ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 3, Summer 1989

ON TRACE FORMS OF ALGEBRAIC FUNCTION FIELDS

ALEXANDER PRESTEL

1. Introduction and results. Let L/K be a finite separable field extension. The *trace form* of L/K is the following symmetric bilinear form over K

$$L \times L \to K, \qquad (x, y) \to \mathrm{Tr}_{L/K}(x \cdot y).$$

This form will be denoted by $T_K(L, 1)$. If P is an ordering of K, it is well-known that

$$\mathrm{sgn}_P T_K(L, 1) = \#\{\text{extensions of } P \text{ to } L\}.$$

Thus every trace form has *totally positive signature over* K, i.e.,

$$\mathrm{sgn}_P T_K(L, 1) \geq 0 \quad \text{for all } P \in X_K.$$

As usual X_K denotes the set of all orderings of K. Therefore every (regular) quadratic form ρ over K which is Witt equivalent to some trace form over K has totally positive signature.

In [**3**] the question has been raised whether, for algebraic number fields K, the converse also holds, i.e., whether in this case every regular quadratic form ρ which has totally positive signature over K is Witt equivalent to a trace form $T_K(L, 1)$ for some finite extension L/K. Conner and Perlis succeeded in proving this in case $K = \mathbf{Q}$. In a recent paper W. Scharlau [**8**] gave a positive answer for all number fields, reducing the general case to the 1-dimensional case already solved in [**5**]. In the 1-dimensional case $\rho = (\beta)$, the condition of totally positive signature just means that β is a sum of squares in K.

The main result of this paper is

MAIN THEOREM. *Let K be an algebraic function field in one variable over a real closed field R. Then every regular quadratic form ρ which*

The result of this paper has been announced at the Conference on Quadratic Forms and Real Algebraic Geometry, Corvallis, July 1986.
Received by the editors on November 17, 1986 and in revised form on April 24, 1987.

has totally positive signature over K is Witt equivalent to some trace form $T_K(L,1)$.

The strategy of the proof is the same as in [**8**]: first reducing the general case to the 1-dimensional case, and then proving the 1-dimensional case.

Scharlau's reduction step used the two facts that algebraic number fields are *hilbertian* (i.e., satisfy Hilbert's Irreducibility Theorem) and have only a finite number of orderings. While the first fact is still true for algebraic function fields, the second no longer holds (except for the case $X_K = \emptyset$). A substitute for this second fact will be that every algebraic function field in one variable over a real closed field R (as well as every algebraic number field) allows *Effective Diagonalization* (ED) of quadratic forms (see [**11**] and [**7**]), i.e., for every quadratic form ρ over K, there is a diagonalization

$$\rho \simeq \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

such that, for each $P \in X_K$, we have

$$d_{i+1} \in P \Rightarrow d_i \in P.$$

This means that - independent of P - the positive elements d_i always are on top of the negative ones.

The first theorem we prove corresponds to Scharlau's reduction step. (Since the fields in the Main Theorem are of characteristic zero, we will restrict ourselves to this case.)

THEOREM 1. *Let K be a hilbertian field of characteristic zero satisfying* ED. *Then every regular quadratic form ρ having totally positive signature over K is isometric to a scaled trace form $T_K(L,\beta)$ for some finite extension L/K with β being a sum of squares in $L^\times$.*

A *scaled trace form $T_K(L,\beta)$* is given by the symmetric bilinear form over K

$$L \times L \to K, \qquad (x,y) \to \mathrm{Tr}_{L/K}(\beta xy),$$

where L/K is a finite (separable) extension and $\beta \in L$. It is easy to prove (see, e.g., [**9**; Chapter 3, Theorem 4.5]) that, for every $P \in X_K$,

$$\begin{aligned}\operatorname{sgn}_P T_K(L,\beta) =& \#\{\text{extensions } P' \text{ of } P \text{ to } L \text{ s.t. } \beta \in P'\} \\ &- \#\{\text{extensions } P' \text{ of } P \text{ to } L \text{ s.t. } -\beta \in P'\}.\end{aligned}$$

Thus, as a consequence, we have

PROPOSITION. *Let $T_K(L,\beta)$ be a scaled trace form with $\beta \in L$. Then β is a sum of squares in L if and only if, for all $P \in X_K$,*

$$\operatorname{sgn}_P T_K(L,\beta) = \#\{\textit{extensions of } P \textit{ to } L\}.$$

The second theorem we prove will correspond to Corollary 1 of [**5**]. We will say that a field L satisfies the *Norm Theorem* (NT) of [**5**] if, for every sum of squares $\beta \in L$ which is not a square in L, there is a natural number m such that $-m$ is a Norm of $L(\sqrt{\beta})$ over L, i.e., the form $(1, -\beta, m)$ is isotropic over L. By a theorem of Witt (see [**12**] and [**4**]), every totally indefinite quadratic form of dimension ≥ 3 over an algebraic function field L in one variable over a real closed field is isotropic. Thus by taking, e.g., $m = 1$, every such function field L satisfies the Norm Theorem.

