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SOME OBSERVATIONS ON THE SAFF-VARGA WIDTH CONJECTURE

TIMOTHY S. NORFOLK

Dedicated to W.J. Thron on the occasion of his 70th birthday.

ABSTRACT. In this paper, we consider the Saff-Varga Width Conjecture, which relates the order of an entire function to the asymptotic behavior of the zeros of its partial sums. Together with a partial history of the problem, we derive a set of conditions sufficient to prove the conjecture, based on the rate of growth and angular distribution of the zeros of a certain sequence of partial sums. The results presented demonstrate the construction of a sequence having the desired minimal growth rate and indicate the direction that might be taken to complete the proof of the conjecture, in terms of a possible modification of a theorem of Erdös and Turán.

For each nonnegative integer, n, let $s_n(z;f) := \sum_{k=0}^{n} a_k z^k$ denote the n-th partial sum of the entire function $f(z)$. Our interest is in determining the asymptotic behavior of the zeros of $s_n(z;f)$. In particular, we wish to describe regions which must contain zeros of the sequence $\{s_n(z;f)\}_{n=1}^{\infty}$.

A critical result in this area is the theorem of Carlson [**2, 3**], which we state in the following form, as proven by Rosenbloom [**13**]:

Theorem A. *Let $f(z)$ be an entire function. If there exists a sector of infinite area with vertex at the origin such that the number Z_n of zeros of $s_n(z;f)$ inside the sector satisfies $\lim_{n\to\infty} Z_n/n = 0$, then $f(z)$ is of order zero.*

This result clearly delineates the entire functions of positive order from those of order zero. The statement of the Saff-Varga Width

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Conjecture is an attempt to express the behavior of the partial sums of an entire function in terms of its order. In one form, the conjecture may be given as follows.

The Saff-Varga Width Conjecture [**14**]. *Let* $0 \leq \tau < \infty$, $K > 0$ *and* $x_0 \geq 0$ *be given, and define the "parabolic" region*

$$S_\tau(K; x_0) := \{z = x + iy \in \mathbf{C} : |y| \leq K x^{1-\frac{\tau}{2}}, x \geq x_0\}.$$

Then if $f(z)$ *is an entire function of order* $\tau < \lambda \leq \infty$, *the region* $S_\tau(K; x_0)$ *contains infinitely many zeros of* $\{s_n(z; f)\}$. *That is, there exists an increasing sequence* $\{n_k\}$ *of positive integers and a corresponding sequence* $\{z_{n_k}\}$ *of complex numbers such that* $z_{n_k} \in S_\tau(K; x_0)$ *and* $s_{n_k}(z_{n_k}; f) = 0$.

The following special cases are known to hold:

(a) Via the theorem of Carlson [**2, 3** and **13**], the uniform convergence of $\{s_n(z; f)\}$ to $f(z)$ on compact subsets of $\mathbf{C}$, and Hurwitz's Theorem (cf. [**4**, pp. 151–155]), the case $\tau = 0$ holds immediately.

(b) If $f(z)$ has infinitely many zeros on the positive real axis, the result again follows from the uniform convergence of the partial sums on compact subsets of $\mathbf{C}$.

(c) Newman and Rivlin [**11, 12**] and Szegö [**15**] implicitly showed that the partial sums of $f_\theta(z) := \exp(e^{i\theta} z)$ exhibit the desired asymptotic behavior for all $0 \leq \theta < 2\pi$.

(d) In [**7**], it was shown by Edrei, Saff and Varga that the partial sums of the Mittag-Leffler functions of order λ, $1 < \lambda$, defined by

$$E_{1/\lambda}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + k/\lambda)},$$

have the desired asymptotic behavior about any ray emanating from the origin. That is, any rotation of $S_\tau(K; x_0)$ contains infinitely many zeros of $\{s_n(z; E_{1/\lambda})\}$ for any $0 \leq \tau < \lambda$, $K > 0$ and $x_0 \geq 0$.

(e) In [**5**] and [**6**], Edrei also showed that the conjecture holds for certain classes of entire functions of infinite order.

