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ON SUMS OF UNISERIAL MODULES

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Let R be any ring. Following [**2**] a module M_R is called a TAG-module if it satisfies the following two conditions.

(I) Every finitely generated submodule of a homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$ provided the composition length $\mathrm{d}(U/W) \leq \mathrm{d}(V/f(W))$.

A module M_R satisfying condition (I) is called a QTAG-module [**9**]. Through a number of papers it has been seen that the structure theory of these modules is similar to that of torsion abelian groups and that these modules occur over any ring. In this paper, in addition to further developing their structure theory, we give some applications of these modules to ring theory. In §2, Proposition 2.3 and Theorem 2.5 give some new characterizations of TAG-modules and QTAG-modules respectively. In §3 we determine when the class of QTAG modules over a ring R is closed under direct sums and use these results to give some characterizations of generalized uniserial rings (Theorem 3.5). Even if the class of QTAG-modules over a ring is closed under direct sums, it need not be closed under extensions. In §4 we determine when, over a commutative ring R, the class of TAG-modules is closed under extensions; these rings are precisely those which do not admit any homomorphic image which is a special ring in the sense defined by Shores [**6**, Theorem 4.9].

1. Preliminaries. All the rings considered here are with unity, and the modules are unital right modules unless otherwise stated. For any module M with finite composition length, $\mathrm{d}(M)$ denotes its (composition) length. For any module M_R, $\mathrm{J}(M)$ and $\mathrm{E}_R(M)$ (or

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simply E(M)) denote the Jacobson radical and the injective hull of M respectively. For the concepts like quasi-injective modules, right or left perfect rings, semi-perfect rings, etc., we refer to Stenström [**11**]. For definition and properties of M-injective modules, we refer to [**1**]. Further, $\operatorname{soc}(M)$ denotes the socle of M, and, for any $n \geq 0$, $\operatorname{soc}^n(M)$ is defined inductively as follows: $\operatorname{soc}^0(M) = 0$ and $\operatorname{soc}^{n+1}(M)/\operatorname{soc}^n(M) = \operatorname{soc}(M/\operatorname{soc}^n(M))$. For any subset X of M, $\operatorname{ann}(X)$ denotes the annihilator of X. A module M_R is said to be serial if its lattice of submodules is linearly ordered under inclusion, and if in addition it has finite length, it is said to be uniserial. For the basic concepts like height, exponent, basic submodules and others in a TAG-module (so in QTAG-modules) we refer to Singh [**7**] and [**8**] (see also [**10**]). For concepts in the theory of commutative rings, we refer to Larsen and McCarthy [**5**]. A commutative ring R is called a ZPI-ring if every ideal of R can be written as a product of prime ideals in R [**5**]; these are precisely the finite direct sums of Dedekind domains and special primary rings [**5**, Theorem 9.10]. For any module M over a commutative ring R, given a maximal ideal P of R, the P-primary component $M_{(P)}$ is $\{x \in M : xP^n = 0 \text{ for some } n \geq 0\}$. Any TAG-module over any commutative ring is the direct sum of its primary components. Finally $A \subset' B$ denotes that A is an essential submodule of the module B.

2. Some general results. It is obvious that any submodule of a homomorphic image of a QTAG-module is a QTAG-module. The following two lemmas were proved in [**10**].

LEMMA 2.1. *Let A and B be two uniserial submodules of a* QTAG-*module M such that $A \cap B = 0$. Let σ be any homomorphism from a submodule W of A into B such that* $\mathrm{d}(A/W) \leq \mathrm{d}(B/\sigma(W))$. *Then σ can be extended to a homomorphism $\bar{\sigma} : A \to B$.*

LEMMA 2.2. *Let A and B be any two uniserial submodules of a* QTAG-*module M such that $A \cap B \neq 0$ and* $\mathrm{d}(A) \leq \mathrm{d}(B)$. *Then there exists a monomorphism $\sigma : A \to B$ which is the identity on $A \cap B$.*

We now prove

PROPOSITION 2.3. *A* QTAG-*module* M_R *is a* TAG-*module if and only if any uniserial submodule of any homomorphic image of* M *is quasi-injective.*