THEOREM 2. *Let L be a hilbertian field of characteristic* 0 *satisfying* NT. *Then, for every sum of squares $\beta \in L^\times$, the* 1*-dimensional form (β) is Witt equivalent to a trace form $T_L(F,1)$ over L for some extension F/L obtained by an irreducible linear trinomial $X^{m+1} + aX + b \in L[X]$ of odd degree.*

From these two theorems the Main Theorem follows at once:

Let K be an algebraic function field in one variable over a real closed field R and let ρ be a regular quadratic form which has totally positive signature over K. Since K is hilbertian and satisfies ED, by Theorem #1 we find a finite extension L/K and a sum of squares $\beta \in L^\times$ such that

$$\rho \simeq T_K(L,\beta).$$

Since L is again an algebraic function field in one variable over R, it is hilbertian and satisfies NT. Thus, applying Theorem 2 to L and β, we obtain a finite extension F/L such that

$$(\beta) \sim T_L(F, 1) \text{ in } W(L).$$

Using the transitivity of the trace and Corollary VII. 1.5 of [**6**], we finally get

$$\rho \sim T_K(F, 1) \text{ in } W(K). \square$$

At the end of the paper we will investigate the property NT for function fields a little closer.

Concerning notations and basic results about quadratic forms we refer the reader to [**6**].

2. Proof of Theorem 1. Since the case $X_K = \emptyset$ is already covered by Scharlau's paper, we concentrate on the case $X_K \neq \emptyset$.

Let K be hilbertian and satisfy ED. Given a regular quadratic form ρ of dimension n over K, we can then assume that ρ is represented by a diagonal matrix

$$\mathbf{D} = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \text{ with } d_i \in K^\times$$

such that, for all $P \in X_K$,

$$d_{i+1} \in P \Rightarrow d_i \in P.$$

If we assume $\operatorname{sgn}_P \rho \geq 0$ for all $P \in X_K$, we know that each d_i with $i \leq [(n+1)/2]$ is a sum of squares in K. As usual $[m/2]$ denotes the integral part of $m/2$. Thus, after rearranging the elements of the diagonal, we can assume that all d_i with odd index $d_1, d_3, d_5, \ldots$ are sums of squares in K. Multiplying by suitable squares, we may in addition assume that

$$d_{2i-1}d_{2i} \neq d_{2j-1}d_{2j}$$

for all $1 \leq i < j \leq [(n+1)/2]$. We require this condition also for the case $j = [(n+1)/2]$ and n odd after setting $d_{n+1} := d_n$.

Using now Scharlau's argument (see [8]) it suffices to find a symmetric matrix $B \in K^{(n,n)}$ such that

(i) the characteristic polynomial $f(X)$ of DB is irreducible over K.

(ii) $f(X)$ has exactly $\mathrm{sgn}_P\rho$ roots in the real closure $(\overline{K,P})$ of K with respect to P.

As it is explained in [8], by (i) there exists a β in $L = K[X]/(f)$ such that

$$\rho = T_K(L, \beta).$$

By (ii) the number of extensions of P to L is

$$\mathrm{sgn}_P T_K(L, \beta).$$

Thus by Proposition, β is a sum of squares in L.

In order to find such a matrix B, let us start with the symmetric matrix

$$\mathbf{B}_0 = \begin{pmatrix} 01 & & & \\ 10 & & & \\ & 01 & & \\ & 10 & & 0 \\ 0 & & \ddots & \end{pmatrix} = (b_{ij})$$

The last square in the diagonal of $\mathbf{B}_0$ is the 2-by-2 matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if n is even, and the 1-by-1 matrix 1 if n is odd.

