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TOPICS ON HOLOMORPHIC CORRESPONDENCES[1]

KARL STEIN

Correspondences, i.e. set valued mappings, occur in a natural way in the field of complex function theory. First of all multivalued functions arise from single valued functions via analytic continuation. Another phenomenon is that meromorphic functions of more than one variable can have points of indeterminacy. As an example, consider the meromorphic function z_1/z_2 on the space C^2 of two complex variables z_1, z_2. Outside the origin of C^2 this defines a mapping into the extended complex plane $\bar{C}$. But for each value in $\bar{C}$ there is a complex line through the origin on which z_1/z_2 is constantly equal to that value. Hence it is natural to assign $\bar{C}$ as the "value" of z_1/z_2 at the origin. In this way we obtain a correspondence which even has the additional property that its graph is an analytic set in $C^2 \times \bar{C}$.

This leads to the notion of holomorphic correspondences. The purpose of the present note is to deal with this concept from several points of view.

First we discuss some definitions of continuity and other topological aspects. Next we consider holomorphic correspondences and their extensions over analytic exceptional sets. Another method of extending holomorphic correspondences is to apply a reflection principle. This is described in the last section. We sketch a new proof of a theorem first shown by Tornehave and discuss applications and generalizations.

1. Topological aspects.

1.1. Let $X, Y, \cdots$ be sets. A *correspondence* from X to Y is a triple $f = (X, G, Y)$ where G, called the *graph* of f and denoted also G_f, is a subset of $X \times Y$. $f = (X, G, Y)$ assigns to any $x \in X$ the subset $f(x) := \{y \in Y : (x, y) \in G\}$ of Y; conversely, if to every $x \in X$ a subset M_x of Y is assigned, then there is a unique correspondence f from X to Y such that $M_x = f(x)$, one has $G_f = \{(x, y) \in X \times Y : y \in M_x\}$. For $A \subset X$ we set $f(A) := \bigcup_{x \in A} f(x)$. f is called empty if $G_f = \emptyset$ or, equivalently, if $f(X) = \emptyset$.

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Instead of $f = (X, G, Y)$ we write $f: X \multimap Y$ in the following. A mapping $\varphi : X \to Y$ is looked upon as a special correspondence from X to Y (we do not distinguish between a set consisting of one element and the element, hence we identify $\varphi(x) \in Y$ and $\{\varphi(x)\} \subset Y$ for $x \in X$). The projection maps of the graph G_f of $f: X \multimap Y$ into X and Y are denoted by $\check{f}: G_f \to X$ and $\hat{f}: G_f \to Y$, then $f(x) = \hat{f}(\check{f}^{-1}(x))$.

The *composition* of two correspondences $f: X \multimap Y$ and $g : Y \multimap Z$ is the correspondence $g \circ f: X \multimap Z$ defined by $(g \circ f)(x) := g(f(x))$. We have the rules $f \circ I_X = f$, $I_Y \circ f = f$, $h \circ (g \circ f) = (h \circ g) \circ f$ (I_X, I_Y denote the identity maps of X, Y). Hence the classes of sets and correspondences of sets form a category, we denote it $\mathfrak{S}$.

For subsets $A \subset X$ and $B \subset Y$ the *restriction* $f \mid A, B : A \multimap B$ of $f : X \multimap Y$ is defined by setting $G_{f|A,B} := G_f \cap (A \times B)$. We have $f \circ I_X{}^A = I_Y{}^B \circ f \mid A, B$ ($I_X{}^A : A \to X$ and $I_Y{}^B : B \to Y$ denote the inclusion maps). If $B = Y$ we write $f \mid A$ instead of $f \mid A, Y$.

$f : X \multimap Y$ is said to be *contained* in $'f$: $'X \multimap {}'Y$ if $X \subset {}'X$, $Y \subset {}'Y$ and $G_f \subset G_{'f}$; we then write $f \subset {}'f$.

The *union* and the *intersection* of correspondences $f_j : X \multimap Y$ ($j \in \mathfrak{J}$; $\mathfrak{J}$ a set), written $\bigcup_j f_j$ and $\bigcap_j f_j$ resp., are defined as the correspondences whose graphs are $\bigcup_j G_{f_j}$ and $\bigcap_j G_{f_j}$ resp.

To every correspondence $f : X \multimap Y$ there is associated the *reciprocal correspondence* $f^{-1} : Y \multimap X$ whose graph is $G_{f^{-1}} := \{(y, x) \in Y \times X : (x, y) \in G_f\}$. One has the rules $(f^{-1})^{-1} = f$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

The *cartesian product* $\prod_j f_j : \prod_j X_j \multimap \prod_j Y_j$ of correspondences $f_j : X_j \multimap Y_j$ ($j \in \mathfrak{J}$) is defined by assigning to any element $\{x_j\}_{j \in \mathfrak{J}}$ ($x_j \in X_j$) of the cartesian product $\prod_j X_j$ the set $\prod_j f_j(x_j) \subset \prod_j Y_j$. If $\mathfrak{J} = \{1, \cdots, n\}$ we write

$$f_1 \times \cdots \times f_n : X_1 \times \cdots \times X_n \multimap Y_1 \times \cdots \times Y_n.$$

We have the rule $(\prod_j f_j)^{-1} = \prod_j f_j{}^{-1}$.

Remark. In general, the reciprocal correspondence $f^{-1} : Y \multimap X$ of $f: X \multimap Y$ is not an inverse of f in $\mathfrak{S}$ in the categorical sense: $f^{-1} \circ f$, $f \circ f^{-1}$ are generally not the identity maps of X, Y (but if $f: X \to Y$ is a surjective map we have $f \circ f^{-1} = I_Y$).

Furthermore, the cartesian product of sets is generally not a direct product in the category $\mathfrak{S}$: If correspondences $q_j : Z \multimap X_j$ ($j \in \mathfrak{J}$) are given, then there is in general not a unique correspondence $\alpha : Z \multimap \prod_j X_j$ such that $q_j = p_j \circ \alpha$ for any $j \in \mathfrak{J}$ ($p_i : \prod_j X_j \to X_i$ denotes the projection map). For instance, if there is an element $z \in Z$ such that $q_{j_1}(z) = \emptyset$ but $q_{j_2}(z) \neq \emptyset$ ($j_1, j_2 \in \mathfrak{J}$), then an

$\alpha : Z \multimap \prod_j X_j$ with $q_j = p_j \circ \alpha$ cannot exist. Nevertheless, the category $\mathfrak{S}$ admits (arbitrary) direct products. The disjoint union $\coprod_j X_j$ together with the projection correspondences

$$'p_i : \coprod_j X_j \multimap X_i \qquad (i \in \mathfrak{J})$$

defined by

$$'p_i(x) := x \quad \text{if } x \in X_i \quad \text{and} \quad 'p_i(x) := \emptyset \quad \text{if } x \notin X_i$$

is obviously a direct product of the X_j in $\mathfrak{S}$. Hence one has in $\mathfrak{S}$ also direct products of morphisms.

1.2. Let $X, Y, \cdots$ now denote topological spaces.

A correspondence $f: X \multimap Y$ is said to be *weakly c-continuous at* $x \in X$ if, given a neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$. f is called *weakly c-continuous* if f is weakly c-continuous at any $x \in X$.[2]

PROPOSITION 1.2.1. *$f: X \multimap Y$ is weakly c-continuous if and only if f^{-1} is closed (in the sense that the images of closed sets are closed).*

PROOF. (a) Let f be weakly c-continuous, assume that f^{-1} is not closed. Choose a closed set $N \subset Y$ such that $f^{-1}(N)$ is not closed, then there exists a point $x \in \overline{f^{-1}(N)} \cap (X - f^{-1}(N))$. From $x \in X - f^{-1}(N)$ it follows $f(x) \subset Y - N$; hence $Y - N$ is a neighborhood of $f(x)$. Since f is weakly c-continuous at x, there is a neighborhood U of x such that $f(U) \subset Y - N$. Hence $U \subset X - f^{-1}(N)$, but this contradicts $x \in \overline{f^{-1}(N)}$. (b) Assume that f^{-1} is closed. Choose some point $x \in X$ and an open neighborhood V of $f(x)$. Then $f^{-1}(X - V)$ is closed and $x \notin f^{-1}(X - V)$. Therefore $U := X - f^{-1}(X - V)$ is a neighborhood of x such that $f(U) \subset V$. Hence f is weakly c-continuous at x.