PROOF. "Only if" is obvious. Conversely, let any uniserial submodule of any homomorphic image of M_R be quasi-injective. Let U and V be any two uniserial submodules of a homomorphic image of M and, for some submodule W of U, let $f : W \to V$ be a non-zero homomorphism such that $\mathrm{d}(U/W) \leq \mathrm{d}(V/f(W))$. We need to show that f can be extended to the homomorphism $\bar{f} : U \to V$. In view of Lemma 2.1 we may assume that $U \cap V \neq 0$. Let $\mathrm{d}(U) \leq \mathrm{d}(V)$. By Lemma 2.2, U embeds in V. As V is quasi-injective, V is U-injective. Hence, in this case, f can be extended to a homomorphism $\bar{f} : U \to V$. Now, let $\mathrm{d}(U) > \mathrm{d}(V)$. By Lemma 2.2 there exists an embedding $g : V \to U$. Now $g \circ f : W \to U$ can be extended to a homomorphism $j : U \to U$. Now

$$\begin{aligned} \ker(f) &= \ker(j), \\ \mathrm{d}(j(U)) &= \mathrm{d}(U/W) + \mathrm{d}(W/\ker f) \\ &\leq \mathrm{d}(V/f(W)) + \mathrm{d}(f(W)) \\ &= \mathrm{d}(V) = \mathrm{d}(g(V)). \end{aligned}$$

Thus $j(U) \subset g(V)$. We get $g^{-1} \circ j : U \to V$ where g^{-1} is defined on $g(V)$. This mapping extends f. Hence M is a TAG-module. □

COROLLARY 2.4. *Over commutative rings,* QTAG-*modules are* TAG-*modules.*

PROOF. Since any uniserial module over a commutative ring is quasi-injective, the result follows from Proposition 2.3. □

We now establish a characterization of QTAG-modules.

Theorem 2.5. *A module M_R is a* QTAG-*module if and only if it satisfies the following conditions.*

(i) *Any cyclic submodule of M is a sum of uniserial submodules.*

(ii) *For any pair U, V of uniserial submodules of a homomorphic image of M, $U + V$ is a direct sum of uniserial submodules.*

Proof. A QTAG-module clearly satisfies (i) and (ii). Let M be a module satisfying (ii), and let U, V be two uniserial submodules of a homomorphic image of M. Suppose that $d(U) \leq d(V)$. We first show that V is an absolute summand of $U + V$. Let K be a complement of V in $U + V$. Thus $(V \oplus K)/K \subset' (U + V)/K$. This shows that $(U + V)/K$ is uniform. Since $(U + V)/K$ is a sum of two uniserial submodules of M/K, by (ii), $(U + V)/K$ is uniserial. Further, since $\mathrm{d}[(U+K)/K] \leq \mathrm{d}[(V+K)/K]$ we get $(U+V)/K = (V+K)/K$. Hence $U+V = V \oplus K$. This proves that V is an absolute summand of $U+V$. We now show by induction that the sum of any n uniserial submodules of a homomorphic image of M is a direct sum of uniserial submodules. It follows from above that the sum of two uniserial submodules of a homomorphic image of M is a direct sum of (not more than two) uniserial modules. To apply induction let $n > 2$ and suppose that, given any sum $L = V_1 + V_2 + \cdots + V_{n-1}$ of $n - 1$ uniserial submodules of a homomorphic image of M, any V_i of largest composition length is a summand of L and that L is a direct sum of k uniserial submodules for some $k \leq n - 1$. Consider $N = U_1 + U_2 + \cdots + U_n$, where all the U_i's are uniserial submodules of a homomorphic image of M. Without loss of generality we may suppose that $\mathrm{d}(U_i) \leq \mathrm{d}(U_n)$ for $1 \leq i \leq n-1$. Then $U_{n-1} + U_n = K \oplus U_n$ and

$$N/K = \sum_{i \neq n-1} (U_i + K)/K.$$

So, by the induction hypothesis,

$$N/K = L/K \oplus [(U_n \oplus K)/K]$$

for some submodule L of N containing K. Then

$$N = L \oplus U_n.$$

Since L is a sum of $n-1$ uniserial submodules, the induction hypothesis gives that N is a direct sum of $\leq n$ uniserial submodules. Hence if, in addition, M satisfies (i), we get that any finitely generated submodule of a homomorphic image of M, being a sum of uniserial submodules, is a direct sum of uniserial submodules. Hence the result follows. □

We now give an example to show that conditions (i) and (ii) in Theorem 2.5 are independent of each other.