Forming the characteristic polynomial

$$f_0(X) = \det_n(\mathbf{D}\mathbf{B}_0 - X\mathbf{I}_n)$$

we find that

$$f_0(X) = \prod_{1 \leq i \leq [n/2]} (d_{2i-1}d_{2i} - X^2) \cdot l(X),$$

where $l(X) = 1$ if n is even and $l(X) = d_n - X$ if n is odd. By our choice of the $d_i \in K$ we see immediately that

$$\mathrm{sgn}_P\rho = \#\{\text{zeros of } f_0 \text{ in } (\overline{K,P})\}$$

for each $P \in X_K$. Thus f_0 satisfies (ii).

Since, for a fixed ordering P, all the zeros of f_0 in $(\overline{K,P})$ are simple, any matrix $\mathbf{B}_P$ which has its elements very close to that of $\mathbf{B}_0$ in $(\overline{K,P})$ yields a polynomial

$$f_P = \det_n(\mathbf{D}\mathbf{B}_P - X\mathbf{I}_n)$$

which has the same number of zeros in $(\overline{K,P})$ as f_0. Actually, for a fixed $P \in X_K$ we find some $a_P \in P^\times$ such that, for all $\epsilon_{ij} \in (\overline{K,P})$ with $0 \leq_P \epsilon_{ij} \leq_P 1/a_P^2$ and $\epsilon_{ij} = \epsilon_{ji}$, the symmetric matrix

$$\mathbf{B}_{P,\epsilon} = \mathbf{B}_0 + \epsilon = (b_{ij} + \epsilon_{ij})$$

yields a polynomial

$$f_{P,\epsilon} = \det_n(\mathbf{D}\mathbf{B}_{P,\epsilon} - X\mathbf{I}_n)$$

having the same number of zeros in $(\overline{K,P})$ as f_0. Thus, in particular,

$$\mathrm{sgn}_P\rho = \#\{\text{zeros of } f_{P,\epsilon} \text{ in } (\overline{K,P})\}.$$

As we will see there are always choices of $\epsilon_{ij} \in K$ which make $f_{P,\epsilon}$ irreducible over K. This gives a positive solution to (i). But now (ii) can be guaranteed only for the ordering P which we fixed. Thus our problem is to find some $\epsilon_{ij} \in K$ which do the job simultaneously for all $P \in X_K$. This can be achieved in the following way.

For every $P \in X_K$ we choose $a_P \in K^\times$ as above and consider the subset U_P of X_K consisting of those $Q \in X_K$ such that

$$\mathrm{sgn}_Q\rho = \#\{\text{zeros of } f_{Q,\epsilon} \text{ in } (\overline{K,Q})\}$$

for all $\epsilon_{ij} \in (\overline{K,Q})$ satisfying $0 \leq_Q \epsilon_{ij} \leq_Q 1/a_P^2$ and $\epsilon_{ij} = \epsilon_{ji}$. Clearly $P \in U_P$, since it is well-known X_K is a compact space with respect to the topology generated by the subsets

$$H(c) = \{P \in X_K \mid c \in P\}, \qquad c \in K.$$

The sets U_P are open in this topology. This is a consequence of Tarski's Theorem on the Elimination of Quantifiers over real closed fields. In fact, it is not difficult to write down a formula $\varphi(x_1, \ldots, x_n, y)$ in the language of ordered fields such that

$$U_P = \{Q \in X_K \mid (\overline{K,Q}) \text{ satisfies } \varphi(d_1, \ldots, d_n, a_P)\}.$$

By Tarski's Theorem there are polynomials

$$p_{ij}, q_i \in \mathbf{Z}[X_1, \ldots, X_n, Y]$$

such that $\psi(d_1, \ldots, d_n, a_p)$ is equivalent to

$$\vee_{i=1}^{r}(q_i(\overline{d}, a_P) = 0 \wedge \wedge_{j=1}^{s} p_{ij}(\overline{d}, a_P) > 0)$$

in all real closures $(\overline{K,Q})$. Assuming w.l.o.g. that $q_i(\overline{d}, a_P) = 0$ for all i, we thus have

$$U_P = \cup_{i=1}^{r} \cap_{j=1}^{s} H(p_{ij}(\overline{d}, a_P)).$$

Hence U_P is open in X_K. By compactness we can therefore find a finite cover $U_{P_1} \cup \cdots \cup U_{P_m}$ of X_K. If we now let $a = a_{P_1}^2 + \cdots + a_{P_m}^2$, then the choice $\epsilon_{ij} = 1/(a + y_{ij}^2)$ with $y_{ij} = y_{ji} \in K$ obviously satisfies