PROPOSITION 1.2.2. *Let $f: X \multimap Y$, $g : Y \multimap Z$ be weakly c-continuous. Then $g \circ f: X \multimap Z$ is weakly c-continuous.*

PROOF. Let N be a closed set in Z. Then $(g \circ f)^{-1}(N) = (f^{-1} \circ g^{-1})(N) = f^{-1}(g^{-1}(N))$ is closed in X, hence $(g \circ f)^{-1}$ is closed.

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[2]There are many versions of the concept of continuity for correspondences in the literature and these notions are denoted very differently. Compare e.g. [4], [8], [10], [13], [16], [25], [26], [30]. Instead of *weakly c-continuous* the notation *upper semicontinuous (u.s.c.)* is often used, other notations are *upper continuous, continuous.*

The cartesian product of two weakly c-continuous correspondences need not be weakly c-continuous. This is shown by the following example: Choose $X := \{0\}$ $(0 \in \boldsymbol{R})$, $Y := [0,1) \subset \boldsymbol{R}$; let $f: X \multimap Y$ be the correspondence such that $f(0) = Y$. Then $f^{-1}: Y \to X$ is closed, but $(f \times I_Y)^{-1} = f^{-1} \times I_Y : Y \times Y \to X \times Y$ is not closed: The set $D := \{(y,y') \in Y \times Y = [0,1) \times [0,1) : y + y' \geqq 1\}$ is closed in $Y \times Y$, but $(f^{-1} \times I_Y)(D) = \{0\} \times (0,1)$ is not closed in $\{0\} \times [0,1)$. Hence f is weakly c-continuous and $f \times I_Y$ is not.

With regard to this example we define (in analogy with Bourbaki's definition of proper maps, cf. [**6**]):

The correspondence $f: X \multimap Y$ is called *c-continuous* if the correspondence $f \times I_W : X \times W \multimap Y \times W$ is weakly c-continuous, for every topological space W.[3]

A c-continuous correspondence f is particularly weakly c-continuous (viz. choose in the definition above W as a topological space consisting of one point). A continuous map is obviously a c-continuous correspondence.

PROPOSITION 1.2.3. *Let $f: X \multimap Y$, $g: Y \multimap Z$ be c-continuous. Then $g \circ f: X \multimap Z$ is c-continuous.*

PROOF. We have

$$(g \circ f) \times I_W = (g \times I_W) \circ (f \times I_W)$$

for any topological space W, hence $(g \circ f) \times I_W$ is weakly c-continuous by Proposition 1.2.2.

PROPOSITION 1.2.4. *Let $f: X \multimap Y$, $f' : X' \multimap Y'$ be c-continuous. Then $f \times f' : X \times X' \multimap Y \times Y'$ is c-continuous.*

PROOF. We have

$$f \times f' = (f \times I_{Y'}) \circ (I_X \times f'),$$

hence $f \times f'$ is c-continuous by Proposition 1.2.3.

Propositions 1.2.3 and 1.2.4 imply that the classes of topological spaces and of c-continuous correspondences form a category which contains the category $\mathfrak{T}$ of topological spaces and continuous mappings and which has the property that finite cartesian products of morphisms (in the sense defined above) are again morphisms. We denote this category by $\mathfrak{T}_c$. It can be easily shown that $\mathfrak{T}_c$ even admits arbitrary cartesian products.

[3]Such correspondences (equivalently defined) are called *semi-continue supérieurement* in [**4**]. In [**22**] the notation *continuous* was used. Compare also [**31**], [**23**], [**24**].

PROPOSITION 1.2.5. *$f: X \multimap Y$ is c-continuous if and only if $\check{f}^{-1}: X \multimap G_f$ is c-continuous.*

PROOF. Assume first that $\check{f}^{-1}$ is c-continuous. One has $f = \hat{f} \circ \check{f}^{-1}$. Now $\hat{f}: G_f \to Y$ is a c-continuous correspondence since it is a continuous mapping, hence f is c-continuous because of Proposition 1.2.3.

Assume now that f is c-continuous. We have to show that $(\check{f}^{-1} \times I_Z)^{-1} = \check{f} \times I_Z$ is closed for any topological space Z. Let N be a closed set in $G_f \times Z$, then there is a closed subset $\tilde{N} \subset X \times Y \times Z$ such that $\tilde{N} \cap (G_f \times Z) = N$. Hence $N = (I_{X \times Y \times Z}^{G_f \times Z})^{-1}(\tilde{N})$ ($I_{X \times Y \times Z}^{G_f \times Z}: G_f \times Z \to X \times Y \times Z$ denotes the inclusion map) and $(\check{f} \times I_Z)(N) = (I_{X \times Y \times Z}^{G_f \times Z} \circ (\check{f} \times I_Z)^{-1})^{-1}(\tilde{N})$.

Thus it is enough to show that $(I_{X \times Y \times Z}^{G_f \times Z} \circ (\check{f} \times I_Z)^{-1})^{-1}$ is closed or, equivalently, that $I_{X \times Y \times Z}^{G_f \times Z} \circ (\check{f} \times I_Z)^{-1}$ is weakly c-continuous. One has the identity

$$I_{X \times Y \times Z}^{G_f \times Z} \circ (\check{f} \times I_Z)^{-1} = (P_X^{X \times Z} \times f \times I_Z) \circ \Delta_{X \times Z}$$

where $P_X^{X \times Z}: X \times Z \to X$ denotes the projection map and $\Delta_{X \times Z}: X \times Z \to X \times Z \times X \times Z$ the diagonal map. Now $P_X^{X \times Z}$ and $f \times I_Z$, by Proposition 1.2.4, are c-continuous; hence $P_X^{X \times Z} \times f \times I_Z$ is c-continuous by Proposition 1.2.4; thus it is weakly c-continuous. Since $\Delta_{X \times Z}$ is also weakly c-continuous it follows (Proposition 1.2.2) that $(P_X^{X \times Z} \times f \times I_Z) \circ \Delta_{X \times Z}$ is weakly c-continuous.

The condition "$\check{f}^{-1}: X \multimap G_f$ is a c-continuous correspondence" means that $\check{f}: G_f \to X$ is a proper map. Recall that a continuous map $\varphi: Z \to Z'$ is proper if and only if φ is closed and $\varphi^{-1}(z')$ is quasicompact for any $z' \in Z'$. Furthermore, if φ is proper then $\varphi^{-1}(K')$ is quasicompact for any quasicompact set $K' \subset Z'$. We conclude

PROPOSITION 1.2.6. *A weakly c-continuous correspondence $f: X \multimap Y$ is c-continuous if and only if $f(x)$ is quasicompact for any $x \in X$.*

PROOF. Let f be c-continuous, then $\check{f}: G_f \to X$ is a proper map and $\check{f}^{-1}(x)$ is quasicompact for any $x \in X$. Since $f(x)$ is homeomorphic to $\check{f}^{-1}(x)$, it is quasicompact. Assume now that $f(x)$ is quasicompact for any $x \in X$, then $\check{f}^{-1}(x)$ is quasicompact. Let $W \supset \check{f}^{-1}(x)$ be open in G_f. We can cover $\check{f}^{-1}(x)$ by a finite number of sets of the form $(U_j \times V_j) \cap G_f$ with U_j open in X, V_j open in Y, $x \in U_j$ and $(U_j \times V_j) \cap G_f \subset W$. Put $V := \bigcup_j V_j$, then $f(x) \subset V$. Since f is weakly c-continuous at x, there is a neighborhood U' of x such that $f(U') \subset V$. Put $U := (\bigcap_j U_j) \cap U'$, then $\check{f}^{-1}(U) \subset W$. Thus $\check{f}^{-1}$ is weakly c-continuous in x and, since $x \in X$ is arbitrary, it is weakly c-continuous. Therefore $\check{f}: G_f \to X$ is a proper map which means that

$\check{f}^{-1}: X \multimap Y$ is c-continuous. Hence, by Proposition 1.2.5, f is c-continuous.

A correspondence $f: X \multimap Y$ is called *c-continuous at* $x \in X$ if f is weakly c-continuous at x and $f(x)$ is quasicompact (cf. [31]). Proposition 1.2.6 implies

PROPOSITION 1.2.7. *$f: X \multimap Y$ is c-continuous if and only if f is c-continuous at any $x \in X$.*

REMARK. The c-continuity of $f: X \multimap Y$ at $x \in X$ can also be characterized by the following property: Let $(x_j)_{j \in J}$ and $(y_j)_{j \in J}$ (J a directed set) be nets in X and Y resp. such that $(x_j)_{j \in J}$ converges to x and $y_j \in f(x_j)$ for every $j \in J$; then $(y_j)_{j \in J}$ has a point of accumulation on $f(x)$.