On the basis of the proofs in the known cases, and on the results which follow, it seems reasonable to restate the conjecture in terms of the following sufficient conditions:

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be entire, of order $0 < \lambda \le \infty$. Then there exists a sequence $\{z_{n_k}\}$ of zeros of $s_{n_k}(z; f)$ satisfying

I. $\lim_{k\to\infty} |z_{n_k}| = \infty$,

II. $\lim_{k\to\infty} \log|z_{n_k}|/\log n_k \le \frac{1}{\lambda}$,

III. $\limsup_{k\to\infty} \log|\arg z_{n_k}|/\log n_k \le -\frac{1}{2}$.

To see that these conditions are indeed sufficient, suppose that such a sequence exists and write

$$z_{n_k} = x_{n_k} + iy_{n_k} = n_k^{1/\lambda+\varepsilon_k} \exp\left(\pm i n_k^{-(1-\delta_k)/2}\right),$$

where $\lim_{k\to\infty} \varepsilon_k \le 0$, $\limsup_{k\to\infty} \delta_k \le 0$ and, by condition I, $1/\lambda + \varepsilon_k > 0$.

Rewriting, using $\alpha_k := 1/\lambda + \varepsilon_k$ and $\beta_k := -(1-\delta_k)/2$, we obtain

$$n_k = x_{n_k}^{1/\alpha_k} \sec^{1/\alpha_k}(n_k^{\beta_k})$$

and

$$\begin{aligned} |y_{n_k}| &= n_k^{\alpha_k} \sin(n_k^{\beta_k}) \le n_k^{\alpha_k} n_k^{\beta_k} \\ &= x_{n_k}^{1+\beta_k/\alpha_k} \left[\sec(n_k^{\beta_k})\right]^{1+\beta_k/\alpha_k}. \end{aligned}$$

Since

$$\limsup_{k\to\infty} \frac{\beta_k}{\alpha_k} = \limsup_{k\to\infty} \frac{-(1-\delta_k)}{2(1/\lambda+\varepsilon_k)} \le -\frac{\lambda}{2} < 0,$$

given $0 \le \tau < \lambda$, $K > 0$ and $x_0 \ge 0$, there exists a positive integer k_0 such that $z_{n_k} \in S_\tau(K; x_0)$ for all $k \ge k_0$.

The existence of a sequence satisfying conditions I and II is known. In fact, Tsuji [**16**] proved that the set of entire functions having a sequence of partial sums whose largest zeros satisfy conditions I and II is precisely the set of entire functions of order at least λ.

Our aim is to construct a natural sequence of partial sums whose largest zeros satisfy the given growth condition and also provide a special sequence of partial sums whose existence was demonstrated by

Rosenbloom [**13**]. Since we wish to consider the asymptotic behavior of the partial sums, we may assume without loss of generality that our entire functions satisfy $f(0) = 1$. We will begin by defining a dominating sequence of partial sums for any entire function and estimating the growth rate of the corresponding zeros.

Theorem 1. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be a nonpolynomial entire function satisfying* $f(0) = 1$*. Define an increasing sequence* $\{n_k\}$ *of nonnegative integers via*

$$n_0 := 0, \quad |a_{n_k}|^{1/n_k} = \max_{n_{k-1}<n} |a_n|^{1/n}$$

and a sequence of normalizations via

$$R_k := \max_{0\le j<k} \left(\frac{|a_{n_j}|}{|a_{n_k}|}\right)^{\frac{1}{n_k-n_j}}.$$

Then the largest zero of $s_{n_k}(R_k z; f)$ *has modulus bounded by* 2.

Proof. The fact that $f(z)$ is entire implies that $\lim_{n\to\infty} |a_n|^{1/n} = 0$, which in turn means that the sequence $\{n_k\}$ is well defined. From the definitions of $\{n_k\}$ and $\{R_k\}$, it follows that the largest coefficient (in modulus) of $s_{n_k}(R_k z; f)$ is $a_{n_k} R_k^{n_k}$, as can be seen by the following:

Given $0 \le i < n_k$, there exists $0 < j \le k$ such that $n_{j-1} < i \le n_j$. Consequently, by the definitions of $\{n_k\}$ and $\{R_k\}$, we have

$$\begin{aligned} |a_i| R_k^i &\le |a_{n_j}|^{i/n_j} R_k^i \\ &\le \left(|a_{n_k}| R_k^{n_k-n_j}\right)^{i/n_j} R_k^i = (|a_{n_k}| R_k^{n_k})^{i/n_j} \\ &\le |a_{n_k}| R_k^{n_k}, \end{aligned}$$

since $i \le n_j$ and $|a_{n_k}| R_k^{n_k} \ge |a_0| = 1$.