$$0 \leq_Q \epsilon_{ij} \leq_Q \frac{1}{a_{P_\nu}^2} \quad \text{and} \quad \epsilon_{ij} = \epsilon_{ji}$$

for all $Q \in U_P$ and all $1 \leq \nu \leq m$. Thus, if we set

$$\mathbf{B}_Y = \mathbf{B}_0 + \left(\frac{1}{a + y_{ij}^2}\right) \text{ with } Y_{ij} = Y_{ji}$$

and

$$f_Y = \det_n(\mathbf{D}\mathbf{B}_Y - X\mathbf{I}_n),$$

by the definition of U_P, we obtain

$$\operatorname{sgn}_Q \rho = \#\{\text{zeros of } f_y \text{ in } (\overline{K,Q})\}$$

for all substitutions $y_{ij} \in K$ and all $Q \in X_K$. Thus f_y satisfies (ii). In addition we can choose $y_{ij} \in K$ such that f_y is also irreducible, thus also satisfying (i). In fact,

$$f_y = \frac{g(X, \overline{Y}_{ij})}{\prod_{1 \leq i,\ j \leq n}(a + y_{ij}^2)} \quad \text{with } g \in K[X, \overline{Y}_{ij}].$$

As we will show in the next section, g is irreducible over K. Thus, by the assumption on K being hilbertian, we find $y_{ij} \in K$ such that $g(X, \overline{y}_{ij}) \in K[X]$ is irreducible. This finishes the proof of Theorem 1. □

Looking carefully at the proof of Theorem 1 we can see that after having used ED the rest of the proof actually yields

ADDITION LEMMA. *Let K be a hilbertian field of characteristic zero. If $\rho_i \simeq T_K(L_i, \beta_i)$ for some extensions L_i/K and some sums of squares β_i of L_i (for $i = 1, 2$), then $\rho_1 \perp \rho_2 \simeq T_K(L, \beta)$ for some extension L/K and some sum of squares β* of L.

In fact, by Scharlau's argument we find symmetric matrices $\mathbf{B}_i$ such that (i) and (ii) holds for $\mathbf{D}_i\mathbf{B}_i$ where $\mathbf{D}_i$ are symmetric matrices representing ρ_i (for $i = 1, 2$). Considering now the matrix

$$\mathbf{B}_0 = \begin{pmatrix} \mathbf{B}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{B}_2 \end{pmatrix}$$

we can follow the proof of Theorem 1 in order to obtain the Addition Lemma. □

As a consequence we get

COROLLARY. *Every closed and open subset of the order space X_K of a hilbertian field K is the image under the restriction map of some finite extension L/K.*

PROOF. Let $A \subset X_K$ be open and closed. Then

$$A = \cup_{i=1}^{n} \cap_{j=1}^{m_i} H(a_{ij}), \qquad a_{ij} \in K^{\times}.$$

Clearly, the sets $B_i = \cap_{j=1}^{m_i} H(a_{ij})$ are the images under the restriction map for the fields

$$L_i = K(\sqrt{a_{i1}}, \dots, \sqrt{a_{im_i}})$$

for $1 \leq i \leq n$. Since the trace forms $\rho_i = T_K(L_i, 1)$ have non-vanishing signature exactly on B_i, the corollary follows from the Addition Lemma. □

This corollary generalizes the corresponding result of Andradas and Gamboa for real function fields [**1**, Theorem 4.1].

3. An irreducibility result. The aim of this section is to prove

LEMMA. *Assume that K is a formally real field. For $1 \le i,\ j \le n$ let $a_{ij} \in K$ and $c_{ij} \in K^\times$. Then, for any sum of squares $a \in K^\times$, the characteristic polynomial f_n of the matrix*

$$\mathbf{A}_n = \left(a_{ij} + \frac{c_{ij}}{a + Y_{ij}^2} \right) \quad \textit{with } Y_{ij} = Y_{ji}$$

is a quotient of an irreducible polynomial $g_n \in K[X, \overline{Y}_{ij}]$ and $\prod_{1 \le i,\ j \le n} (a + Y_{ij}^2)$.

This lemma applied to the case

$$a_{ij} = d_i b_{ij} \quad \text{and} \quad c_{ij} = d_i$$

yields the result used in the proof of Theorem 1.