PROPOSITION 1.2.8. *Let $f: X \multimap Y$ be c-continuous. Then $f(K)$ is quasicompact for any quasicompact set $K \subset X$.*

PROOF. Given a covering of $f(K)$ by open sets V_j; then, for each $x \in K$, there are finitely many V_j which cover $f(x)$ since $f(x)$ is quasicompact. Let $\tilde{V}_x$ be the union of those V_j. f is particularly weakly c-continuous at x, hence there is a neighborhood U_x of x such that $f(U_x) \subset \tilde{V}_x$. Now finitely many U_x, say $U_{x_1}, \cdots, U_{x_n}$ cover K, hence $\tilde{V}_{x_1} \cup \cdots \cup \tilde{V}_{x_n} \supset f(K)$, therefore finitely many V_j cover $f(K)$.

PROPOSITION 1.2.9. *If $f: X \multimap Y$ is c-continuous and Y is a Hausdorff space, G_f is closed in $X \times Y$.*

PROOF. Let (x, y) be a point of $X \times Y - G_f$, then $y \notin f(x)$. Since Y is Hausdorff and $f(x)$ compact, there is a neighborhood V of $f(x)$ and a neighborhood W of y such that $V \cap W = \emptyset$. Furthermore, there is a neighborhood U of x such that $f(U) \subset V$, then $f(U) \cap W = \emptyset$. Hence $U \times (f(U) \cap W) = (U \times f(U)) \cap (U \times W) = \emptyset$ which particularly means that the neighborhood $U \times W$ of (x, y) does not meet G_f. Hence G_f is closed in $X \times Y$.

REMARK. If Y is not Hausdorff, the assertion of Proposition 1.2.9 is false in general, even if f is a mapping. (Example: The graph of the identical map $I_X : X \to X$ is closed in $X \times X$ if and only if X is Hausdorff.)

1.3. Bourbaki's definition of proper maps is extended to correspondences as follows: A c-continuous correspondence $f: X \multimap Y$ is called *proper* if the cartesian product $f \times I_W : X \times W \multimap Y \times W$ of f with the identical map I_W of any topological space W is closed.

PROPOSITION 1.3.1. *$f: X \multimap Y$ is proper is and only if f and $\check{f}^{-1}$ are c-continuous.*

This follows directly from the definitions.

PROPOSITION 1.3.2. *Let* $f : X \multimap Y$, $g : Y \multimap Z$, $f' : X' \multimap Y'$ *be proper correspondences. Then* $g \circ f : X \multimap Z$ *and* $f \times f' : X \times X' \multimap Y \times Y'$ *are proper correspondences.*

PROOF. Since the correspondences $g \circ f$, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, $f \times f'$, $(f \times f')^{-1} = f^{-1} \times f'^{-1}$ are c-continuous (Propositions 1.2.3 and 1.2.4), Proposition 1.3.1 implies that $g \circ f$ and $f \times f'$ are c-continuous.

1.4. We finish this section with some remarks about another notion of continuity for correspondences.

A correspondence $f: X \multimap Y$ is said to be *o-continuous* if $f^{-1}: Y \multimap X$ is open (in the sense that the images of open sets are open).[4]

An o-continuous correspondence need not be c-continuous or weakly c-continuous, and vice versa. As examples can serve the reciprocals of mappings which are open but not closed, or of mappings which are proper or closed but not open. Consider also the following example (which generalizes the example given in the introduction): Denote by $\bar{f}: \boldsymbol{C}^{n+1} \multimap \boldsymbol{P}_n$ ($\boldsymbol{P}_n$ the complex projective space of complex dimension n; $n \geqq 1$) the correspondence such that $\bar{f} | (\boldsymbol{C}^{n+1} - \{0\})$ is the canonical map $f: \boldsymbol{C}^{n+1} - \{0\} \to \boldsymbol{P}_n$ and $\bar{f}(0) = \boldsymbol{P}_n$; then $\bar{f}$ is c-continuous but not o-continuous.

The composition $g \circ f: X \multimap Z$ of two o-continuous correspondences $f: X \multimap Y$ and $g: Y \multimap Z$ is obviously o-continuous; furthermore, the cartesian product of two o-continuous correspondences is always o-continuous. Hence the classes of topological spaces and of o-continuous correspondences form a category, denoted $\mathfrak{T}_0$, which contains $\mathfrak{T}$ and which has the property that finite cartesian products of morphisms are morphisms.

2. Holomorphic correspondences. Extension theorems.

2.1. Let $X, Y, \cdots$ now denote reduced complex spaces (for the definition and related concepts compare for instance [**14**]). Complex spaces are always assumed Hausdorff.

A correspondence $f: X \multimap Y$ is said to be *a-holomorphic* if its graph G_f is an analytic subset of $X \times Y$. f is called *c-a-holomorphic* (*o-a-holomorphic* resp.) if f is a-holomorphic and c-continuous (o-continuous resp.).[5] In that which follows we will be concerned with a-

[4]Other notations in the literature are *lower semicontinuous* (*l.s.c.*), *lower continuous, skew continuous.*

[5]Instead of *a-holomorphic, c-a-holomorphic* the notations *weakly holomorphic, holomorphic* were used in [**23**]. Compare also [**22**], [**24**], [**31**].

holomorphic and c-a-holomorphic correspondences.

We have the following statements.

PROPOSITION 2.1.1. *Let $f: X \multimap Y$ be a-holomorphic. Then* (i) *f^{-1} is a-holomorphic.* (ii) *For locally analytic sets $A \subset X$ and $B \subset Y$, the restriction $f \mid A, B$ is a-holomorphic.* (iii) *$f(x)$, $f^{-1}(y)$ are analytic sets in Y, X resp. for any $x \in X$, $y \in Y$.*

PROOF. (i) $G_{f^{-1}}$ is analytic in $Y \times X$, since it is the image of G_f with respect to the biholomorphic map $X \times Y \to Y \times X$ which changes the order of the spaces X, Y. (ii) We have $G_{f|A,B} = G_f \cap (A \times B)$, hence $G_{f|A,B}$ is analytic in $A \times B$. (iii) We have $\check{f}^{-1}(x) = G_f \cap (\{x\} \times Y)$, hence $\check{f}^{-1}(x)$ is contained in $\{x\} \times Y$ as an analytic subset. Now $\{x\} \times Y$ is biholomorphically mapped onto Y by the projection map $p_Y : X \times Y \to Y$, and $\check{f}^{-1}(x)$ is mapped onto $f(x)$ by p_Y. Thus $f(x)$ is an analytic subset of Y. The analogous statement holds for $f^{-1}(y) \subset X$ because of (i).

The set $f^{-1}(y) \subset X$ is called the fibre of f over $y \in Y$.

PROPOSITION 2.1.2. *Let $f: X \multimap Y$ be c-a-holomorphic and A' an analytic subset of Y. Then $f^{-1}(A')$ is an analytic subset of X.*

PROOF. We have $f^{-1}(A') = \check{f}(\hat{f}^{-1}(A'))$. $\hat{f}^{-1}(A')$ is analytic in G_f since $\hat{f}: G_f \to Y$ is an holomorphic mapping. $\check{f}: G_f \to X$ is a proper holomorphic mapping by Proposition 1.2.5. Hence Remmert's mapping theorem [17] (see also [14] implies that $\check{f}(\hat{f}^{-1}(x))$ is analytic in X.

PROPOSITION 2.1.3. *Let $f: X \multimap Y$, $g: Y \multimap Z$, $f': X' \multimap Y'$ be c-a-holomorphic correspondences. Then $g \circ f: X \multimap Z$ and $f \times f'$: $X \times X' \multimap Y \times Y'$ are c-a-holomorphic.*

PROOF. $g \circ f$ and $f \times f'$ are c-continuous by Propositions 1.2.3 and 1.2.4. Therefore, we have to show that the graphs are analytic sets.