EXAMPLE 2.6. (i) does not imply (ii). Let R be any commutative local ring with $J = \mathrm{J}(R)$, $J^2 = 0$ and $\dim_{R/J} J > 1$. Now $J = \oplus \sum_{i \in \Gamma} U_i$, where U_i are minimal ideals of R. Notice that any cyclic R-module which is also uniform is uniserial and is of composition length ≤ 2. Let S be a simple R-module and $E = \mathrm{E}(S)$. As $J^2 = 0, E = \mathrm{soc}^2(E)$, and any cyclic submodule of E, being uniform, is uniserial. So E satisfies condition (i). For each $i \in \Gamma$ let $K_i = \sum_{j \neq i} U_j$. Then $J = U_i \oplus K_i, N_i = R/K_i$ are mutually non-isomorphic uniserial R-modules, each of length two. Since $S \approx \mathrm{soc}(N_i)$, there exists an embedding $\sigma_i : N_i \to E$. Consider any two distinct elements $i, j \in \Gamma$. Then $\sigma_i(N_i) + \sigma_j(N_j)$ is a uniform module, but, being a sum of two non-isomorphic uniserial modules each of length two, it is not a direct sum of uniserial modules. Hence E satisfies (i) but not (ii).

Any non-torsion, abelian group satisfies (ii) but not (i).

3. Direct sums. In general the class of TAG-modules and the class of QTAG-modules over a ring R need not be closed under direct sums. This is evident from Example 2.6 where the module $\mathrm{E}(S)$ is a sum of TAG-modules, but $\mathrm{E}(S)$ is not a TAG-module and is not even a QTAG-module because of Corollary 2.4.

PROPOSITION 3.1. *For any ring R, the class of* QTAG-*modules is closed under direct sums if and only if every finite length R-module is a direct sum of uniserial modules*; *in that case every* QTAG-*module over R is a* TAG-*module.*

Proof. Let the class of QTAG-modules over R be closed under direct sums. Then the sum of QTAG-submodules of an R-module is a QTAG-module. Let M_R be any finite length module. Consider the case when M is uniform. If M is not uniserial, there exists a positive integer k such that $\operatorname{soc}^k(M)$ is uniserial but $\operatorname{soc}^{k+1}(M)$ is not uniserial. We can find $x, y \in \operatorname{soc}^{k+1}(M)$ such that xR and yR are simple modulo $\operatorname{soc}^k(M)$ and neither of them contains the other. Then $xR \cap yR = \operatorname{soc}^k(M)$. So $xR + yR$ is not a QTAG-module, however each of xR and yR being uniserial is a QTAG-module. This gives a contradiction. Hence M is uniserial. In general M is a subdirect sum of finitely many finite length uniform (hence QTAG)-modules. Consequently M is a QTAG-module, and, by the definition, M is a finite direct sum of uniserial modules.

The converse is obvious, since, given any two QTAG-modules M_R and N_R, any finitely generated submodule of $M \oplus N$ is of finite length. □

A ring R, over which the class of TAG (QTAG)-modules is closed under direct sums, is called a TAG (respectively QTAG)-ring. Over a commutative ring these concepts are the same. If a commutative ring R has a maximal ideal P such that $P/P^2 \neq 0$ and is not simple as an R-module, we can easily find an ideal A such that $P^2 < A < P$, R/A is of finite length, but R/A is not uniserial. Then R/A is not a TAG-module. Using this, the above proposition gives

Corollary 3.2. *A commutative ring R is a* TAG-*ring if and only if, for any maximal ideal P, there is no ideal between P and P^2.*

The following example shows that a TAG-ring need not be a QTAG-ring.

Example 3.3. Let R be a local ring such that its Jacobson radical J has the properties, $J^2 = 0$, $\dim(J_{R/J}) = 1$ and $\dim({}_{R/J}J) > 1$. Then R_R is not quasi-injective, and hence R_R is not a TAG-module. However R_R is a QTAG-module. Since, by Proposition 3.1, any uniserial module over a QTAG-ring is quasi-injective, we get that R is not a QTAG-ring. However it can be easily seen that, over R, the TAG-modules are precisely the completely reducible modules. So the class of TAG-modules over R is closed under direct sums and hence R is a TAG-ring.