PROOF. We proceed by induction on n. For the case $n = 1$ one obtains

$$g_1 = a_{11}(a + Y_{11}^2) + c_{11} - X(a + Y_{11}^2).$$

As a polynomial in X this is clearly irreducible over $K[Y_{11}]$ since $c_{11} \neq 0$.

Now let us assume by induction that $n \ge 2$ and, for all $m < n$, the polynomial g_m is irreducible and of degree m in X.

Writing for a moment Z_{ij} for $(a + Y_{ij}^2)^{-1}$, we obtain

$$\begin{aligned} f_n &= \det_n \begin{pmatrix} a_{11} + c_{11} Z_{11} - x & a_{12} + c_{12} Z_{12} \cdots & \\ a_{21} + c_{21} Z_{12} & & \\ \vdots & & \end{pmatrix} \\ &= (a_{11} + c_{11} Z_{11} - X) f_{n-1} + \sum_{1 \neq i,j} (a_{1i} + c_{1i} Z_{1i})\ (a_{j1} + c_{j1} Z_{1j}) f_{n-2}^{(i,j)}, \end{aligned}$$

where $f_{n-2}^{(i,j)}$ is (up to sign) the $\det_{n-2}$ of the matrix obtained by cancelling the 1st and ith columns and 1st and jth rows of $A_n - XI_n$. By induction hypothesis we have

$$f_{n-1} = \frac{g_{n-1}}{\Pi_{1<i,\ j\leq n}(a+Y_{ij}^2)}$$

with g_{n-1} irreducible and of degree $n-1$ in X and

$$f_{n-2}^{(\nu,\nu)} = \frac{g_{n-2}^{(\nu)}}{\Pi_{i,j\neq 1,\nu}(a+Y_{ij}^2)}$$

with $g_{n-2}^{(\nu)}$ irreducible and of degree $n-2$ in X. Clearly, in the case $n=2$ we let $f_0^{(2,2)} = g_0^{(2,2)} = -1$. Thus we obtain

$$g_n = (c_{11} + (a_{11} - X)(a + Y_{11}^2))\delta + (a + Y_{11}^2)\gamma$$

with

$$\begin{aligned}\delta &= g_{n-1}(a+Y_{12}^2)^2\cdots(a+Y_{1n}^2)^2\\ \gamma &= \sum_{1\neq i,j} h^{(i,j)} \prod_{\nu\neq 1,\ i}(a+Y_{1\nu}^2)\prod_{\mu\neq 1,j}(a+Y_{1\mu}^2)\end{aligned}$$

for suitable polynomials $h^{(i,j)} \in K[X,\overline{Y}_{ij}]_{i,j\neq 1}$. In particular we find

$$h^{(i,i)} = (c_{1i}+a_{1i}(a+Y_{1i}^2))(c_{i1}+a_{i1}(a+Y_{1i}^2))g_{n-2}^{(i)}(a+Y_{ii}^2)\cdot\prod_{j\neq 1,i}(a+Y_{ij}^2)^2.$$

As a polynomial in Y_{11} we have

$$g_n = \alpha Y_{11}^2 + \beta$$

with $\alpha = \gamma + a_{11}\delta - X\delta$ and $\beta = a\gamma + c_{11}\delta + aa_{11}\delta - aX\delta$. If α and β would have a common divisor, also δ and γ would have one. If $(a+Y_{1i}^2)$, for some $1<i$, would divide γ, then it would also divide $h^{(i,i)}$ and hence c_{1i} or c_{i1} which is impossible. Since g_{n-1} is irreducible and $\deg_X g_{n-1} = n-1 > \deg_X\gamma$, there could only be a common divisor of δ and γ, if $\gamma = 0$. But this would yield

$$0 = \sum_{1\neq i,j}(a_{1i} + c_{1i}Z_{1i})(a_{j1} + c_{j1}Z_{1j})f_{n-2}^{(i,j)}.$$

Observing that

$$\deg_X f_{n-2}^{(i,j)} < \deg_X f_{n-2}^{(\nu,\nu)} = n-2$$

for $i \neq j$ one easily sees that the highest coefficients of X cannot cancel.

If g_n were reducible in Y_{11}, then the highest coefficients of α and β in X would differ by a negative square from the field

$$\mathrm{Quot}(K[\overline{Y}_{ij}]_{(i,j)\neq(1,1)}).$$

This is impossible since they differ by a which is a non-zero sum of squares in K. Thus g_n is irreducible.

