on the boundary, and the univalence gives (1.2). In these cases Theorem 1 provides a necessary and sufficient condition for $P \prec Q$. Note that here the stronger $P(\mathbf{D}) \subset Q(\mathbf{D})$ is replaced by (1.1). The following simple application of Theorem 1 deals with this type of Q.

Corollary. *Let $P \in \mathcal{P}_n$ be such that $P(0) = 0$ and*

$$P(z) \neq \frac{n-1}{n} e^{\frac{2\pi i j}{n-1}}, \quad j = 1, \dots, n-1, \ z \in \mathbf{D}.$$

Then $P \prec z - z^n/n$, and, in particular,

$$|P(z)| \leq \frac{n+1}{n}, \quad z \in \mathbf{D}.$$

An interesting consequence of this Corollary is that if $P \in \mathcal{P}_n$ with $P(0) = 0$ and

$$P(z) \neq \frac{n}{n+1} e^{\frac{2\pi i j}{n}}, \quad j = 1, \dots, n, \ z \in \mathbf{D},$$

then P is the n-th partial sum of the Taylor series expansion about $z = 0$ of an $\omega \in \mathcal{A}$ satisfying $|\omega(z)| \leq 1$ in $\mathbf{D}$.

We note that it is easy to construct univalent polynomials with all zeros of the derivative on $\partial\mathbf{D}$. Following a result of Suffridge [**7**], we have that

$$P(z) = \int_0^z \prod_{j=1}^{n-1} (1 - t e^{i\alpha_j})\, dt$$

is univalent in $\mathbf{D}$ if

$$\frac{2\pi}{n+1} \leq \min\{|\alpha_j - \alpha_k + 2\pi m| : 1 \leq j \leq k \leq n-1, \ m \in \mathbf{Z}\}.$$

Many of these and other univalent polynomials with the given property have been studied in [**1, 2, 6, 7**].

The situation in Theorem A is special in some other sense. Let P, Q_n be as in that theorem. Then one can show that

$$P \in \overline{\text{co}}\{Q_n(xz) : x \in \partial\mathbf{D}\}.$$

Here $\overline{\text{co}}$ stands for the closed convex hull. A corresponding result does not hold, in general, for the polynomials subordinate to some Q in Theorem 1, as can easily be seen from the case discussed in the Corollary above. It is an open problem to determine to which of the cases in Theorem 1 this special property of the polynomials Q_n extends.

Our proof of Theorem 2 makes use of a result which is somewhat reminiscent of Alternant Theorems in real approximation theory. Since this may have other interesting applications as well, we wish to state it here as a theorem.

Theorem 3. *Let $R(z) = \prod_{j=1}^{m}(z - z_j)$, and let $\varnothing \neq \mathfrak{H}$ be a finite subset of $\partial\mathbf{D}\backslash\{1, z_1, \dots, z_m\}$. Assume that, for every $V \in \mathcal{P}_m$ with $V(1) = 0$, we have*

$$0 \in \text{co}\{R(z)\overline{V(z)} : z \in \mathfrak{H}\}. \tag{1.3}$$

Then $z_j \in \partial\mathbf{D}$, $j = 1, \dots, m$.

2. Proof of Theorem 2. This proof has two parts. First, we show, by a fairly straightforward argument, that the theorem holds when $P' \equiv$ const. or P' has all of its zeros on $\partial\mathbf{D}$. In the second, much more involved, part, we reduce the general case to Part 1.

Part 1. We make use of the following Lemma.

Lemma 1. (Jack [**3**], Miller and Mocanu [**4**]) *Let $\omega \in \mathcal{A}$ be nonconstant, $\omega(0) = 0$, and $z_0 \in \mathbf{D}$ such that $|\omega(z_0)| \geq |\omega(z)|$ for all z satisfying $|z| \leq |z_0|$. Then $(z_0\omega'(z_0))/\omega(z_0) \in \mathbf{R}$ and*

$$\text{Re}\left(\frac{z_0\omega''(z_0)}{\omega'(z_0)} + 1\right) \geq \frac{z_0\omega'(z_0)}{\omega(z_0)} \geq 1. \tag{2.1}$$

Furthermore, equality in (2.1) can occur only if $\omega(z) \equiv az$, where a is some constant.