(1) $g \circ f$. Consider the fibre product $H := G_f \times_Y G_g = \{(\xi, \eta) \in G_f \times G_g : \hat{f}(\xi) = \check{g}(\eta)\}$, denote by $\varphi_1 : H \to G_f$ and $\varphi_2 : H \to G_g$ the projection maps. φ_1 is a proper map since $\check{g}$ is proper, hence $\check{f} \circ \varphi_1 : H \to X$ is proper. This implies, as it can be checked easily, that

$$\Phi := ((\check{f} \circ \varphi_1) \times (\hat{g} \circ \varphi_2)) \circ \Delta_H : X \to X \times Z$$

is a proper map. One has $\Phi(H) = G_{g \circ f}$. The analyticity of $G_{g \circ f}$ follows now from Remmert's mapping theorem.

(2) $f \times f'$. We have $G_{f \times f'} = \alpha(G_f \times G_{f'})$ where $\alpha : (X \times Y) \times (X' \times Y') \to (X \times X') \times (Y \times Y')$ denotes the biholomorphic map such that $\alpha((x, y), (x', y')) = (x, x', y, y')$. Hence $G_{f \times f'}$ is analytic in

$(X \times X') \times (Y \times Y')$ since $G_f \times G_{f'}$ is analytic in $(X \times Y) \times (X' \times Y')$.

An a-holomorphic correspondence $f: X \multimap Y$ is called *reducible* resp. *irreducible* if G_f is reducible resp. irreducible. If f is reducible and $G_f = \bigcup_j G^{(j)}$ is the decomposition of G_f into irreducible components, the a-holomorphic correspondences f_j given by $G_{f_j} := G^{(j)}$ are called the *irreducible components* of f, we have $f = \bigcup_j f_j$. Note that the f_j are c-a-holomorphic if f is c-a-holomorphic.

The *dimension* of an a-holomorphic correspondence $f: X \multimap Y$, written $\dim f$, is by definition the complex dimension $\dim G_f$ of G_f. If G_f is pure k-dimensional, f is called *pure k-dimensional*. Assume that f is not empty, consider a fibre $\mathfrak{F}_y := f^{-1}(y)$ which is not empty, let $\mathfrak{F}_y = \bigcup_{j_y} \mathfrak{F}_y^{(j_y)}$ be the decomposition of $\mathfrak{F}_y$ into irreducible components. We define

$$\operatorname{fmd} f := \min_{y, j_y} \dim \mathfrak{F}_y^{(j_y)}$$

where y runs over all points of Y such that $\mathfrak{F}_y \neq \emptyset$. If f is empty we set $\operatorname{fmd} f = -1$. $\operatorname{fmd} f$ is called the *fibre-minimal dimension* of f.

A c-a-holomorphic correspondence f from X to Y is called a *meromorphic mapping*, written $f: X \rightharpoonup Y$, if the following holds:

(i) Let X be irreducible. Then f is irreducible, and there is an open set $U \neq \emptyset$ in X such that $f \mid U$ is a map (in the usual sense).

(ii) Let X be reducible and $X = \bigcup_j X^{(j)}$ the decomposition of X into irreducible components. Then there are c-a-holomorphic correspondences $f_j: X \multimap Y$ such that (1) $f_j \mid X^{(j)}$ is a meromorphic mapping, (2) $f_j \mid X - X^{(j)}$ is empty and (3) $f = \bigcup_j f_j$.

Every holomorphic map $X \rightarrow Y$ is obviously a meromorphic mapping. Note that a meromorphic mapping is generally not a mapping in the strong sense. (Example: The correspondence $\bar{f}: \mathbf{C}^{n+1} \multimap \mathbf{P}_n$ defined in 1.4.)

A point $x \in X$ is called a *singularity* of the meromorphic mapping $f: X \rightharpoonup Y$ if x does not have an open neighborhood U such that $f \mid U$ is a holomorphic map. The set $S(f)$ of singularities of f is always a nowhere dense analytic set in X [22]; f is a holomorphic map if and only if $S(f) = \emptyset$.

2.2. We consider the following situation: Let A be a nonempty nowhere dense analytic set in X and $f: X - A \multimap Y$ a c-a-holomorphic correspondence. We ask for conditions under which there exists a c-a-holomorphic correspondence $\tilde{f}: X \multimap Y$ such that $f = \tilde{f} \mid X - A$.

One can always define a topological extension $\bar{f}: X \multimap Y$ of f in the following way: The graph G_f of f is a subset of $(X - A) \times Y$; form the closure $\bar{G}_f$ of G_f in $X \times Y$, let $\bar{f}: X \multimap Y$ be the correspondence such that $G_{\bar{f}} = \bar{G}_f$. It is obvious that, if there is at all a correspondence $\tilde{f}$ as desired, $\bar{f}$ is *c-a*-holomorphic and contained in any such $\tilde{f}$. Hence the question is whether $\bar{f}$ is *c-a*-holomorphic. The following statement gives a sufficient condition:

THEOREM 2.2.1. *Let* $f: X - A \multimap Y$ *be a c-a-holomorphic correspondence such that* $\operatorname{fmd} f > \dim A$. *Then* $\bar{f}: X \multimap Y$ *is c-a-holomorphic. If f is particularly a meromorphic mapping then* $\bar{f}$.

If $f: X - A \to Y$ is a holomorphic map with $\operatorname{fmd} f > \dim A$, then $\bar{f}: X \multimap Y$ is, by Theorem 2.2.1, indeed a meromorphic mapping. However, $\bar{f}$ need not be a holomorphic map as it is shown by the example of the canonical mapping $f: \boldsymbol{C}^{n+1} - \{0\} \to \boldsymbol{P}_n$ considered above. But we have

THEOREM 2.2.2. *Let X be, in addition to the earlier assumptions, an irreducible complex manifold, A an irreducible analytic set in X and* $f: X - A \to Y$ *a holomorphic map. Then*

(1) *If* $\operatorname{fmd} f > \dim A + 1, \bar{f}$ *is a holomorphic map.*

(2) *If* $\operatorname{fmd} f = \dim A + 1$, *then* $\bar{f}$ *is either a holomorphic map or* $\bar{f}$ *is a meromorphic mapping and* $\bar{f}(a) = \bar{f}(X)$ *for every* $a \in A$.

As to the proofs of Theorems 2.2.1 and 2.2.2 we refer to [**24**]. In the proofs essential use is made of the generalization of Thullen's extension theorem for analytic sets ([**27**], [**18**]), of Remmert's mapping theorem [**17**] and its extension by Kuhlmann [**12**] and Whitney [**29**], and of a theorem of Grauert and Remmert on modifications of complex manifolds ([**9**], [**11**]).

We apply Theorem 2.2.2 to the case where $\dim A = \dim X - 2$ and $Y = \boldsymbol{C}$, hence f is in this case a holomorphic function. If f is constant, then $\bar{f}$ is of course again a holomorphic function. If f is not constant, we have $\operatorname{fmd} f = \dim X - 1 = \dim A + 1$. Hence, by Theorem 2.2.2, $\bar{f}$ is then either a holomorphic function or $\bar{f}$ is a meromorphic function with $\bar{f}(a) = \bar{f}(X)$ for every $a \in A$. But $\bar{f}(a)$ is a nonempty connected and compact analytic subset of $\boldsymbol{C}$ and consequently a set consisting of exactly one point; this implies $\bar{f}(a) \neq \bar{f}(X)$ since $\bar{f}(X)$ contains more than one point for a nonconstant f. Therefore $\bar{f}$ is always a holomorphic function. This statement is known as the "second Riemann theorem on removable singularities."

3. **Tornehave's theorem. Generalizations.**

3.1. We consider here another type of extension theorem for correspondences different from that dealt with in §2. The first theorem of this kind (stated below as Theorem 3.1.1) is due to H. Tornehave.

Let $D := \{z \in \boldsymbol{C} : |z| < 1\}$ be the unit disk and $\bar{D}$ its closure in

C. Let $f: D \multimap D$ be a correspondence, denote by $\bar{G}_f$ the closure of its graph G_f in $\bar{D} \times \bar{D}$. It is easy to show that f is c-continuous if and only if $(\bar{G}_f - G_f) \cap (D \times \bar{D}) = \emptyset$ (this implies particularly that G_f is closed in $D \times D$). Hence f is proper if and only if $(\bar{G}_f - G_f) \cap (D \times \bar{D} \cup \bar{D} \times D) = \emptyset$ or, in other words, if and only if $\partial G_f = \bar{G}_f - G_f$ is empty or contained in $\partial D \times \partial D$.