4. Proof of Theorem 2. Let $f \in L[X]$ be an irreducible linear trinomial

$$f(X) = X^{m+1} + aX + b$$

of odd degree. The trace form $T_L(F,1)$ of the extension $F = L[X]/(f)$ turns out to be (for a computation see [**3**] or [**10**])

$$T_L(F,1) \sim (1, m, -md) \text{ in } W(L),$$

where

$$d \equiv m^m a^{m+1} + (m+1)^{m+1} b^m \bmod L^2.$$

If we knew in addition that $d \equiv \beta \bmod L^2$ and that $-m$ is a norm from $L(\sqrt{\beta})$ over L (assuming that β is not a square in L), then the form $(1, -\beta, m)$ would be isotropic. Hence the 2-fold Pfister form $(1, -\beta, m, -\beta m)$ would be zero in $W(L)$. Thus we would get

$$T_L(F,1) \sim (\beta) \text{ in } W(L).$$

We are therefore looking for such a trinomial.

Let us first assume that β is not a square in L, since otherwise we may take $F = L$. Next let us assume w.l.o.g. that $-m$ is a norm from $L(\sqrt{\beta})$ and m is even. Now let

$$r = \frac{(m+1)^{m+1}}{m^m}, \qquad \beta_1 = \frac{\beta}{m^m}.$$

The polynomial

$$f(X,Y) = X^{m+1} + (\beta_1 Y^2 - r)X + (\beta_1 Y^2 - r)$$

is irreducible in $K[X,Y]$. This follows at once by Eisenstein's Criterion if we consider f as a polynomial in X over $L[Y]$. Since L is hilbertian, we find $y \in L^\times$ such that

$$f(X) = X^{m+1} + (\beta_1 y^2 - r)X + (\beta_1 y^2 - r)$$

is irreducible in $K[X]$.

If we now set $a = b = \beta_1 y^2 - r$ we have found the desired linear trinomial of odd degree. In fact, we have (observing that m is even) $\bmod L^2$

$$\begin{aligned} d &\equiv m^m a^{m+1} + (m+1)^{m+1} b^m \\ &\equiv m^m a + (m+1)^{m+1} \\ &\equiv \beta y^2 \\ &\equiv \beta. \end{aligned}$$

This finishes the proof of Theorem 2. □

It may be interesting to observe that the converse of Theorem 2 also holds, i.e., assuming that, in L, every sum of squares β is Witt equivalent to a trace form $T_L(F,1)$ for some F obtained by a linear trinomial $X^{m+1} + aX + b$ with m even, then L satisfies NT.

In fact, by this assumption we have

$$(\beta) \sim (1, m, -md) \text{ in } W(L)$$

with d as above. From this Witt equivalence we obtain the isometry

$$(\beta, 1, -1) \simeq (1, m, -md)$$

which clearly implies

$$\beta \equiv d \bmod L^2.$$

In particular we obtain that the 2-fold Pfister form $(1, m, -m\beta, -\beta)$ is zero in $W(L)$. But this implies that $(1, m, -\beta)$ is isotropic. Thus, in case β is not a square in L, this tells us that $-m$ is a norm from $L(\sqrt{\beta})$.

5. More about NT. The remark at the end of the last section shows that NT is in some sense essential for the result of the Main Theorem. It will be thus interesting to ask in general which function fields are satisfying this property. For rational function fields we can give a complete answer.

THEOREM. *Let k be a formally real field. Then the rational function field $k(t)$ satisfies* NT *if and only if k is hereditarily pythagorean, i.e., k and all its finite formally real extensions are pythagorean.*

PROOF. In [**2**; Chapter III, Theorem 4], it is shown that if k is hereditarily pythagorean, in $k(t)$ every sum of squares β is equal to a sum of 2 squares. Thus the form $(1, -\beta, 1)$ is isotropic over $k(t)$.

