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REMOVING THE JUMP-KATO'S DECOMPOSITION

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ABSTRACT. A simple proof, using the adjoint operator and Hahn-Banach theorem, is given of Kato's Decomposition which removes the jump at the origin in the nullity (or defect) of a semi-Fredholm operator by subtracting a finite dimensional summand.

Let X be a Banach space over the complex field and let $B(X)$ denote the Banach algebra of bounded linear operators on X. For $T \in B(X)$ set $n(T) = \dim \ker(T)$ and $d(T) = \operatorname{codim} T(X)$. Define the *generalised kernel* $\mathbf{K}(T)$ and the *generalised range* $\mathbf{R}(T)$ of T to be the subspaces

$$\mathbf{K}(T) = \bigcup_1^\infty \ker(T^n), \quad \mathbf{R}(T) = \bigcap_1^\infty T^n(X).$$

Write

$$\Phi_+(X) = \{T \in B(X) : n(T) < \infty \text{ and } T(X) \text{ is closed in } X\},$$
$$\Phi_-(X) = \{T \in B(X) : d(T) < \infty \text{ and } T(X) \text{ is closed in } X\}.$$

$\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X)$ is the set of semi-Fredholm operators in $B(X)$,while $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$ is the set of Fredholm operators in $B(X)$. If $T \in \Phi_\pm(X), i(T) = n(T) - d(T)$, a finite or infinite integer, is the index of T. X^* denotes the dual space of X and T^* the adjoint operator of T.

If $T \in \Phi_\pm(X)$, then $\mathbf{R}(T)$ is a closed subspace of T, and if $T_R = T|\mathbf{R}(T)$ denotes the restriction operator, then it is well known [**3**] that $n(T_R) \leq n(T), n(T_R + \lambda) = n(T + \lambda)$ for $\lambda \neq 0, d(T_R) = 0$ and $T_R \in \Phi(\mathbf{R}(T))$. This result is important in that it reduces properties of semi-Fredholm operators to those of Fredholm operators.

If $T \in \Phi_+(X)$ then $\exists \varepsilon > 0$ such that $n(T + \lambda)$ is constant $(\leq n(T))$ for $0 < |\lambda| < \varepsilon$, while if $T \in \Phi_-(X)$ the same is true of $d(T + \lambda)$. Therefore we can define the *jump* of T

$$j(T) = n(T) - n(T + \lambda), \quad 0 < |\lambda| < \varepsilon, \text{ for } T \in \Phi_+(X)$$

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and

$$j(T) = d(T) - d(T+\lambda), \quad 0 < |\lambda| < \varepsilon, \quad \text{for } T \in \Phi_-(X).$$

Continuity of the index ensures that the jump is unambiguously defined for $T \in \Phi(X)$.

Now let $T \in B(X)$. The first two lemmas are entirely elementary.

LEMMA 1.

$$\begin{aligned} \ker(T) \subset T^n(X) &\Leftrightarrow \ker(T^2) \subset T^{n-1}(X) \\ &\Leftrightarrow \dots \\ &\Leftrightarrow \ker(T^n) \subset T(X). \end{aligned}$$

LEMMA 2.

$$\begin{aligned} &\mathbf{K}(T) \subset \mathbf{R}(T), \\ &\Leftrightarrow \ker(T) \subset T^n(X), \quad n = 1, 2, \dots, \\ &\Leftrightarrow \ker(T^n) \subset T(X), \quad n = 1, 2, \dots. \end{aligned}$$

PROPOSITION 3. *If* $T \in \Phi_\pm(X)$ *then*

$$j(T) = 0 \Leftrightarrow \mathbf{K}(T) \subset \mathbf{R}(T).$$

PROOF. First let $T \in \Phi_+(X)$. Then

$$\begin{aligned} &\mathbf{K}(T) \subset \mathbf{R}(T), \\ &\Leftrightarrow \ker(T) = \ker(T_R), \ \text{(Lemma 2)} \\ &\Leftrightarrow n(T) = n(T_R). \end{aligned}$$

But $n(T+\lambda) = n(T_R+\lambda)$ for $\lambda \neq 0$ since, for these values of λ, $\ker(T+\lambda) \subset \mathbf{R}(T)$, therefore $n(T+\lambda) = n(T_R+\lambda)$ for all values of λ. Now $d(T_R) = 0$

$$\begin{aligned} &\Rightarrow d(T_R+\lambda) = 0, \quad |\lambda| < \varepsilon, \\ &\Rightarrow n(T_R+\lambda) \text{ is constant}, \quad |\lambda| < \varepsilon, \\ &\Rightarrow n(T+\lambda) \text{ is constant}, \quad |\lambda| < \varepsilon. \end{aligned}$$