LEMMA 3.4. *Let R be any semi-primary* QTAG-*ring. Then R is a generalized uniserial ring.*

PROOF. Now $R = e_1R \oplus e_2R \oplus \cdots \oplus e_tR$ for some orthogonal, indecomposable idempotents $e_1, e_2, \ldots, e_t$. As R is semi-primary, each e_iR is a local module. Let $J = \mathrm{J}(R)$. Suppose that, for some $i, e_iJ \neq 0$ and e_iJ/e_iJ^2 is not simple. We can find a right ideal A such that $e_iJ^2 \subset A \subset e_iJ$ and e_iJ/A is a direct sum of two simple modules. Then $M = e_iR/A$ is a finite length module, which is not a direct sum of uniserial modules. This is a contradiction. Thus, for each $i, e_iJ/e_iJ^2$ is either zero or simple. This in turn yields that each e_iR is a uniserial module. Consequently R_R is a TAG-module, and hence so is every finitely generated R-module. Hence, by [**4**, Theorem 1.3], R is a generalized uniserial ring. (See also [**9**, Theorem 4.1]).

THEOREM 3.5. *The following are equivalent for a ring R:*

(a) *R is generalized uniserial.*

(b) *R is a left perfect right* QTAG-*ring.*

(c) *R is a right perfect right* QTAG-*ring.*

(d) *R is a semi-perfect* QTAG-*ring with* $\mathrm{J}(R)$ *a* TAG-*module.*

PROOF. That (a) implies (b), (c) and (d) is obvious. Let R satisfy (b) or (c). Let $J = \mathrm{J}(R)$. For any positive integer n, if $J^n \neq 0$, then $J^n \neq J^{n+1}$. We show that J is nilpotent. Let $J^n \neq 0$ for every n, and $A = \cap_n J^n$. Write $R = e_1R \oplus e_2R \oplus \cdots \oplus e_tR$ for some orthogonal, indecomposable idempotents e_i. Then $A = \sum e_iA$ with $e_iA = \cap_n e_iJ^n$. Consider the ring $\bar{R} = R/A$. $\bar{R}$ is a QTAG-ring which is left or right perfect and is not generalized uniserial. So, without loss of generality, we take $A = 0$. There exists at least one i such that $e_iJ^n \neq e_iJ^{n+1}$ for every n. By Lemma 3.4, e_iR/e_iJ^n is a uniserial module of length n. Write

$$\mathrm{soc}(e_iR) = A_n \oplus [\,\mathrm{soc}(e_iR) \cap e_iJ^n].$$

Suppose for some value of n, say, for $n = k$, we have $A_k \neq 0$. Since e_iR/e_iJ^k is uniserial, we get that $\operatorname{soc}(e_iR/e_iJ^k)$ is a simple module. Consequently A_k is a minimal right ideal contained in e_iR and $A_k \not\subset e_iJ^k$. Then $(A_k + e_iJ^{k+1})/e_iJ^{k+1} = \operatorname{soc}(e_iR/e_iJ^{k+1}) = e_iJ^k/e_iJ^{k+1}$. Consequently $A_k \subset e_iJ^k$. This is a contradiction. Hence $\operatorname{soc}(e_iR) \subset \cap_n e_iJ^n = 0$. Similarly, for some j, $\operatorname{soc}(Re_j) = 0$. Since R is left or right perfect, either every non-zero right R-module or every non-zero left R-module has non-zero socle. This gives a contradiction. Hence each of (b) and (c) implies (a). Let (d) hold. Now $\mathrm{J}(R)$ is semi-artinian and, by [**11**; Chapter VIII, Property 2.6], $J = J(R)$ is T-nilpotent on the right R-module J. Consequently, J itself is left T-nilpotent. Hence, by [**11**; Chapter VIII, Property 5.1], R is left perfect. Hence (d) implies (b). □

It is obvious from Corollary 3.2 that any Prüfer domain is a QTAG-ring. Using [**5**, Theorem 9.10] it follows that any noetherian commutative TAG-ring is a ZPI-ring. However the structure of non-commutative QTAG-rings, even those satisfying the ascending chain condition, is not yet clear.