Conversely, let us assume that there is a finite formally real extension k_1 of k which is not pythagorean. Then there is some $\alpha \in k_1$ such that $\gamma = 1 + \alpha^2$ is not a square in k_1. We consider the extensions

$$k_2 = k_1(\sqrt{\gamma}) \quad \text{and} \quad k_3 = k_2\Big(\sqrt{\sqrt{\gamma} - \gamma}\Big).$$

Observing that $(\sqrt{\gamma} - \gamma)(-\sqrt{\gamma} - \gamma) = \gamma^2 - \gamma = \gamma(\gamma - 1) = \gamma\alpha^2$ we see that k_3/k_1 is cyclic with the automorphism

$$\sigma\Big(\sqrt{\sqrt{\gamma} - \gamma}\Big) = \sqrt{-\sqrt{\gamma} - \gamma}$$

generating the Galois group. The unique extension of k_1 of degree 2 in k_3 is k_2. Since k_2 is formally real, we find that, for every $m \in \mathbf{N}$, $\sqrt{-m} \notin k_3$. On the other hand k_3 is not formally real. In fact, $\gamma > 1$ implies $\gamma > \sqrt{\gamma}$. Denoting by $f(t)$ the irreducible polynomial of some generator of k_3 over k, we thus find polynomials $f_1, \dots, f_r \in k[t]$ such that

$$-1 \equiv f_1^2 + \cdots + f_r^2 \bmod f$$

and $\deg f_i < \deg f$. If we now assume that $k(t)$ satisfies NT, we could find some $m \in \mathbf{N}$ and $g_1, g_2, h \in k[t]$ such that

$$-mh^2 = g_1^2 - \Big(1 + \sum_{i=1}^{r} f_i^2\Big) g_2^2.$$

Since f divides $1+\sum f_i^2$, we may assume (after cancelling) that f does not divide h or g_1. Thus, computing $\bmod f$, we would get that $-m$ is a square in k_3. This contradiction proves the theorem. □

This theorem in particular shows that the field $\mathbf{Q}(t)$ does not satisfy Theorem 2. More precisely, considering the fields $k = k_1 = \mathbf{Q}$, $k_2(\sqrt{2})$, $k_3 = \mathbf{Q}((\sqrt{2}-2)^{\frac{1}{2}})$ we see that $f = t^4 + 4t^2 + 2$ and

$$-1 \equiv (t^2)^2 + (2t)^2 + 1^2 \bmod f.$$

Thus we find that the sum of squares

$$\beta = t^4 + 4t^2 + 1 \in \mathbf{Q}(t)$$

is not Witt equivalent to any trace form of a finite extension of $\mathbf{Q}(t)$ given by some linear trinomial of odd degree.

According to [**4**] and [**11**] a formally real function field in one variable over a field k satisfies ED if and only if k is hereditarily euclidean, i.e., k and all its finite formally real extensions are pythagorean and have just one ordering. Thus the main theorem already holds for algebraic function fields in one variable over a hereditarily euclidean field, since also Witt's Theorem on totally indefinite quadratic forms of dimension ≥ 3 holds for such function fields (see [**4**]).

REFERENCES

1. C. Andradas and J.M. Gamboa, *On projections of real algebraic varieties*, Pacific J. Math. **121** (1986), 281-291.

2. E. Becker, *Hereditarily pythagorean fields and orderings of higher level*, Monografias de Matemática **29**, IMPA, Rio de Janeiro, 1978.

3. P.E. Conner and R. Perlis, *A survey of trace forms of algebraic number fields*, Lecture Notes on Pure Math. **2**, World Scientific, Singapore, 1984.

4. R. Elman, Y.T. Lam and A. Prestel, *On some Hasse principles over formally real fields*, Math. Z. **134** (1973), 291-301.

5. D.R. Estes and J. Hurrelbrink-R. Perlis, *Total positivity and algebraic Witt classes*, Comment. Math. Helvetici **60** (1985), 284-290.

6. T.Y. Lam, *The algebraic theory of quadratic forms*, Benjamin, Reading, Mass. 1973.

7. A. Prestel and R. Ware, *Almost isotropic quadratic forms*, J. London Math. Soc. (2) **19** (1979), 241-244.

8. W. Scharlau, *On trace forms of algebraic number fields*, Math. Z. **196** (1987), 125-127.

9. ———, *Quadratic and hermitian forms*, Springer, Berlin-Heidelberg-New York-Tokyo, 1985.

10. J.-P. Serre, *L'invariante de Witt de la forme* $\mathrm{Tr}(x^2)$, Comment. Math. Helvetici **59** (1984), 651-676.

11. R. Ware, *Hasse principles and the u-invariant over formally real fields.* Nagoya Math. J. **61** (1976), 117-125.

12. E. Witt, *Theorie der quadratischen Formen in beliebigen Körpern*, J. reine angew. Math. **176** (1937), 31-44.

ALEXANDER PRESTEL, FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT KONSTANZ, WEST GERMANY