We associate to $f: D \multimap D$ a correspondence $Tf: \bar{C} \multimap \bar{C}$ ($\bar{C}$ the extended complex plane) in the following way: Denote by $\alpha_D : \bar{C} \to \bar{C}$ the reflection map defined by $\alpha_D(z) = 1/\bar{z}$, let $\bar{f}: \bar{D} \multimap \bar{D}$ be the correspondence such that $G_{\bar{f}} = \overline{G_f}$. Then we define $Tf(z) := \bar{f}(z)$ if $z \in \bar{D}$ and $Tf(z) = \alpha_D(f(\alpha_D(z)))$ if $x \in \bar{C} - \bar{D}$. Obviously, $Tf: \bar{C} \multimap \bar{C}$ is proper and one has $\alpha_D \circ Tf = Tf \circ \alpha_D$. Moreover, if f is c-continuous, then $I_{\bar{C}}{}^D \circ f = Tf \mid D$. If f is proper, then $Tf(\partial D) \subset \partial D$.

One has now

THEOREM 3.1.1. *Let $f: D \multimap D$ be a proper c-a-holomorphic correspondence. Assume that f is pure* 1*-dimensional. Then Tf: $\bar{C} \multimap \bar{C}$ is c-a-holomorphic and pure* 1*-dimensional. If f is irreducible then so is Tf.*

REMARK. Every analytic set in $\bar{C} \times \bar{C}$ is algebraic by Chow's theorem [7], therefore every a-holomorphic correspondence φ: $\bar{C} \multimap \bar{C}$ is algebraic in the classical sense. Hence Theorem 3.1.1 states that the correspondence $f: D \multimap D$ can be continued to an algebraic correspondence $Tf: \bar{C} \multimap \bar{C}$ (in the sense that $I_{\bar{C}}{}^D \circ f = Tf \mid D$) by means of a generalized Schwarz reflection principle.

A proof of the theorem above (in an equivalent version) was given by H. Tornehave [28]. Furthermore, a proof is due to B. Shiffman [20]. (Actually, B. Shiffman proved a more general theorem on the continuation of analytic curves which implies Tornehave's theorem. See also [21].)

We will sketch here another proof:

The assumption on f implies that $f(z)$ is a nonempty compact analytic subset of D for every $z \in D$, hence $f(z)$ is particularly a finite set. Let $c_z(f)$ be the cardinal of $f(z)$, set $c(f) := \sup_{z \in D} c_z(f)$. Then $c(f)$ is also finite ($c(f)$ is the mapping degree of $\check{f}: G_f \to D$); furthermore, the set $\mathfrak{B}(f) := \{z \in D : c_z(f) < c(f)\}$ is discrete in D. We claim that $\mathfrak{B}(f)$ is finite. If this is shown, we get the assertion of Theorem 3.1 as follows:

Let $z^{(0)}$ be a point of ∂D. We can choose an open neighborhood U of $z^{(0)}$ in C such that $U \cap D$ is simply connected, $\alpha_D(U) = U$ and, since $\mathfrak{B}$ is assumed finite, such that $U \cap \mathfrak{B} = \emptyset$. Then there are meromorphic functions $f_j : U \cap D \to \bar{C}$, $j = 1, \cdots, c(f)$, with the property that $f(z) = \bigcup_j f_j(z)$ for every $z \in U \cap D$. The properness of f implies that $|f_j(z)|$ tends towards 1 if $z \in U \cap D$ tends towards ∂D. Therefore, we can apply the Schwarz reflection principle for f_j; it provides that there is a meromorphic continuation $\tilde{f}_j : U \to C$ of f_j for each j. We have then $Tf(z) = \bigcup_j \tilde{f}_j(z)$ for every $z \in U$. It follows that the graph G_{Tf} of T_f is analytic in a neighborhood of $\partial D \times \partial D$. On the other hand, G_{Tf} is certainly analytic in $\bar{C} \times \bar{C} - \partial D \times \partial D$. Hence G_{Tf} is an analytic subset of $\bar{C} \times \bar{C}$ which means that Tf is a-holomorphic and thus c-a-holomorphic. It is obvious that G_{Tf} is pure 1-dimensional. Furthermore, if f is irreducible, then there is a unique irreducible component $G^{(0)}$ of G_{Tf} which contains G_f. Then $G_f = G^{(0)} \cap (D \times D)$, this implies that any irreducible component of G_{Tf} is contained in $(\bar{C} - \bar{D}) \times (\bar{C} - D)$. But a 1-dimensional analytic subset of $\bar{C} \times \bar{C}$ cannot be contained in $(\bar{C} - D) \times (\bar{C} - D)$, hence $G_{Tf} = G^{(0)}$; thus Tf is irreducible.

So it remains to prove that $\mathfrak{B}(f)$ is finite.

We set

$$\delta(f) := \sum_{z \in D} (c(f) - c_z(f)).$$

Note that exactly the points of $\mathfrak{B}(f)$ provide nonzero contributions to the sum, hence it is enough to show that $\delta(f)$ is finite. In fact one has

Proposition 3.1.2. *Let $f : D \multimap D$ be proper, c-a-holomorphic and pure* 1*-dimensional. Then*

$$\delta(f) \leqq (c(f) - 1) \cdot c(f^{-1}).$$

Proof. We can assume $c(f) > 1$, for if $c(f) = 1$ the correspondence f is a holomorphic mapping and the assertion is obvious in this case.

Every point $w \in f(z)$, $z \in D$, has a multiplicity $m_f(w)$ which is, by definition, the order of the holomorphic function $I_C{}^D \circ \check{f} : G_f \to C$ at $\zeta := (z, w) \in G_f$ (the order of $I_C{}^D \circ \check{f}$ at ζ is defined by using local parametric representations of the irreducible components of G_f at ζ). We can write $f(z) = \{w_1, \cdots, w_{c(f)}\}$ where every $w \in f(z)$ is counted according to its multiplicity. Now we associate to f the correspondence $f^{(2)} : D \multimap D \times D$ defined by $f^{(2)}(z) = \{(w_\lambda, w_\mu)\}$ $(z \in D)$ with $1 \leqq \lambda, \mu \leqq c(f)$ and $\lambda \neq \mu$ (since $c(f) > 1$, there are always those (w_λ, w_μ)). $f^{(2)}$ can also be described as follows: Consider the proper c-a-holomorphic and pure 1-dimensional correspondence $F := (f \times f) \circ \Delta_D : D \multimap D \times D$ $(\Delta_D : D \to D \times D$

denotes the diagonal map). F contains the correspondence $F' := \Delta_D \circ f : D \multimap D \times D$. On the other hand, F has irreducible components F_j which are not contained in F' since $c(f) > 1$. Then $f^{(2)}$ is the union of those F_j. This shows that $f^{(2)}$ is again proper, c-a-holomorphic and pure 1-dimensional.

We use now the fact that a point $z \in D$ belongs to $\mathfrak{B}(f)$ (hence contributes to $\delta(f)$) if and only if $f^{(2)}(z)$ meets the diagonal D^* of $D \times D$. Let $s(G_{f^{(2)}}, D \times D^*)$ be the intersection number (in the sence of algebraic topology) of $G_{f^{(2)}}$ and $D \times D^*$ in $D \times (D \times D)$ where $G_{f^{(2)}}$, $D \times D^*$ and $D \times (D \times D)$ are taken with their natural oreintations. We claim that $\delta(f) \leqq s(G_{f^{(2)}}, D \times D^*)$. To show this we note that a point $(z, w, w) \in D \times D^*$ is on $G_{f^{(2)}}$ if and only if $w \in f(z)$ and $m_f(w) > 1$. Let (z, w, w) be such a point. One checks easily that $G_{f^{(2)}}$ has at least $m_f(w) - 1$ irreducible components at (z, w, w). Hence the contribution to $s(G_{f^{(2)}}, D \times D^*)$ at (z, w, w) is at least $m_f(w) - 1$, since the intersection number of analytic sets of complementary dimension which intersect in isolated points is always positive. It follows that the contribution to $s(G_{f^{(2)}}, D \times D^*)$ provided by the point z is at least $c(f) - c_z(f)$, therefore $\delta(f) \leqq s(G_{f^{(2)}}, D \times D^*)$.