The case of equality is not covered in [**3**] or [**4**], but can easily be deduced from the Julia-Wolff Theorem (see, for instance, Pommerenke [**5**, p. 306]).

Now assume that $P = F\circ\omega$ and, to begin with, that P' is nonconstant and has all its zeros on $\partial\mathbf{D}$, which implies that

$$\operatorname{Re}\left(\frac{zP''(z)}{P'(z)}+1\right) < \frac{n+1}{2}, \quad z \in \mathbf{D}. \tag{2.2}$$

If the Theorem does not hold for P, then there exists $z_0 \in \mathbf{D}$ such that $|\omega(z_0)| = 1$. In fact, we may assume that z_0 is one of the points with smallest modulus which have this property. Then $|\omega(z)| \le |\omega(z_0)| = 1$ holds for $|z| \le |z_0|$. We have

$$P'(z_0) = F'(\omega(z_0))\omega'(z_0), \tag{2.3}$$

and we note that $\omega(z_0) \notin \mathfrak{C}$ by assumption. We take logarithmic derivatives in (2.3), then real parts and use (2.1) to obtain

$$\begin{aligned}\operatorname{Re}\left(\frac{z_0P''(z_0)}{P'(z_0)}+1\right) &= \frac{z_0\omega'(z_0)}{\omega(z_0)}\operatorname{Re}\left(\frac{\omega(z_0)F''(\omega(z_0))}{F'(\omega(z_0))}+1\right)\\ &\quad+\left[\operatorname{Re}\left(\frac{z_0\omega''(z_0)}{\omega'(z_0)}+1\right)-\frac{z_0\omega'(z_0)}{\omega(z_0)}\right].\end{aligned} \tag{2.4}$$

Here, the left-hand side turns out to be $< (n+1)/2$ by (2.2), while the right-hand side is $\ge (n+1)/2$ by Lemma 1 and the assumption on F. This gives the desired contradiction.

If $P \equiv$ const., then $\omega \equiv 0$ and we are done. If $P' \equiv$ const. $\neq 0$, then the left-hand side of (2.4) is 1, while the right-hand side is > 1 except in the case that equality occurs in (2.1). That, however, implies $\omega(z) = az$, and the relation $P = F \circ \omega$ shows that F must be linear. This contradicts the assumptions on F.

Part 2. Let $\mathcal{F}_1$ denote the set of functions ω which satisfy the assumptions of Theorem 2 for the given F. If Theorem 2 is false, then there exist $\omega_1 \in \mathcal{F}_1$ and $z_0 \in \mathbf{D}$ with $|\omega_1(z_0)| > 1$. The function $\omega_2 \in \mathcal{F}_1$ defined by $\omega_2(z) := \omega_1(zz_0)$ is analytic in $\overline{\mathbf{D}}$, and we have

$$|\omega_2(1)| > 1. \tag{2.5}$$

Proposition 1. *Assume Theorem 2 is false and ω_2 as above. Then there exists a domain Ω with the following properties*:

(a) $\omega_2(\overline{\mathbf{D}}) \subset \Omega$.

(b) *$\partial\Omega$ is a finite collection of Jordan curves.*

(c) $\overline{\Omega}$ is compact and $\mathfrak{C} \cap \overline{\Omega} = \varnothing$.

(d) *The set*

$$\mathcal{F} := \{\omega \in \mathcal{F}_1 : \omega(\mathbf{D}) \subset \Omega\}$$

is nonempty and compact. Each function $\omega \in \mathcal{F}$ extends analytically into $\overline{\mathbf{D}}$.

(e) *Each function $\omega \in \mathcal{F}$ has at most a finite number of points of contact with $\partial\Omega$, i.e., the sets*

$$\mathfrak{H}_\omega := \{z \in \partial\mathbf{D} : \omega(z) \in \partial\Omega\}$$

are finite. Furthermore, $\omega'(z) \neq 0$ for $z \in \mathfrak{H}_\omega$.