It then follows that the largest zero is bounded by the *Cauchy Inclusion Radius* (Henrici [**9**, pp. 457–458], Marden [**10**, p. 122]), which is in turn bounded by 2 (cf. [**10**, pp. 122–126]). □

It is to be remarked that the sequence $\{s_{n_k}(z; f)\}$ constructed above must contain the subsequence of partial sums of minimal growth rate since $s_n(z;f)$ has a zero of modulus at least $|a_n|^{-1/n}$ if $a_n \neq 0$. It thus behooves us to construct a subsequence of $\{R_k\}$ with the desired minimal growth rate, for which we will require the following lemmas.

Lemma 1. *If $0 < x < y$, then $0 < (x \log y/x)/(y - x) < 1$.*

Proof. Let $t := x/y$, and consider the behavior of the function $f(t) := (-t \log t)/(1 - t)$ for $0 < t < 1$. □

Lemma 2. *Let $\{n_k\}$ be an increasing sequence of positive integers, and let $\{\gamma_k\}$ be a sequence of real numbers satisfying $\liminf_{k\to\infty} \gamma_k = 0$. Then*

$$\liminf_{k\to\infty} \max_{0\le j<k} \left\{ \frac{n_j}{n_k - n_j} (\gamma_k \log n_k - \gamma_j \log n_j) \right\} \le 0.$$

Proof. We proceed by contradiction. Given $k \ge 2$, define an integer-valued function h such that the maximum in the above is achieved at $1 \le h(k) < k$. If the limit inferior of the aforementioned quantity is positive, there exist $B > 0$ and $k_0 \ge 1$ such that

$$\frac{n_{h(k)}}{n_k - n_{h(k)}} \left(\gamma_k \log n_k - \gamma_{h(k)} \log n_{h(k)}\right) \ge B \quad \text{for } k \ge k_0.$$

Consequently,

$$\gamma_k \log n_k \ge B \left(\frac{n_k - n_{h(k)}}{n_{h(k)}} \right) + \gamma_{h(k)} \log n_{h(k)} \quad \text{for } k \ge k_0. \tag{1}$$

Given $k \ge 2$, we may define a finite decreasing set $\{p_n\}$ of nonnegative integers via

$$p_0 := k, \quad p_1 := h(k) \quad \text{and} \quad p_n := h(p_{n-1}),$$

stopping when some $p_n = 1$. Since this sequence is decreasing, there exists a maximal integer $1 \le m = m(k) \le n_k - n_{k_0}$ such that $p_m \le k_0$. Thus, applying (1) m times, we obtain

$$\gamma_k \log n_k \ge B \sum_{i=1}^{m} \frac{n_{p_{i-1}} - n_{p_i}}{n_{p_i}} + \gamma_{p_m} \log n_{p_m}. \tag{2}$$

We define

$$A := \min_{1 \le j \le k_0} \gamma_j \log n_j, \tag{3}$$

and so, by (2) and (3), we have

$$\begin{aligned}\gamma_k \log n_k &\ge B\left(\sum_{i=1}^{m} \frac{n_{p_{i-1}}}{n_{p_i}} - m\right) + A \\ &\ge Bm\left[\left(\frac{n_{p_0}}{n_{p_m}}\right)^{1/m} - 1\right] + A\end{aligned}$$

(by the arithmetic-geometric mean inequality)

$$\ge Bm\left[\frac{\log n_k - \log n_{k_0}}{m}\right] + A,$$

so that

$$\gamma_k \ge B\left[\frac{\log n_k - \log n_{k_0} + A/B}{\log n_k}\right] \quad \text{for } k \ge k_0.$$

Consequently, we have $\liminf_{k\to\infty} \gamma_k \ge B > 0$, which is the desired contradiction, proving our assertion.