Furthermore, we claim $s(G_{f^{(2)}}, D \times D^*) \leqq (c(f) - 1) \cdot c(f^{-1})$. Denote $L_t := \{(w, w') \in D \times D : w = t \cdot w',\ t \in \mathbf{C}\}$, then $L_1 = D^*$. Since $f^{(2)}$ is proper, there is a neighborhood U of $D \times \partial D$ in $\mathbf{C} \times \mathbf{C}$ such that $f^{(2)}(D) \cap U = \emptyset$; this implies that $G_{f^{(2)}} \cap (D \times L_t)$ is compact for any t with $|t| < 1$. Hence the intersection number $s_t := (G_{f^{(2)}}, D \times L_t)$ is finite for those t; moreover, it follows that s_t is constant for $|t| < 1$. Now s_0 is the number of points $(z, 0, w') \in G_{f^{(2)}}$ counted with multiplicities according to their order with respect to the holomorphic function $I_C{}^D \circ \mathrm{pr}_1 \circ f^{(2)} : G_{f^{(2)}} \to \mathbf{C}$ ($\mathrm{pr}_1 : D \times D \to D$ denotes the projection map of $D \times D$ onto the first factor), therefore s_0 equals the mapping degree of $p_1 \circ \check{f}^{(2)}$ which is $(c(f) - 1) \cdot c(f^{-1})$. Thus $s_t = (c(f) - 1) \cdot c(f^{-1})$ if $|t| < 1$. Furthermore, one has $s(G_{f^{(2)}}, D \times D^*) = s_1 \leqq \mathrm{Sup}_{|t|<1} s_t$, hence $s(G_{f^{(2)}}, D \times D^*) \leqq (c(f) - 1) \cdot c(f^{-1})$.

It follows $\delta(f) \leqq (c(f) - 1) \cdot c(f^{-1})$ as asserted which completes also the proof of Theorem 3.1.1.

REMARK. The assertion of Proposition 3.1.2 remains valid if f is a proper c-a-holomorphic and pure 1-dimensional correspondence from an arbitrary noncompact connected Riemann surface R to D ($c(f)$, $c(f^{-1})$ and $\delta(f)$ are then defined analogously). Moreover, one has a generalization to the following situation: Let $f : R \multimap \bar{\mathbf{C}}$ be a c-holomorphic and pure 1-dimensional correspondence from the noncompact Riemann surface R to $\bar{C}$. Assume that f is "proper mod ∂D" which means that $f^{-1} \mid D$ and $f^{-1} \mid \bar{C} - \bar{D}$ are c-continuous (hence c-a-holomorphic), or equivalently that $f(z)$ tends towards ∂D if $z \in R$ tends towards ∂R. Define $c^{(i)}(f^{-1}) := c(f^{-1} \mid D)$, $c^{(e)}(f^{-1}) := c(f^{-1} \mid \bar{C} - \bar{D})$. Then

$$\delta(f) \leqq (c(f) - 1) \cdot (c^{(i)}(f^{-1}) + c^{(e)}(f^{-1}));$$

this can be obtained in an analogous manner.

Furthermore, one can show by a similar method:

Let B be a domain in $\mathfrak{C}$ such that ∂B contains an open arc $\mathfrak{C}$ of ∂D. Let $f: B \multimap \bar{C}$ be a c-a-holomorphic and pure 1-dimensional correspondence. Assume that f is "proper mod ∂D in $\mathfrak{C}$" which means that $f(z)$ tends towards ∂D if $z \in B$ tends towards a point of $\mathfrak{C}$. Then every point of $\mathfrak{C}$ has a neighborhood U in C such that $\delta(f \mid B \cap U)$ is finite.

3.2. We discuss some applications of the results of the preceding section.

(1) If X is a complex space (compare e.g. [**14**]) with finite Betti numbers $p_j(X)$ $(j = 0, 1, \cdots)$, we denote by

$$e(X) = \sum_j (-1)^j p_j(X)$$

its Euler characteristic.

Consider a proper c-a-holomorphic and pure 1-dimensional correspondence $f: D \multimap D$ as in Theorem 3.1.1 and Proposition 3.1.2. One has the Hurwitz formula

$$e(G_f) = c(f) \cdot e(D) - \delta(f).$$

Now $e(D) = 1$, hence Proposition 3.1.2 implies

$$e(G_f) \geqq c(f) - (c(f) - 1) \cdot c(f^{-1}).$$

Assume now f to be irreducible, then $p_0(G_f) = 1$ and $p_j(G_f) = 0$ for $j \geqq 2$; it follows $e(G_f) = 1 - p_1(G_f)$ and

$$\begin{aligned} p_1(G_f) &\leqq -c(f) + (c(f) - 1) \cdot c(f^{-1}) + 1 \\ &= (c(f) - 1) \cdot (c(f^{-1}) - 1). \end{aligned}$$

If $(\tilde{G}_f, \nu)$ is a normalization of G_f (compare [**14**]), then $p_1(\tilde{G}_f) \leqq p_1(G_f)$, hence

$$(*) \qquad p_1(\tilde{G}_f) \leqq (c(f) - 1) \cdot (c(f^{-1}) - 1).$$

$\tilde{G}_f$ is a Riemann surface such that there exists a compact bordered

Riemann surface (cf. [2]) whose interior is $\tilde{G}_f$; we call a Riemann surface with this property a Riemann surface *of type F*. It can be shown that every Riemann surface of type F can occur as a normalized graph $\tilde{G}_f$ of a proper *c-a*-holomorphic irreducible correspondence $f: D \multimap D$. But if $c(f)$ and $c(f^{-1})$ are prescribed, the possible $\tilde{G}_f$ are restricted by relation (*). For instance, if $c(f) = c(f^{-1}) = 2$, one has $p_1(\tilde{G}_f) \leqq 1$, hence $\tilde{G}_f$ is in this case biholomorphically equivalent to the unit disk or to an annulus.

(2) Consider two proper *c-a*-holomorphic pure 1-dimensional correspondences $f_j : R \multimap D$ $(j = 1, 2)$ of a noncompact Riemann surface R to the unit disk. We claim f_1, f_2 are algebraically dependent in the following sense: There exists a complex polynomial $p \neq 0$ of two variables such that for every $\xi \in R$ all points of $f_1(\xi) \times f_2(\xi)$ are zeros of p.

As to the proof we consider the correspondence

$$f_2 \circ f_1{}^{-1} : D \multimap D$$

which is again proper, *c-a*-holomorphic and pure 1-dimensional. Hence, by Tornehave's theorem, $f_2 \circ f_1{}^{-1}$ is a restriction of a *c-a*-holomorphic pure 1-dimensional correspondence $Tf: \bar{C} \multimap \bar{C}$ whose graph is an algebraic set, by Chow's theorem. This implies the assertion.

(3) We introduce first some notations with respect to Riemann surfaces.

Let P be a relatively compact domain in a Riemann surface R. We call P a *distinguished polyhedral domain* in R if there is a nonconstant meromorphic function $\varphi : R \rightarrow \bar{C}$ such that P coincides with a connected component of the set $\{\xi \in R : |\varphi(\xi)| < 1\}$; φ is called a *defining function* of P. By a theorem of Bishop [5] every noncompact Riemann surface can be exhausted by distinguished polyhedral domains whose defining functions are holomorphic.

Obviously, a distinguished polyhedral domain in a Riemann surface is a Riemann surface of type F (see (1) above). Conversely, let R_0 be a Riemann surface of type F and $R_0{}^*$ a bordered Riemann surface whose interior is R_0. $R_0{}^*$ has a double $\hat{R}_0$ (cf. [2]); then $R_0{}^*$ is the closure $\bar{R}_0$ of R_0 in $\hat{R}_0$. We call $\hat{R}_0$ also a double of R_0; note that $\hat{R}_0$ is uniquely determined up to biholomorphic equivalence if R_0 is given. Particularly, $\bar{C}$ is a double of D. By a theorem of Ahlfors [1] there is always a meromorphic function $\varphi : \hat{R}_0 \rightarrow \bar{C}$ such that $R_0 = \{\xi \in \hat{R}_0 : |\varphi(\xi)| < 1\}$; hence R_0 is a distinguished polyhedral domain in $\hat{R}_0$. Furthermore, there exists a unique bijective involutory and antiholomorphic mapping $\alpha_{R_0} : \hat{R}_0 \rightarrow \hat{R}_0$ such that

$\alpha_{R_0}(R_0) = \hat{R}_0 - \bar{R}_0$ and $\alpha_{R_0}(\xi_0) = \xi_0$ for every $\xi_0 \in \partial R_0 = \bar{R}_0 - R_0$. One has $\alpha_D \circ \varphi = \varphi \circ \alpha_{R_0}$ for any meromorphic function $\varphi : \hat{R}_0 \to \bar{C}$ with $R_0 = \{\xi \in \hat{R}_0 : |\varphi(\xi)| < 1\}$ (as in 3.1, $\alpha_D : \bar{C} \to \bar{C}$ is the map defined by $\alpha_D(z) = 1/\bar{z}$).