(f) *The set*

$$\Delta := \{\omega(1) : \omega \in \mathcal{F}\}$$

is compact, and there exists $\zeta \in \partial\Delta \backslash \partial\Omega$ with $|\zeta| > 1$.

Proof. We can find a bounded domain Ω_2 such that

- $\partial\Omega_2$ is a finite collection of Jordan curves,
- $\omega_2(\overline{\mathbf{D}}) \subset \Omega_2$,
- $\mathfrak{C} \cap \overline{\Omega_2} = \varnothing$,
- F maps $\partial\Omega_2$ into a finite number of line segments.

Let

$$\mathcal{F}_2 := \{\omega \in \mathcal{F}_1 : \omega(\mathbf{D}) \in \Omega_{\mathbf{2}}\}.$$

We note that, by construction, F is analytic in $\overline{\Omega_2}$ and F' is bounded away from zero in the same set: there exist constants $\mu_1 < \infty$, $\mu_2 > 0$ such that

$$|F(w)| \leq \mu_1, \quad |F'(w)| \geq \mu_2, \quad \text{for } w \in \overline{\Omega_2}.$$

Let $\omega \in \mathcal{F}_2$ and $P = F \circ \omega$. Then we have $|P(z)| \leq \mu_1$ for $z \in \mathbf{D}$, and thus, by Bernstein's theorem, $|P'(z)| \leq n\mu_1$ in $\mathbf{D}$. This gives

$$|\omega'(z)| = \frac{|P'(z)|}{|F'(\omega(z))|} \leq \frac{n\mu_1}{\mu_2}, \quad z \in \mathbf{D}. \tag{2.6}$$

This implies, in particular, that the length of the curve

$$\gamma_\omega := \{\omega(t) : 0 \le t \le 1\}$$

is at most $M := n\mu_1/\mu_2$.

Let $J \subset \Omega_2 \backslash (\omega_2(\overline{\mathbf{D}}) \cup \overline{\mathbf{D}})$ be a Jordan domain which satisfies

- F maps ∂J into finitely many line segments,
- there exists a point $\tilde{w} \in \partial J$ such that every analytic curve connecting the origin with $\tilde{w}$ in $\Omega_2 \backslash J$ is of length larger than M.

We now set $\Omega := \Omega_2 \backslash \overline{J}$ and verify the assertions in Proposition 1(a), (b), (c) are obvious. The first part of (d) follows from (2.6), noting that $\mathcal{F} \subset \mathcal{F}_2$ and that $\mathcal{F}$ is closed. The second part can be deduced from the fact that $F \circ \omega \in \mathcal{P}_n$ and $F' \neq 0$ in $\overline{\Omega}$. Each point of contact of $\omega \in \mathcal{F}$ with $\partial\Omega$ corresponds to a contact of the corresponding polynomial P with one of the finitely many straight line segments in $F(\partial\Omega)$. Since P has at most n contacts with any given straight line, the conclusion follows. Furthermore, $P' \neq 0$ in the points of contact, which, therefore, holds for ω' as well; this gives (e). To prove (f), we first note that Δ has points outside $\overline{\mathbf{D}}$ (see (2.5)). On the other hand, $\tilde{w} \notin \partial\Delta$ by construction (compare the necessary length of analytic curves to connect 0 with $\tilde{w}$ in Ω with the length of the curves γ_ω). This shows that there must be a boundary point ζ of Δ in $\Omega \backslash \overline{\mathbf{D}}$. □

Proposition 2. *Let $\mathcal{F}, \Delta, \zeta, \mathfrak{H}_\omega$ be as in Proposition 1. Then, for $\omega \in \mathcal{F}$ with $\omega(1) = \zeta$ and $P = F \circ \omega$, we have*

$$\max_{z \in \mathfrak{H}_\omega} \operatorname{Re}\left[zP'(z)\overline{U(z)}\right] \ge 0 \tag{2.7}$$

holds for every $U \in \mathcal{P}_n$ with $U(0) = U(1) = 0$.