4. Extensions. In general the class of QTAG-modules over a ring R is not closed under extensions. For example the ring R in Example 2.6 as an R-module is an extension of $\mathrm{J}(R)$ by $R/\mathrm{J}(R)$, both of which are QTAG-modules. However R_R is not a QTAG-module. In general the structure of rings over which the class of QTAG-modules is closed under extensions is not known. In this section we determine which commutative rings have this property. As defined by Shores [**6**] a commutative ring R is called a special ring if R is a non-noetherian semi-prime ring such that $\operatorname{soc}(R)$ is a maximal ideal. In Theorem 4.9 we show that the class of QTAG-modules over a commutative ring R is closed under extensions if and only if R is a QTAG-ring having no homomorphic image a special ring. We start with some general results.

LEMMA 4.1. *Let R be a* QTAG-*ring and let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules such that A and C are* QTAG-*modules. Then the following are equivalent*:

(1) *B is a* QTAG-*module.*

(2) *For any finitely generated submodule K of B, $A \cap K$ is finitely generated.*

(3) *For any cyclic submodule xR of B, $xR \cap A$ is finitely generated.*

(4) *For any cyclic submodule xR of B, $xR + A$ is a* TAG-*module.*

The lemma follows from the fact that any finite length R-module is a QTAG-module by Proposition 3.1.

LEMMA 4.2. *Let R be a* QTAG-*ring, and let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules, where A and C are* QTAG-*modules, but B is not a* QTAG-*module. There exists a cyclic extension yR in B of a non-zero submodule A' of A such that yR is not a* QTAG-*module, yR/A' is uniserial and A' is not finitely generated. Further, there exists a cyclic extension zR of A' by a uniserial module such that A' is essential in zR and zR is not a* QTAG-*module.*

PROOF. By Lemma 4.1 there exists $x \in B$ such that $xR + A$ is not a QTAG-module. Clearly xR is not a QTAG-module. Since $xR/xR \cap A$ is a QTAG-module,

$$xR/xR \cap A = \oplus \sum_{i=1}^{n} \bar{x}_i R$$

for some uniserial submodules $\bar{x}_i R$. At least one of the $x_i R$, say $x_1 R$, is not a QTAG-module. Clearly $x_1 R$ is an extension of $x_1 R \cap A$ and $x_1 R / x_1 R \cap A$ is uniserial. Further, if $x_1 R \cap A$ is finitely generated, we get $x_1 R$ is of finite length and that in turn gives $x_1 R$, a QTAG-module; this leads to a contradiction. Hence $x_1 R \cap A$ is not finitely generated. Let K be a complement of $A' = x_1 R \cap A$ in $x_1 R$. Then $x_1 R/K$ is a desired extension of A'. □

Henceforth we consider commutative rings, for which the concepts of TAG-modules and QTAG-modules (and also the concepts of TAG-rings and QTAG-rings) are the same.

LEMMA 4.3. *Let R be a commutative* TAG-*ring and let* $0 \to A \to B \to C \to 0$ *be an exact sequence of R-modules, where C is any* TAG-*module and A is a serial* TAG-*module. Then B is a* TAG-*module.*

PROOF. If A is uniserial (more generally of finite length), by Lemma 4.1, B is a TAG-module. Thus we may suppose that A is of infinite length. In other words A is h-divisible. Let B not be a TAG-module. By Lemma 4.2 there exists $x \in B$ such that xR is not a TAG-module and $xR/xR \cap A$ is uniserial. As $xR \cap A$ is not finitely generated, we get $xR \cap A = A$. Hence $A \subset xR$. By Lemma 4.2 there exists a cyclic, essential extension zR of A such that zR/A is uniserial. If, for some $u \in zR \backslash A, uR \cap A$ is finitely generated, by Proposition 3.1, uR is a TAG-module. Consequently $uR + A$ is a TAG-module. As A is h-divisible, by [**10**, Remark 3.6], A is a summand of $uR + A$. This contradicts the fact that $A \subset' zR$. Hence $A \subset uR$. Consequently zR is a serial module. So without loss of generality we can take zR/A to be a simple module. Let $I = \operatorname{ann}(z)$. Then $zR \approx R/I$ and R/I is a valuation ring with maximal ideal M/I. Since A is h-divisible, $A(M/I) = A$. Further, $A \approx M/I$ as R/I-modules. However A contains a uniserial submodule K of length 2. Clearly $K(M/I) \neq 0$ while $K(M/I)^2 = 0$. Thus $M/I \neq (M/I)^2$ and M/I is a principal ideal. This in turn gives that A is cyclic. This is a contradiction. □