Consider now two Riemann surfaces R_1, R_2 of type F with doubles $\hat{R}_1, \hat{R}_2$ and closures $\bar{R}_1, \bar{R}_2$ in $\hat{R}_1, \hat{R}_2$ respectively. Let $f: R_1 \multimap R_2$ be a correspondence. As in 3.1 we associate to f a correspondence $Tf: \hat{R}_1 \multimap \hat{R}_2$: Denote by $\bar{G}_f$ the closure of G_f in $\bar{R}_1 \times \bar{R}_2$, let $\bar{f}: \bar{R}_1 \multimap \bar{R}_2$ be the correspondence such that $G_{\bar{f}} = \overline{G_f}$; define $Tf(\xi) := \bar{f}(\xi)$ for $\xi \in \bar{R}_1$ and $Tf(\xi) := \alpha_{R_2}(\bar{f}(\alpha_{R_1}(\xi)))$ for $\xi \in \hat{R}_1 - \bar{R}_1$. Then $Tf \circ \alpha_{R_1} = \alpha_{R_2} \circ Tf$; moreover, if f is c-continuous, then $I_{\hat{R}_2}{}^{R_2} \circ f = Tf \mid R_1$. We have the following generalization of Theorem 3.1.1:

THEOREM 3.2.1. *Let R_1, R_2 be Riemann surfaces of type* F *with doubles $\hat{R}_1$, $\hat{R}_2$; let $f : R_1 \multimap R_2$ be a proper c-a-holomorphic correspondence. Assume that f is pure* 1*-dimensional. Then $Tf: \hat{R}_1 \multimap \hat{R}_2$ is c-a-holomorphic and pure* 1*-dimensional. If f is irreducible then so is Tf.*

PROOF. Take meromorphic functions $\varphi_j : \hat{R}_j \to \bar{C}$ $(j = 1, 2)$ such that $R_j = \{\xi \in \hat{R}_j : |\varphi_j(\xi)| < 1\}$; denote by $\varphi_j' : \hat{R}_j \to D$ the maps such that $\varphi_j \mid \hat{R}_j = I_{\bar{C}}{}^D \circ \varphi_j'$. The φ_j' are proper holomorphic maps. It follows that the correspondence

$$\tilde{f} := \varphi_2' \circ f \circ \varphi_1'^{-1} : D \multimap D$$

is proper, c-a-holomorphic and pure 1-dimensional. Hence by Theorem 3.1.1, $\tilde{f}$ can be continued to the c-a-holomorphic and pure 1-dimensional correspondence $T\tilde{f}: \bar{C} \multimap \bar{C}$. Define now $'G := (\varphi_1 \times \varphi_2)^{-1}(G_{T\tilde{f}})$, then $'G$ is a pure 1-dimensional analytic subset of $R_1 \times R_2$ which contains G_f; furthermore one has $'G \cap \partial(R_1 \times R_2) \subset \partial R_1 \times \partial R_2$. Hence G_f can be analytically extended across $\partial R_1 \times \partial R_2$. The map $\varphi_1 \times \varphi_2 : R_1 \times R_2 \to \bar{C} \times \bar{C}$ is locally biholomorphic in every point of $\partial R_1 \times \partial R_2$. It follows that the extension of G_f across $\partial R_1 \times \partial R_2$ is obtained by reflection with respect to $\partial R_1 \times \partial R_2$ (i.e. by applying the map $\alpha_{R_1} \times \alpha_{R_2} : \hat{R}_1 \times \hat{R}_2 \to \hat{R}_1 \times \hat{R}_2$), since the extension of $G_{\tilde{f}}$ across $\partial D \times \partial D$ is obtained by reflection with respect to $\partial D \times \partial D$. On the other hand, one has, by the definition of Tf, $G_{Tf} \cap ((\hat{R}_1 - R_1) \times (\hat{R}_2 - R_2)) = (\alpha_{R_1} \times \alpha_{R_2})(\bar{G}_f)$, furthermore $(\alpha_{R_1} \times \alpha_{R_2})(G_f)$ is an analytic subset of $(\hat{R}_1 - \bar{R}_1) \times (\hat{R}_2 - \bar{R}_2)$. Therefore G_{Tf} is an analytic subset of $\hat{R}_1 \times \hat{R}_2$ which is obviously pure 1-dimensional, hence Tf has the asserted properties. If f is irreducible, the irreducibility of Tf

can be shown similarly as in the proof of Theorem 3.1.1.

There is still another application of Tornehave's theorem to the extension of correspondences between special Riemann surfaces.

PROPOSITION 3.2.2. *Let P_j be distinguished polyhedral domains in the Riemann surfaces R_j $(j = 1, 2)$ resp.; let $f: P_1 \multimap P_2$ be proper irreducible c-a-holomorphic correspondences of dimension* 1. *Then there is an irreducible a-holomorphic correspondence $'f: R_1 \multimap R_2$ of dimension* 1 *such that $f \subset {'f}$. $'f$ is uniquely determined.*

PROOF. We proceed similarly as in the proof of Theorem 3.2.1: Let $\varphi_j : R_j \to \bar{C}$ be defining functions of P_j resp., then $\varphi_j(P_j) \subset D$ and there are proper holomorphic maps $\varphi_j' : P_j \to D$ such that $\varphi_j \mid P_j = I_{\bar{C}}^D \circ \varphi_j'$. The correspondence

$$\tilde{f} := \varphi_2' \circ f \circ \varphi_1'^{-1} : D \multimap D$$

satisfies the assumptions of Theorem 3.1.1; hence the extension $T\tilde{f}$: $\bar{C} \multimap \bar{C}$ is c-a-holomorphic and pure 1-dimensional. It follows that $'G := (\varphi_1 \times \varphi_2)^{-1}(G_{T\tilde{f}})$ is a pure 1-dimensional analytic subset of $R_1 \times R_2$ which contains G_f. Since G_f is irreducible, there is also an irreducible component $'G_0$ of $'G$ with $G_f \subset {'G_0}$. Thus the correspondence $'f: R_1 \multimap R_2$ defined by $G_{'f} := {'G}$ has the asserted property. If $''f: R_1 \multimap R_2$ is any correspondence with this property, then $'G_0 \subset G_{'f} \cap G_{''f}$, hence $G_{'f} = G_{''f}$ and $'f = {''f}$.

REMARK. In general $I_{R_2}^{P_2} \circ f$ does not coincide with $'f \mid P_1$, even if f is a biholomorphic mapping; this is shown by simple examples.

3.3. We give two extensions of Tornehave's theorem to higher dimensions.

Let $D^n := \{z = (z_1, \cdots, z_n) \in C^n : |z_\nu| < 1 \quad (\nu = 1, \cdots, n)\}$ be the unit polycylinder in C^n $(n \geqq 1)$. We write also $D^n = D_1 \times \cdots \times D_n$ where $D_\nu := \{z_\nu \in C : |z_\nu| < 1\}$.

(1) THEOREM 3.3.1. *Let $f: D^n \multimap D^n$ $(n \geqq 1)$ be a proper c-a-holomorphic correspondence. Assume that f is pure n-dimensional. Then there is a unique pure n-dimensional c-a-holomorphic correspondence $'f: \bar{C}^n \multimap \bar{C}^n$ such that $I_{\bar{C}^n}^{D^n} \circ f = {'f} \mid D^n$ and that $'f(z)$ is finite for every $z \in D^n$. If f is irreducible then so is $'f$.*

REMARK. $f(z)$ is finite for every $z \in D^n$ since $f(z)$ is a compact analytic subset of D^n.