Conversely suppose that $n(T+\lambda)$ is constant for $|\lambda|<\varepsilon$. Then

$$n(T_R) \le n(T) = n(T+\lambda) = n(T_R+\lambda), \quad 0<|\lambda|<\varepsilon.$$

But

$$\begin{aligned} &n(T_R) \ge n(T_R+\lambda), \quad 0<|\lambda|<\varepsilon,\\ &\Rightarrow n(T_R) = n(T),\\ &\Rightarrow \mathbf{K}(T) \subset \mathbf{R}(T). \end{aligned}$$

Suppose now that $T \in \Phi_-(X)$. Then $T^* \in \Phi_+(X^*)$ and, for $0<|\lambda|<\varepsilon$,

$$j(T) = d(T) - d(T+\lambda) = n(T^*) - n(T^*+\lambda) = j(T^*),$$

so $j(T)=0 \Leftrightarrow j(T^*)=0$. Now

$$\begin{aligned} &\mathbf{K}(T^*) \subset \mathbf{R}(T^*),\\ \Leftrightarrow &\ker(T^{*n}) \subset T^*(X^*),\ n=1,2,\dots \quad \text{(Lemma 2)},\\ \Leftrightarrow &T^n(X)^\perp \subset \ker(T)^\perp,\ n=1,2,\dots,\\ \Leftrightarrow &T^n(X) \supset \ker(T),\ \text{since all subspaces are closed},\\ \Leftrightarrow &\mathbf{R}(T) \supset \mathbf{K}(T). \ \square \end{aligned}$$

If $T \in \Phi_\pm(X)$, Kato's decomposition is nontrivial $\Leftrightarrow j(T) \ne 0 \Leftrightarrow \mathbf{K}(T) \not\subset \mathbf{R}(T)$, and in this case $\exists$ a smallest integer ν such that

$$\ker(T^{\nu-1}) \subset T(X), \text{ but } \ker(T^\nu) \not\subset T(X).$$

PROPOSITION 4. *With these hypotheses we can choose a cascade $y, Ty, \dots, T^{\nu-1}y$ satisfying*

$$\begin{gathered} y \in \ker(T^\nu)\backslash T(X),\\ Ty \in \ker(T^{\nu-1})\backslash T^2(X),\\ \dots\dots\dots.\\ T^{\nu-1}y \in \ker(T)\backslash T^\nu(X),\\ T^\nu y = 0; \end{gathered}$$

further, $y, Ty, \ldots, T_y^{\nu-1}$ *are linearly independent modulo* $T^\nu(X)$.

PROOF. If $\nu = 1$ we are done. So assume $\nu \geq 2$ and $y \in \ker(T^\nu)\backslash T(X)$. Then $Ty \in \ker(T^{\nu-1})$, so suppose that $Ty \in T^2(X)$. Then

$$T^{-1}(Ty)\bigcap T(X) = \{y + \ker(T)\}\bigcap T(X) \neq \varnothing;$$

thus $\exists z \in \ker(T)$ such that $y + z \in T(X)$. Now $z \in \ker(T) \subset \ker(T^{\nu-1}) \subset T(X)$ by hypothesis, so $y \in T(X)$ which is false, and this process can be continued.

If $\exists$ complex numbers $\alpha_k,\ 0 \leq k \leq \nu - 1$ such that

$$\sum_0^{\nu-1} \alpha_k T^k y \in T^\nu(X),$$

apply $T^{\nu-1}$ to this inclusion to get

$$\alpha_0 T^{\nu-1} y \in T^{2\nu-1}(X) \subset T^\nu(X),$$

which gives $\alpha_0 = 0$. A similar argument gives $\alpha_k = 0,\ k = 1, \ldots, \nu - 1$. □

PROPOSITION 5. *Under the same hypotheses we can choose* $f \in \ker(T^{*\nu})$ *such that*

$$T^{*i} f(T^{\nu-j-1}y) = \delta_{ij}, \quad 0 \leq i, \quad j \leq \nu - 1.$$

PROOF. Since $y, Ty, \ldots, T^{\nu-1}y$ are linearly independent modulo the closed subspace $T^\nu(X)$, by the Hahn-Banach Theorem

$$\exists f \in T^\nu(X)^\perp = \ker(T^{*\nu})$$

such that

$$f(T^{\nu-1}y) = 1,$$

and

$$f(T^j y) = 0, \quad 0 \leq j \leq \nu - 2.$$

Now $T^*f(T^{\nu-1}y) = T^{*\nu}f(y) = 0$, and, for $0 \le j \le \nu - 2$,

$$T^*f(T^j y) = f(T^{j+1}y),$$

so $T^*f(T^{\nu-2}y) = 1$, and $T^*f(T^j y) = 0,\ 0 \le j \le \nu - 3$. Continuing the process proves the proposition. Observe that

$$f \in \ker(T^{*\nu})\backslash T^*(X^*),$$

$$T^*f \in \ker(T^{*\nu-1})\backslash T^{*2}(X^*),$$

$$\cdots\cdots\cdots$$

$$T^{*\nu-1} \in \ker(T^*)\backslash T^{*\nu}(X^*),$$

$$T^{*\nu}f = 0,$$

forms a *cascade* adjoint to that in Proposition 4. □

The construction of the decomposing projection of Kato's theorem combines the two cascades. Let Y denote the subspace spanned by $y, Ty, \ldots, T^{\nu-1}y$ of Proposition 4.