COROLLARY 4.4. *Let R be a commutative* TAG-*ring and let* $0 \to A \to B \to C \to 0$ *be an exact sequence of R-modules, where A and C are* TAG-*modules. Then any finite rank h-divisible submodule of A is a summand of B.*

PROOF. Let D be a finite rank h-divisible submodule of A. Write $D = D_1 \oplus D_2 \oplus \cdots \oplus D_n$, where D_i are infinite length serial modules. For $n = 1, D = D_1$ and $A = A_1 \oplus D$. Since $(B/A_1)/(A/A_1) \approx C$, B/A_1 is an extension of the serial TAG-module D by the TAG-module C. Consequently, by Lemma 4.3, B/A_1 is a TAG-module. As A/A_1 is h-divisible, we get

$$B/A_1 = (A/A_1) \oplus (E/A_1)$$

for some submodule E of B containing A_1. This in turn gives $B = D \oplus E$, since $A = D \oplus A_1$. Hence the result holds for $n = 1$. To apply induction, let $n > 1$ and the result hold for $n-1$. So

$$B = (D_1 \oplus D_2 \oplus \cdots \oplus D_{n-1}) \oplus B'$$

for some submodule B' of B. Then

$$A = (D_1 \oplus D_2 \oplus \cdots \oplus D_{n-1}) \oplus (A \cap B'),$$
$$D = (D_1 \oplus D_2 \oplus \cdots \oplus D_{n-1}) \oplus (D \cap B').$$

Now $D \cap B'$ is a rank one h-divisible submodule of $A \cap B'$. Further, B' is an extension of $A \cap B'$ by C, so, by the case when the rank is one, $D \cap B'$ is a summand of B'. Consequently D is a summand of B. □

THEOREM 4.5. *Let R be a commutative* TAG-*ring having a maximal ideal M which is a* TAG-*module. Then*:

(1) *For any maximal ideal $P \neq M$, the P-primary component of M is a summand of R.*

(2) *Every primary component of M is uniserial.*

(3) *For the prime radical N of R, R/N is a Von-Neumann regular ring.*

PROOF. Let P be a maximal ideal of R. First we show that the P-primary component $M_{(P)}$ is bounded. Let the contrary hold. Then M would have a non-zero P-primary, h-divisible, serial homomorphic image, say M/K. By (4.4), M/K is a summand of R/K, and hence M/K is finitely generated. This is a contradiction. Thus $M_{(P)}$ is bounded. So there exists a positive integer n such that $M_{(P)} \cdot P^n = 0$. Now $M = M_{(P)} \oplus L_P$, where $L_P = \sum_{Q \neq P} M_{(Q)}, Q \in \max(R)$, the set of all maximal ideals of R. Then $MP^n = L_P$ gives $L_P \subset P^n$. Let $P \neq M$. Then $R = M + P^n = M_{(P)} + P^n = M_{(P)} \oplus P^n$, since $M_{(P)} \cap P^n = M_{(P)}P^n = 0$. This proves (1). Since R/P^n is uniserial, we get that $M_{(P)}$ is uniserial. Also, $(M_{(P)} \cap P)^n = 0$ gives $M_{(P)} \cap P \subset N$, the prime radical of R. Let $P = M$. Then $M^{n+1} = L_M$ and $M_{(M)}^{n+1} = 0$. Consequently $M_{(M)} \subset N$. In this case R/M^{n+1} has maximal ideal M/M^{n+1}, which is isomorphic to $M_{(M)}$ as an R-module. Consequently $M_{(M)}$ is also uniserial. This proves (2).