To prove this, we employ the following lemma [**1**, Lemma 2.3].

Lemma 2. *Let $P \in \mathcal{P}_n$, $z_0 \in \partial\mathbf{D}$, with $P'(z_0) \neq 0$. Then, for every $\delta \in (0, \pi/2)$, there exist $\varepsilon_0 > 0, \gamma > 0$ such that*

$$P(z) - \varepsilon e^{i\psi} zP'(z) \in P(\mathbf{D})$$

for each $z = z_0 e^{i\phi}$ with $|\phi| \le \gamma$, $0 < \varepsilon < \varepsilon_0$, $|\psi| < \delta$.

Proof of Proposition 2. First, we can rule out the possibility that $P' \equiv$ const. because, for such a P, Part 1 applies and proves that the corresponding ω takes values only in $\overline{\mathbf{D}}$, which contradicts $\omega(1) = \zeta$. Second, we observe that $\mathfrak{H}_\omega \neq \varnothing$ since, otherwise, the functions $\omega_\rho(z) := \omega(\rho z)$, with $|\rho - 1|$ small, belong to $\mathcal{F}$ and can be used to show that ζ is an interior point of Δ.

Now assume that there exists $U \in \mathcal{P}_n$ with $U(0) = U(1) = 0$ which does not fulfill (2.7). For $\varepsilon > 0$, we define $P_\varepsilon := P + \varepsilon U$. The analytic functions $\omega_\varepsilon := F^{-1} \circ P_\varepsilon$ are, in a neighborhood of $z = 0$, uniquely defined by fixing $\omega_\varepsilon(0) = 0$. It is clear that, for ε small, they can be analytically extended to the whole of $\mathbf{D}$ (they are continuous variations of ω). We wish to show that $\omega_\varepsilon \in \mathcal{F}$ and that, in fact,

$$\omega_\varepsilon(\overline{\mathbf{D}}) \subset \Omega \tag{2.8}$$

holds for ε small. Once this has been done, it becomes clear that, for all ρ with $|\rho - 1|$ small, we have $\omega_{\varepsilon,\rho}(z) := \omega_\varepsilon(\rho z) \in \mathcal{F}$. Since $\omega_\varepsilon(1) = \omega(1) = \zeta$, this will prove that ζ is an interior point of Δ, a contradiction.

It remains to prove (2.8). Let $z_0 \in \mathfrak{H}_\omega$. Then, from the properties of U, we deduce the existence of an open arc $I'_{z_0} \subset \partial\mathbf{D}$ which contains z_0, and of a $\delta < \pi/2$ such that

(i) $m := \inf_{z \in I'_{z_0}} |zP'(z)/U(z)| > 0$,

(ii) $|\arg(-zP'(z)\overline{U(z)})| \leq \delta,\ z \in I'_{z_0}$.

By Lemma 2 we find an open subarc $I_{z_0} \subset I'_{z_0}$ containing z_0 such that

$$P(z) - \varepsilon e^{i\psi} zP'(z) \in P(\mathbf{D}) \subset F(\omega(\mathbf{D})) \tag{2.9}$$

for $0 < \varepsilon' < \varepsilon'_{z_0}$, $|\psi| < \delta$, $z \in I_{z_0}$. In view of (i), (ii), the relation (2.9) holds for the choices

$$\varepsilon' e^{i\psi} := -\varepsilon \frac{U(z)}{zP'(z)}, \quad 0 < \varepsilon < \varepsilon_{z_0} := m\varepsilon'_{z_0},\ z \in I_{z_0},$$

in other words,

$$P_\varepsilon(z) \in F(\omega(\mathbf{D})), \quad 0 < \varepsilon < \varepsilon_{z_0},\ z \in I_{z_0}.$$

Since $F'(\omega(z_0))\omega'(z_0) \neq 0$ we see (possibly after making ε_{z_0} and I_{z_0} still somewhat smaller) that

$$\omega_\varepsilon(z) \in \Omega, \quad 0 < \varepsilon < \varepsilon_{z_0}, \ z \in I_{z_0}. \tag{2.10}$$

If $z_0 \in \partial\mathbf{D}\backslash\mathfrak{H}_\omega$, then $P(z_0) \in F(\Omega)$, and we can find an open arc $I_{z_0} \subset \partial\mathbf{D}$ containing z_0 such that $P_\varepsilon(z) \in F(\Omega)$ for $0 < \varepsilon < \varepsilon_{z_0}$, $z \in I_{z_0}$. In the same way as above, we deduce that (2.10) also holds in this case.