PROPOSITION 6.

$$P = \sum_{j=0}^{\nu-1} T^{*j}f \otimes T^{\nu-j-1}y$$

is a projection in $B(X)$ with range Y which commutes with T; $T|Y$ is nilpotent and $j(T|Y) = 1$.

PROOF. P is a projection in $B(X)$ by Proposition 5.

$$TP = \sum_{j=0}^{\nu-1} T^{*j}f \otimes T^{\nu-j}y = \sum_{j=1}^{\nu-1} T^{*j}f \otimes T^{\nu-j}y,$$

$$PT = \sum_{j=0}^{\nu-1} T^{*j+1}f \otimes T^{\nu-j-1}y = \sum_{j=0}^{\nu-2} T^{*j+1}f \otimes T^{\nu-j-1}y,$$

hence $PT = TP$. Clearly $T|Y$ is nilpotent, $n(T|Y) = 1$, and, Y being finite dimensional, $j(T|Y) = 1$. □

Our main result is a special case of Kato's decomposition theorem, [**1**, Theorem 4].

THEOREM 7. *If $T \in \Phi_{\pm}(X)$, then $T = T_1 \oplus T_2$, where T_1 is a finite dimensional nilpotent direct summand $T_2 \in \Phi_{\pm}(X)$, and $j(T_2) = 0$.*

PROOF. Let $T \in \Phi_{\pm}(X)$. If $j(T) = 0$ there is nothing to prove. If $j(T) > 0$ let P be the non-zero finite rank projection of Proposition 6 commuting with T. Clearly $T|\ker(P)$ is semi-Fredholm and $j(T|\ker(P)) = j(T) - 1$. Continuing the process a finite number of times reduces the jump of the residual operator to zero. □

Let $T \in B(X)$ and let K be a compact set contained in the *semi-Fredholm domain* of T, that is the set of complex numbers $\{\lambda : \lambda + T \in \Phi_{\pm}(X)\}$. The points $\mu \in K$ such that $j(\mu + T) > 0$ form an isolated and therefore a finite set. Thus, by repeating Kato's Decomposition a finite number of times, we can simultaneously remove all jumps in K. □

THEOREM 8. *If $T \in B(X)$ and K is a compact subset of the semi-Fredholm domain of T, then $T = T_1 \oplus T_2$, where T_1 is a finite dimensional nilpotent direct summand and $j(\mu + T_2) = 0$ for $\mu \in K$.*

Applications of Kato's Decomposition are given in [**2**, **3** and **4**]. Here we use it to examine the increasing sequence of finite dimensional subspaces $\ker(T^k)$ where $T \in \Phi_+(X)$ (dually the decreasing sequence of subspaces $T^k(X)$ of finite codimension where $T \in \Phi_-(X)$).

Let $T \in \Phi_+(X)$ where ν is the least positive integer such that

$$\ker(T^\nu) \not\subset T(X)$$

and $j(T)$ is the jump of T. Kato's decomposition gives

$$
\begin{aligned}
X &= X_1 \bigoplus X_2, \\
T &= T_1 \bigoplus T_2, \\
\ker(T^k) &= \ker(T_1^k) \bigoplus \ker(T_2^k), \\
n(T^k) &= n(T_1^k) + n(T_2^k), \\
\mathbf{K}(T) &= \mathbf{K}(T_1) \bigoplus \mathbf{K}(T_2), \\
\mathbf{R}(T) &= \mathbf{R}(T_1) \bigoplus \mathbf{R}(T_2),
\end{aligned}
$$

where $\dim(X_1) < \infty, T_1$ is nilpotent of order exactly $\nu, j(T) = j(T_1)$ and $j(T_2) = 0$. Thus, to examine $\ker(T^k)$, it suffices to examine the sequences $\ker(T_1^k)$ and $\ker(T_2^k)$, combining the results.