It follows from the above paragraph that

$$K = M_{(M)} \oplus \sum_{P \neq M} M_{(P)} \cap P \subset N.$$

Now

$$M/K \approx \oplus \sum_{P \neq M} M_{(P)}/(M_{(P)} \cap P).$$

However, for $P \neq M$ in $\max(R)$, $R = M_{(P)} + P$ gives $R/P \approx M_{(P)}/(M_{(P)} \cap P)$. Hence M/K is a direct sum of fields. Consequently R/K is a Von-Neumann regular ring. Hence $K = N$ and (3) follows. □

REMARK 4.6. The examples of rings considered in the above theorem can be easily constructed. Consider any prime number p and any family $\{z/\langle p^{n_i}\rangle : i \in I\}$ of rings, such that $\sup\{n_i : i \in I\} = n$, a positive integer. Consider the ring

$$S = \oplus \sum Z/\langle p^{n_i}\rangle.$$

S is also a $Z/\langle p^n\rangle$-module. So we can form the ring

$$R = Z/\langle p^n\rangle \times S$$

in which addition is defined component-wise and the multiplication is given by

$$(\bar{k}, x)(\bar{l}, y) = (\bar{k}\bar{l}, ky + lx + xy).$$

R satisfies the hypothesis of Theorem 4.5.

THEOREM 4.7. *Let A be a* TAG-*module with finitely many nonzero primary components, over a commutative* TAG-*ring R. Then any extension of A by a* TAG-*module is a* TAG-*module.*

PROOF. Let B be an extension of A such that B/A is a TAG-module. If B is not a TAG-module, by Lemma 4.2, there exists a cyclic essential extension xR of a submodule A' of A such that xR/A' is uniserial and xR is not a TAG-module. Let $K_i, i = 0, 1, 2, \ldots, n$, be the finite chain of all submodules of xR containing A' with $K_0 = A'$ and $K_n = xR$. Let K_j be the largest, among the K_i's, which is a TAG-module. Clearly $j < n$. Since A has only finitely many primary components, so has K_j. Now K_{j+1}/K_j is simple. Let $y \in K_{j+1} \backslash K_j$. Then $yR + K_j = K_{j+1}$ and yR is not a TAG-module. Let $I = \operatorname{ann}(y)$. Then R/I is a TAG-ring. It has a maximal ideal isomorphic to $yR \cap K_j$. Since $yR \cap K_j$ is a TAG-module, by Theorem 4.5, every primary component of $yR \cap K_j$ is uniserial. However $yR \cap K_j$ has only finitely many primary components. Consequently $yR \cap K_j$ is finitely generated. Hence, by Lemma 4.1, yR is a TAG-module. This is a contradiction. □

Finally we determine the structure of a commutative ring R over which the class of TAG-modules is closed under extensions; such a ring is called a strongly TAG-ring. We start with the following lemma, which can be easily proved.

LEMMA 4.8. *Let R be a commutative* TAG-*ring. Then the following conditions are equivalent.*

(1) *R is a strongly* TAG-*ring.*

(2) *Any cyclic extension of a* TAG-*module over R by a* TAG-*module over R is a* TAG-*module.*

(3) *Any cyclic extension of a* TAG-*module over R by a simple R-module is a* TAG-*module.*

THEOREM 4.9. *A commutative ring R is a strongly* TAG-*ring if and only if it is a* TAG-*ring admitting no special ring as a homomorphic image.*

PROOF. Notice that if R is a special ring then R is Von Neumann regular. So R is a TAG-ring. R_R is an extension of the TAG-module $\operatorname{soc}(R)$ by the TAG-module $R/\operatorname{soc}(R)$, however R_R itself is not a TAG-module. So a special ring is not a strongly TAG-ring. Consequently if R is a strongly TAG-ring, it has no homomorphic image which is a special ring.

Conversely, let R be a TAG-ring having no homomorphic image a special ring. Let R be not a strongly TAG-ring. By Lemma 4.8, there exists a cyclic module xR having a maximal submodule K which is a TAG-module, but xR is not a TAG-module. Let $I = \text{ann}(x)$. Then $S = R/I$ is a TAG-ring having a maximal ideal M which is a TAG-module. Then M is not finitely generated and, by Theorem 4.5, for any maximal ideal P of S, the P-primary component $M_{(P)}$ of M is uniserial; if, further, $P \neq M$, then $M_{(P)}$ is a summand of R. If N is the prime radical of S, then, by Remark 4.6, S/N is a special ring. Obviously S/N is a homomorphic image of R. This is a contradiction. □

There exist many Prüfer domains each of whose non-zero element is contained in only finitely many maximal ideals (see [**3**]); all such Prüfer domains are strongly TAG-rings.

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