The sets I_{z_0}, $z_0 \in \partial\mathbf{D}$ form an open covering of $\partial\mathbf{D}$. We choose a finite subcovering and an $\varepsilon_0 > 0$ that works, in the sense of (2.10), in all the cases of the finite subcovering. (2.10) then gives

$$\omega_\varepsilon(z) \in \Omega, \quad 0 < \varepsilon < \varepsilon_0, \ z \in \partial\mathbf{D}.$$

This proves (2.8). □

Now we can complete the proof of Theorem 2 (assuming the truth of Theorem 3, which will be established in the next section). Let ω, P be as in Proposition 2. In the notation of Theorem 3, we set $R = P'$, $V = U/z$, $\mathfrak{H} = \mathfrak{H}_\omega$, $m = \deg(P')(\neq 0)$. Then Propositions 1, 2 say that all assumptions of Theorem 3 are satisfied. Hence, P' has all its roots on $\partial\mathbf{D}$. But then Part 1 of this proof applies, and we obtain $\omega(\mathbf{D}) \subset \mathbf{D}$, a contradiction to $|\omega(1)| = |\zeta| > 1$. □

3. Proof of Theorem 3. First, note that $\mathfrak{H}$ has at least $m+1$ elements since, otherwise, every polynomial $V^* \in \mathcal{P}_m$ with $V^*(1) = 0$ and $V^*(z) = R(z)$, $z \in \mathfrak{H}$, contradicts (1.3). The next step is to extend (1.3) to the apparently trivial cases where V has zeros in $\mathfrak{H}$. Let V_0 be a polynomial of exact degree m which has all of its zeros (they may be multiple) in $\mathfrak{H}$, except for a simple zero in the point 1, and let $\mathfrak{A}$ be the set of the zeros different from 1. We claim that

$$0 \in \mathrm{co}\{R(z)\overline{V_0(z)}, \ z \in \mathfrak{H}\backslash\mathfrak{A}\}. \tag{3.1}$$

Indeed, let $z^* \in \mathfrak{H}\backslash\mathfrak{A}$ be fixed and assume, without loss of generality, that $V_0(z^*) = R(z^*)$. Then, for $\rho > 0$, use Hermite interpolation to construct the polynomials $V_\rho \in \mathcal{P}_m$ which are uniquely determined by

$$\begin{aligned} V_\rho(1) &= 0, \\ V_\rho(z^*) &= V_0(z^*), \\ V_\rho(z) &= \rho R(z), \\ V_\rho^{(j)}(z) &= 0, \quad j = 1, \ldots, j(z) - 1, \end{aligned}$$

for $z \in \mathfrak{A}$ and with $j(z)$ the multiplicity of the zero z of V_0. Then $\lim_{\rho \to 0} V_\rho = V_0$ and, since $\mathfrak{H}$ is finite, $V_\rho(z) \neq 0$, for $z \in \mathfrak{H}$ and ρ small. Hence, for ρ small, we have

$$0 < R(z)\overline{V_\rho(z)} < R(z^*)\overline{V_\rho(z^*)}, \quad z \in \mathfrak{A},$$

and from (1.3) it follows that

$$\operatorname{co}\{R(z)\overline{V_\rho(z)} : z \in \mathfrak{H}\} = \operatorname{co}\{R(z)\overline{V_\rho(z)} : z \in \mathfrak{H} \backslash \mathfrak{A}\}.$$

Taking the limit $\rho \to 0$, we arrive at (3.1).