PROOF OF THEOREM 3.3.1. By a theorem of H. Rischel [**19**] f is a cartesian product of proper c-a-holomorphic correspondences between unit disks; more precisely: There are proper c-a-holomorphic and pure 1-dimensional correspondences $f_\nu : D_\nu \multimap D_\nu$ and a biholomorphic mapping $b : D^n \to D^n$ given by a permutation of the coordinates $z_1, \cdots, z_n$ such that

$$f = b \circ (f_1 \times \cdots \times f_n) : D_1 \times \cdots \times D_n \multimap D_1 \times \cdots \times D_n.$$

Each correspondence f_ν admits a continuation, according to Theorem 3.1.1, to the proper c-a-holomorphic and pure 1-dimensional correspondence $Tf_\nu : \bar{C} \multimap \bar{C}$. Define

$$'f := {}'b \circ (Tf_1 \times \cdots \times Tf_n) : \bar{C}^n \multimap \bar{C}^n$$

where $'b : \bar{C}^n \multimap \bar{C}^n$ denotes the holomorphic extension of b. Then $'f$ is c-a-holomorphic and pure n-dimensional, one has $I_{\bar{C}^n}^{D^n} \circ f = {}'f \mid D^n$ and $'f(z)$ is finite for every $z \in \bar{C}^n$.

It follows that every irreducible component $G^{(i)}$ of G_f is contained in a (unique) n-dimensional irreducible analytic subset $\tilde{G}^{(i)}$ of $\bar{C}^n \times \bar{C}^n$. Let $\tilde{G}$ be the union of those $\tilde{G}^{(i)}$ and $\tilde{f} : \bar{C}^n \multimap \bar{C}^n$ the correspondence such that $G_{\tilde{f}} = \tilde{G}$, then $\tilde{f}$ is contained in $'f$. Furthermore, let $''f : \bar{C}^n \multimap \bar{C}^n$ be any c-a-holomorphic pure n-dimensional correspondence such that $I_{\bar{C}^n}^{D^n} \circ f = {}''f \mid D^n$ and that $''f(z)$ is finite for every $z \in \bar{C}^n$; then we have also $\tilde{f} \subset {}''f$. Consider any irreducible component $''G$ of $G_{''f}$. Since $''\check{f}^{-1}(z)$ is finite for $z \in \bar{C}^n$, Remmert's mapping theorem implies $''\check{f}(''G) = \bar{C}^n$, hence $''G \cap (D^n \times \bar{C}^n) \neq \emptyset$. One even has $''G \cap (D^n \times D^n) \neq \emptyset$ because of $I_{\bar{C}^n}^{D^n} \circ f = {}''f \mid D^n$, therefore $''G$ contains at least one $G^{(i)}$. It follows that $''G = \tilde{G}^{(i)}$ and $''f \subset \tilde{f}$. Hence $\tilde{f} = {}''f$ and $\tilde{f} = {}'f$ which shows that $'f$ is uniquely determined. Moreover, if f is irreducible, then $\tilde{f} = {}'f$ is also irreducible.

(2) Consider a pure n-dimensional c-a-holomorphic correspondence $f : D^n \multimap D$ $(n \geqq 1)$. If $n > 1$, f cannot be proper, since every fibre of f is a pure n-dimensional analytic set in D^n which approaches the boundary of D^n. But f can satisfy a weaker hypothesis which coincides for $n = 1$ with the properness of f: Denote $B(D^n) := \partial D_1 \times \cdots \times \partial D_n$ ($B(D^n)$ is Bergman's distinguished boundary of D^n). Then f is called *proper in* $B(D^n)$ if, for every compact set K in D, there is a neighborhood U_K of $B(D^n)$ in D^n such that $f^{-1}(K) \cap U_K = \emptyset$; an equivalent condition is that $f(z)$ tends towards ∂D if $z \in D^n$ tends towards $B(D^n)$ (compare 3.1). One has

THEOREM 3.3.2. *Let $f : D^n \multimap D$ be an irreducible n-dimensional c-a-holomorphic correspondence. Assume that f is proper in $B(D^n)$. Then there is a unique irreducible c-a-holomorphic correspondence $'f : \bar{C}^n \multimap \bar{C}$ such that $I_{\bar{C}}^{D} \circ f = {}'f \mid D^n$. $'f$ is n-dimensional.*

We sketch the main steps of the proof.

(a) Denote $E := D^n \cup B(D^n) \cup (\bar{C} - \bar{D})^n$. A correspondence

$f' : E \multimap \bar{C}$ is defined in the following way: We set $f'(z) := f(z)$ for $z \in D^n$ and $f'(z) := \alpha_D \circ f \circ (\alpha_{D_1} \times \cdots \times \alpha_{D_n})(z)$ for $z \in (\bar{C} - \bar{D})^n$. Furthermore, if $z^{(1)} \in B(D^n)$ is given, let $L_{z^{(1)}}$ be the complex line in C^n through $z^{(1)}$ and the origin $0 \in C^n$, denote by $\bar{L}_{z^{(1)}}$ the closure of $L_{z^{(1)}}$ in $\bar{C}^n$. $\bar{L}_{z^{(1)}}$ is a complex subspace of $\bar{C}^n$ and biholomorphically equivalent to $\bar{C}$ such that $R_{z^{(1)}} := \bar{L}_{z^{(1)}} \cap D^n$ corresponds to D; particularly, $\bar{L}_{z^{(1)}}$ is a double of $R_{z^{(1)}}$. Consider the correspondence $f_{z^{(1)}} := f \mid R_{z^{(1)}} : R_{z^{(1)}} \multimap D$ which is proper, *c-a*-holomorphic and pure 1-dimensional. One has the continuation $Tf_{z^{(1)}} : \bar{L}_{z^{(1)}} \multimap \bar{C}$ of $f_{z^{(1)}}$. Define now $f'(z^{(1)}) := Tf_{z^{(1)}}(z^{(1)})$. Then f' is *c*-continuous, moreover, $G_{f'} \cap D^n$ and $G_{f'} \cap (\bar{C} - \bar{D})^n$ are analytic sets in D^n and $(\bar{C} - \bar{D})^n$ resp. (b) One shows (by means of Cauchy's integral formula): There is a neighborhood

$$U_\epsilon = \{z = (z_1, \cdots, z_n) \in C^n : 1 - \epsilon < |z_\nu| < 1 + \epsilon \ (0 < \epsilon < 1)\}$$

of $B(D^n)$ and a pure n-dimensional *c-a*-holomorphic correspondence $f'': U_\epsilon \multimap \bar{C}$ such that $f' \mid E \cap U_\epsilon = f'' \mid E \cap U_\epsilon$. Define $f^* : E \cup U_\epsilon \multimap \bar{C}$ by $f^*(z) := {}'f(z)$ for $z \in E$ and $f^*(z) := f''(z)$ for $z \in U_\epsilon$, then f^* is *c-a*-holomorphic and pure n-dimensional. (c) f^* can be extended to a pure n-dimensional *c-a*-holomorphic correspondence $f^{**} : \bar{C}^n \multimap \bar{C}$ such that $I_{\bar{C}}{}^D \circ f = f^{**} \mid D^n$; this is shown by applying statements on envelopes of meromorphy. f^{**} has an irreducible component ${}'f : \bar{C}^n \multimap \bar{C}$ which contains f, then $I_{\bar{C}}{}^D \circ f = {}'f \mid D^n$ and $\dim {}'f = n$. Any irreducible *c-a*-holomorphic correspondence ${}''f : \bar{C}^n \multimap \bar{C}$ such that $I_{\bar{C}}{}^D \circ f = {}''f \mid D^n$ coincides with ${}'f$ because there exists at most one irreducible analytic subset ${}''G$ of $\bar{C}^n \times \bar{C}^n$ such that $G_f = {}''G \cap (D^n \times D^n)$.

Remark. Theorem 3.3.2 extends also a result of A. Pfister [15]: Assume that $f: D^n \to D$ is a holomorphic map and that f is proper in $B(D^n)$. Then Theorem 3.3.2 implies that there is a continuation of f to a meromorphic function ${}'f: \bar{C}^n \to \bar{C}$ which is, by a theorem of Hurwitz-Weierstrass (cf. [3]), a rational function. ${}'f$ has the form

$${}'f(z_1, \cdots, z_n) = \frac{\sum_{(i_1, \cdots, i_n) = (0, \cdots, 0)}^{(m_1, \cdots, m_n)} a_{i_1, \cdots, i_n} z_1^{i_1} \cdots z_n^{i_n}}{\sum_{(i_1, \cdots, i_n) = (0, \cdots, 0)}^{(m_1, \cdots, m_n)} \bar{a}_{m_1 - i_1, \cdots, m_n - i_n} z_1^{i_1} \cdots z_n^{i_n}}.$$

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UNIVERSITY OF MUNICH, MUNICH, WEST GERMANY