First let us consider the operator T_1 on the finite dimensional space X_1. It is clear from the proof of Theorem 7 that X_1 is a direct sum reducing T_1,

$$
\begin{aligned}
X_1 &= \bigoplus_{i=1}^{j(T)} Y_i, \\
T_1 &= \bigoplus_{i=1}^{j(T)} S_i,
\end{aligned}
$$

where each S_i is a cyclic nilpotent operator of order $\nu_i, \dim(Y_i) = \nu_i$ and at least one ν_i (say $\nu_{j(T)}$) equals ν, thus

$$1 \leq \nu_i \leq \nu_{j(T)} = \nu.$$

Now

$$\ker(S_i^{\nu_i - 1}) \subset S_i(Y_i),$$

but

$$Y_i = \ker(S_i^{\nu_i}) \not\subset S_i(Y_i) \quad \text{for each } i.$$

In fact if $y_i \in \ker(S_i^{\nu_i}) \backslash S_i(Y_i), S_i$ can be represented as a $\nu_i \times \nu_i$ Jordan block matrix with ones on the leading subdiagonal and zeros elsewhere relative to the basis $\{y_i, S_i y_i, \ldots, S_i^{\nu - 1} y_i\}$. Clearly $n(S_i) = d(S_i) = j(S_i) = 1$ for each i. Further

$$\dim(X_1) = \sum_{i=1}^{j(T)} \dim(Y_i) = \sum_{i=1}^{j(T)} \nu_i,$$

and, since at least one ν_i equals ν, we have the following bounds on $\dim(X_1)$,

$$j(T) + \nu - 1 \le \dim(X_1) \le j(T)\nu.$$

Clearly any such direct sum of S_i's is a possible candidate for T_1.

We now turn to examine T_2 noting that, since $j(T_2) = 0$, we have $\mathbf{K}(T_2) \subset \mathbf{R}(T_2)$ by Proposition 3.

PROPOSITION 9. (i) *Let* $T \in \Phi_+(X)$ *then* $j(T) = 0 \Leftrightarrow n(T^k) = kn(T)$ *for each* k;

(ii) *let* $T \in \Phi_-(X)$ *then* $j(T) = 0 \Leftrightarrow d(T^k) = kd(T)$ *for each* k.

PROOF. (i). Let $T \in \Phi_+(X)$, since $j(T) = 0 \Leftrightarrow \mathbf{K}(T) \subset \mathbf{R}(T)$, we may restrict our attention to the restriction T_R of T to the Banach space $\mathbf{R}(T)$, and in this case $T_R \in \Phi(\mathbf{R}(T))$ with $d(T_R) = 0$ [**3**, Corollary 1.8].

Thus it suffices to consider a surjective operator $T \in \Phi(X)$. Then the sequence of mappings on the finite dimensional spaces which are restrictions of T,

$$\ker(T^k) \to \ker(T^{k-1}) \to \cdots \to \ker(T^2) \to \ker(T) \to (0)$$

are all surjective. For clearly

$$T(\ker(T^k)) \subset \ker(T^{k-1}),$$

and if $y \in \ker(T^{k-1})$, since T is surjective, $\exists x \in X$ such that $Tx = y$, but $T^k x = T^{k-1} y = 0$.

The kernel of each of these maps is $\ker(T)$, and using the fact that the sum of the dimensions of the image and kernel is the dimension of the whole space, we see that

$$n(T^k) = n(T^{k-1}) + n(T),$$

thus

$$n(T^k) = kn(T) \quad \text{for each } k.$$

Part (ii) follows at once by duality. □

Thus, reverting to our original situation of $T = T_1 \oplus T_2$ we see that, for $k \geq \nu$, the codimension of $\ker(T^{k-1})$ in $\ker(T^k)$ is constant. In fact we have

$$T = \sum_{i=1}^{j(T)} S_i \bigoplus T_2,$$

$$n(T) = \sum_{i=1}^{j(T)} N(S_i) + n(T_2),$$

$$n(T) = j(T) + n(T_2),$$

hence

$$n(T+\lambda) = n(T_2), \quad 0 < |\lambda| < \varepsilon.$$

Further, for $k \geq \nu$,

$$\begin{aligned} n(T^k) &= n(T_1^k) + n(T_2^k), \\ &= \dim(X_1) + kn(T_2). \end{aligned}$$

Thus, for $k \geq \nu$,

$$n(T^{k+1}) - n(T^k) = n(T_2) = n(T+\lambda), \quad 0 < |\lambda| < \varepsilon.$$

The corresponding results for the decreasing sequence $T^k(X)$, where $T \in \Phi_-(X)$, follow at once by duality. □

THEOREM 10. *Let $T \in \Phi_\pm(X)$ with ν the smallest positive integer such that*

$$\ker(T^\nu) \not\subset T(X).$$

Then, for $k \geq \nu$, for $0 < |\lambda| < \varepsilon$, ε sufficiently small,

$$n(T^{k+1}) - n(T^k) = n(T+\lambda) \quad \textit{for } T \in \Phi_+(X),$$

while

$$d(T^{k+1}) - d(T^k) = d(T+\lambda) \quad \textit{for } T \in \Phi_-(X).$$

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