Now let $e^{i\theta_j}$, $0 \leq \theta_j < 2\pi$, $j = 0, \ldots, m-1$, denote the zeros of $V_0, \theta_0 = 0$. Then, for certain constants A, $A^* \neq 0$, we have

$$V_0(e^{i\theta}) = A \prod_{j=0}^{m-1} (e^{i\theta} - e^{i\theta_j}) = A^* e^{-i\frac{m\theta}{2}} \prod_{j=0}^{m-1} \sin \frac{\theta - \theta_j}{2},$$

and, after a rotation, (3.1) reads

$$0 \in \operatorname{co}\left\{ e^{-i\frac{m\theta}{2}} R(e^{i\theta}) \prod_{j=0}^{m-1} \sin \frac{\theta - \theta_j}{2} : e^{i\theta} \in \mathfrak{H} \backslash \mathfrak{A} \right\}.$$

Without loss of generality, we may assume that $R(1) \geq 0$. We write

$$e^{-i\frac{m\theta}{2}} R(e^{i\theta}) = s(\theta) + it(\theta), \tag{3.2}$$

and note that $t(0) = 0$. To simplify the notation, we write $\theta \in \mathfrak{H}'$ if $e^{i\theta} \in \mathfrak{H}$ and $0 \leq \theta < 2\pi$. Our previous considerations show that, for both choices, $v = s$ and $v = t$, the following is true: for all possible sequences $\mathfrak{A}' = (\theta_1, \dots, \theta_{m-1}) \subset \mathfrak{H}'$, the numbers

$$v(\theta) \prod_{j=1}^{m-1} \sin \frac{\theta - \theta_j}{2}, \quad \theta \in \mathfrak{H}' \backslash \mathfrak{A}', \tag{3.3}$$

cannot be all negative or all positive. We shall prove that this implies that v has at least m zeros (counting multiplicity) in

$$I := [\theta_a, \theta_b], \quad \theta_a = \min \mathfrak{H}', \ \theta_b = \max \mathfrak{H}'.$$

Assuming that there are less than m zeros in I, we construct a suitable sequence $\mathfrak{A}'$ as follows: first put every zero of v which lies in $\mathfrak{H}'$ into the sequence $\mathfrak{A}'$ according to its multiplicity. If any of the open intervals of $I \backslash \mathfrak{H}'$ contains exactly k zeros of v, then we put k copies of the upper limit of that interval into $\mathfrak{A}'$. Clearly, the sequence $\mathfrak{A}'$ now contains exactly the same number of elements as v has zeros in I, namely, $\leq m - 1$. If there are $< m - 1$ elements in $\mathfrak{A}'$, then we fill it up with a corresponding number of copies of θ_b. It is now easy to check that this sequence $\mathfrak{A}'$ leaves all the elements in (3.3) with the same sign (none of them is zero).

The functions s, t can have *at most* m zeros in $[0, 2\pi)$ if they are not identically zero. But t has an additional zero in $\theta = 0$. Hence, $t \equiv 0$. The m zeros of s are, therefore, identified to be, in fact, zeros of R on $\partial \mathbf{D}$, which completes the proof. □

REFERENCES

1. A. Córdova and S. Ruscheweyh, *On maximal ranges of polynomial spaces in the unit disk*, Constr. Approx., to appear.

2. ——— and ———, *On maximal polynomial ranges in circular domains*, Complex Variables, to appear.

3. I.S. Jack, *Functions starlike and convex of order α*, J. London Math. Soc. **3** (1971), 469–474.

4. S.S. Miller and P.T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl. **65** (1978), 289–305.

5. C. Pommerenke, *Univalent functions*, Vandenhoeck and Rupprecht, Göttingen, 1975.

6. T.J. Suffridge, *On univalent polynomials*, J. London Math. Soc. **44** (1969), 496–504.

7. ———, *Starlike functions as limits of polynomials*, in *Advances in complex function theory*, Maryland, 1973/1974, Lecture Notes in Mathematics, **505**, Springer Verlag, 1976, 164–203.

Depto. Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile.

Mathematisches Institut, Universität Würzburg, D-8700 Würzburg, FRG.