In terms of the desired construction of the subsequence $\{k_i\}$ which yields the limit inferior above, we may either define it by

$$\gamma_{k_i} = \min_{1 \le j \le k_i} \gamma_j \quad \text{for } i \ge 1,$$

which will occur, for example, if $\gamma_k > 0$ for all $k \ge 1$, or via

$$\gamma_{k_0} = \min_{1 \le j} \gamma_j \quad \text{and} \quad \gamma_{k_i} = \min_{k_{i-1} < j \le k_i} \gamma_j \quad \text{for } i \ge 2.$$

In either case, it is clear that $\lim_{i\to\infty} \gamma_{k_i} = 0$ and that the required limit inferior will be achieved since the given maximum is an increasing function of γ_k. □

Theorem 2. *Let $f(z) := \sum_{k=0}^{\infty} a_k z^k$ be an entire function of order $0 < \lambda \le \infty$ satisfying $f(0) = 1$, and let $\{n_k\}$ and $\{R_k\}$ be defined as in Theorem 1. Then there exists a subsequence $\{n_{k_i}\}$ of $\{n_k\}$ such that*

(i) $1 \leq \lim_{i\to\infty} |a_{n_{k_i}}|^{1/n_{k_i}} R_{k_i} \leq e^{1/\lambda}$

and the largest zero, $w_{n_{k_i}}$ *of* $s_{n_{k_i}}(z; f)$ *satisfies*

(ii) $\lim_{i\to\infty} \log |w_{n_{k_i}}| / \log n_{k_i} = \frac{1}{\lambda}$.

Proof. Since $f(0) = 1$, it is trivial to show that $s_{n_k}(z; f)$ has a zero of modulus at least $|a_{n_k}|^{-1/n_k}$. However (cf. Boas [**1**, pp. 1–9]), we have that

$$\liminf_{n\to\infty} \frac{\log 1/|a_n|}{n \log n} = \frac{1}{\lambda},$$

so we define a sequence $\{\lambda_k\}$ via

$$\gamma_k := \frac{\log 1/|a_{n_k}|}{n_k \log n_k} - \frac{1}{\lambda} \quad \text{for } k \geq 1$$

(where we set $\gamma_1 := 1$ if $n_1 = 1$).

The definition of our sequence $\{n_k\}$ yields $\liminf_{k\to\infty} \gamma_k = 0$. Thus, considering the conclusion of Theorem 1, it is only required to show that we can generate a subsequence $\{\gamma_{k_i}\}$ with limit 0 and for which $\{R_{k_i}\}$ satisfies condition (i).

From our definitions, we have

$$0 \leq \log |a_{n_k}|^{1/n_k} R_k$$
$$= \max \left\{ 0, \max_{1\leq j<k} \left\{ \frac{n_j}{n_k - n_j} (\gamma_k \log n_k - \gamma_j \log n_j) + \frac{n_j \log(n_k/n_j)}{\lambda(n_k - n_j)} \right\} \right\}.$$

By Lemma 1 and Lemma 2, there exists a sequence $\{k_i\}$ of positive integers such that $\lim_{i\to\infty} \gamma_{k_i} = 0$ and the limit inferior of the quantity above as $i \to \infty$ is at most $1/\lambda$, which is the desired result. □

We remark that the growth condition on the largest zero cannot be replaced, in general, by the statement that $|z_n| = O(n^{1/\lambda})$ as $n \to \infty$.

To see this, for $0 < \rho < \infty$, define the function,

$$g_\rho(z) := 1 + z + \sum_{k=2}^{\infty} \frac{z^k}{k^{k/\rho} (\log k)^k},$$

which is an entire function of order ρ, and for which the largest zero of $s_n(z; g_\rho)$ grows exactly at a rate of $n^{1/\rho} \log n$.

Using the subsequence $\{k_i\}$ constructed in Theorem 2, we also have a constructive proof of the following result of Rosenbloom [**13**]:

Theorem 3. *Let $f(z) := \sum_{k=0}^{\infty} a_k z^k$ be an entire function of order $0 < \lambda \le \infty$ satisfying $f(0) = 1$, and let $\{n_{k_i}\}$ be the sequence of positive integers constructed as in Theorem 1 and Theorem 2. Then, defining*

$$\psi_i(z; f) := \frac{z^{n_{k_i}} s_{n_{k_i}}(R_{k_i}/z; f)}{a_{n_{k_i}} R_{k_i}^{n_{k_i}}},$$

we have that $\psi_i(0; f) = 1$ and that $\{\psi_i(z; f)\}$ is uniformly bounded as $i \to \infty$ on compact subsets of $\{z \in \mathbf{C} : |z| < 1\}$.

To provide justification that the subsequence $\{s_{n_k}(z; f)\}$ of partial sums of $f(z)$ does indeed dominate the asymptotic behavior of the zeros, notice that, for any $\varepsilon > 0$, the sequence $\{\psi_i(z; f)\}$ satisfies the hypotheses of the following theorem of Rosenbloom [**13**], with $r = 1 - \varepsilon$, and $b = e^{-1/\lambda}$:

Theorem B. *Let $p(z) = \sum_{k=0}^{n} c_k z^k$ be given, and suppose that $\max_{|z|=r} |f(z)| =: M_p(r) \le M$ for some $r > 0$, $|c_0| \ge a > 0$, $|c_n| \ge b^n > 0$. Then, given any $0 < \tau \le \pi$ there exist constants $\omega > 0$ and A depending only on M, a, b, r and τ such that $p(z)$ has at least $n\, \omega - A$ zeros in any infinite sector of the form $|\arg z - \alpha| \le \tau$.*

While this result does not prove condition III of our conjecture, it does suggest that the statement might be in the correct form and immediately yields Theorem A of Rosenbloom [**13**] when applied to our sequence $\{\psi_i(z; f)\}$.

We may also use Theorem 3 to show that our sequence $\{R_{k_i}\}$ of normalizations behaves in the general case like the normalizations in the known cases (cf. [**7, 11, 12, 15**]). That is, the number of zeros of a subsequence of $\{s_{n_k}(R_k z; f)\}$ outside any disk of the form $\{z \in \mathbf{C} : 1 + \varepsilon \le |z|\}$ is uniformly bounded in k for each $\varepsilon > 0$.

Theorem 4. *Let $f(z)$, $\{n_{k_i}\}$ and $\{R_{k_i}\}$ be as in the statement of Theorem 2. Then there exists a subsequence of $\{s_{n_{k_i}}(R_{k_i}z; f)\}$ whose zeros have only finitely many limit points in $\{z \in \mathbf{C} : 1+\varepsilon < |z|\}$ for each $\varepsilon > 0$.*

Proof. From the proof of Theorem 3, the sequence $\{\psi_i(z; f)\}$ is uniformly bounded on compact subsets of the open unit disk and satisfies $\psi_i(0; f) = 1$. Thus, by Montel's Theorem (cf. [**4**, pp. 151–155]), this sequence of functions has a subsequence which converges uniformly to a function $\Psi(z)$, analytic on the open unit disk and satisfying $\Psi(0) = 1$.

Hence, $\Psi(z)$ has only finitely many zeros in the disk $\{z \in \mathbf{C} : |z| \leq (1+\varepsilon)^{-1}\}$ which are the limit points of the zeros of the subsequence of $\{\psi_i(z; f)\}$ inside this disk. Thus, since the zeros of $\psi_i(z; f)$ are the reciprocals of those of $s_{n_{k_i}}(R_{k_i}z; f)$, we have the result. □

Theorem 3 would indicate that the remaining condition (III) of our modified conjecture might be proven by applying some modification of the following theorem of Erdös and Turán [**8**].

Theorem C. *Let $p(z) = \sum_{k=0}^{n} c_k z^k$, where $c_0 c_n \neq 0$. For $0 \leq \alpha < \beta \leq 2\pi$, let $Z(\alpha; \beta; p)$ denote the number of zeros of $p(z)$ which satisfy $\alpha \leq \arg z \leq \beta$. Then*

$$|Z(\alpha;\beta;p) - n(\beta-\alpha)/2\pi| < 16\sqrt{n \log \frac{M_p(1)}{\sqrt{|c_0 c_n|}}}.$$

Unfortunately, the theorem in question is too crude to yield the desired result directly and attempts at modification to take into account the structure of the polynomial have, as yet, proved unsatisfactory.

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Department of Mathematical Sciences, The University of Akron, Akron, OH 44325