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# ON THE ANALYTICITY OF THE HOMOLOGY

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**Introduction .** Let $W$ be a real analytic manifold and $\{\alpha\} \in H_P(W, Z_2)$. We shall say that $\{\alpha\}$ is analytic if there exists a compact analytic subset $S$ of $W$ such that: $\{\alpha\} =$ (fundamental class of $S$). The purpose of this short paper is to prove

THEOREM 1. *Let $W$ be a paracompact real analytic manifold. Then any homology class $\{\alpha\} \in H_P(W, Z_2)$ is analytic.*

We remark that a similar result fails to hold in the real algebraic case (see [**2**]).

**1. Definitions and well known facts.** Let $V, W$ be two differentiable (i.e., $\mathbf{C}^\infty$) manifolds. Then, on the set $M(V, W)$ of differentiable maps $f : V \to W$, we shall consider the Whitney topology (see [**5**, p. 42]).

In the following we shall use the known result: if $f \in M(V, W)$, then there exists a neighborhood $U$, in the $\mathbf{C}^O$ topology, of $f$ such that any $g \in U$ is homotopic to $f$ (see [**7**]).

By a real algebraic variety we shall mean an affine real algebraic variety. A regular variety will be called an algebraic manifold. An algebraic map is the restriction of a rational map.

In the following we shall need

LEMMA 1. *Let $V \subset \mathbf{R}^n, W \subset \mathbf{R}^q$ be two real algebraic manifolds and $V \xrightarrow{\varphi} W$ be a differentiable map. If $V$ is compact and bordant to $\phi$, then, for any $\varepsilon > 0$, there exists an algebraic submanifold $V' \subset \mathbf{R}^{n+q}$, an analytic isomorphism $V' \xrightarrow{\pi} V$ and an algebraic map $\varphi' : V' \to W$ such that:*

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(i) $\delta(\varphi(x),\ \varphi' \cdot \pi^{-1}(x)) < \varepsilon, \quad x \in V;$

(ii) $\delta'((d\varphi)(v),\ (d(\varphi' \circ \pi^{-1})(N)) < \varepsilon$

*for any tangent vector* $v$, *to* $V$ *at* $x$.

Here $\delta, \delta'$ are two metrics on $\mathbf{R}^q$ and on the Grassmaniann manifold.

PROOF. See [**8**]. □

LEMMA 2. *Let* $V \subset \mathbf{R}^n, W \subset \mathbf{R}^q$ *be two real algebraic manifolds and* $\varphi : V \to W$ *an algebraic map. Let us suppose that* $V$ *is irreducible and there exists a Zariski open set* $V' \subset V$ *with the property that* $\varphi$ *is injective on* $V'$. *Under these hypotheses, if* $T$ *is the Zariski closure of* $\varphi(V)$ *in* $W$, *we have*:

(i) $T \supset \varphi(V)$, $\dim(T) = \dim(V)$; $T - \varphi(V)$ *is contained in an algebraic set* $S$ *with* $\dim(S) < \dim(V)$,

(ii) (*fundamental class of* $T$) $= \varphi_*$ (*fundamental class of* $V$).

PROOF. Property (i) is proved in [**3**, Lemma 1.1]. Property (ii) follows from the definition of the fundamental class, see [**4**], and the proof of the first part of the lemma. □

Now let $\varphi : V \to W$ be a differentiable map between differentiable manifolds. Let us denote by $\mathcal{E}(V)_x, \mathcal{E}(W)_{\varphi(x)}$ the stalks of the sheaves of the differentiable functions on $V, W$. We recall

DEFINITION. $\varphi$ is called *finite*, in the point $x$, if $\mathcal{E}(V)_x$ is a finite $\varphi^*(\mathcal{E}(V)_{\varphi(x)})$ module.

We have

LEMMA 3. *Let* $\dim(V) \leq \dim(W)$ *and let us suppose* $V$ *is compact. Then the set of differentiable maps that are finite in every point is an* *open dense subset of* $M(V, W)$.

PROOF. See [**1**, p. 96] (see also [**5**, p. 169]). □

**2. Proof of Theorem 1.** Suppose $w$ is a compact, real analytic manifold and $\{\alpha\} \in H_P(W, \mathbf{Z}_2)$. It is known, see [**9**], that we may suppose $W$ is a real algebraic manifold. Moreover there exists a compact differentiable manifold $V$ and a differentiable map $\varphi : V \to W$ such that

$$\{\alpha\} = (\varphi(\text{fundamental class of } V)), \text{ see } [\mathbf{7}].$$

By Lemma 1 we may suppose there exists an algebraic manifold $\hat{V} = V' \cup V''$ and an algebraic map $\hat{\varphi} : \hat{V} \to W$ such that:

(i) $V', V''$ are diffeomorphic to $V$;

(ii) $\{\alpha\} = \{\hat{\varphi}_*$ (fundamental class of $V')\} = \{\hat{\varphi}_*$(fundamental class of $V'')\}$;

(iii) $\hat{\varphi}|_{V'}$ is in general position with respect to $\hat{\varphi}(V'')$.

Moreover, by Lemma 3, we may suppose

(iv) $\hat{\varphi}$ is finite in every $x \in \hat{V}$.

From Lemma 2, finally, we may assume that

(v) if $T$ is the Zariski closure of $\hat{\varphi}(\hat{V})$ in $W$, ]then $T - \hat{\varphi}(\hat{V})$ is contained in an algebraic set $S$ such that $\dim(S) < \dim(V) = p$.

Now let $\tilde{\hat{\varphi}} : \tilde{\hat{V}} \to \tilde{W}$ be a complexification of $\hat{\varphi}$ (such $\tilde{\hat{\varphi}}$ exists, see [**10**]). We may assume $\tilde{\hat{\varphi}}$ is finite in any point of $\tilde{\hat{V}}$, because finiteness is an open condition, see [**5**, p. 168]. We shall suppose $\tilde{\hat{V}} = \tilde{V}' || \tilde{V}''$. The map $\tilde{\hat{\varphi}}$ is finite, hence the image of any analytic germ $\tilde{\hat{V}}_y$ is the germ of a complex analytic set of $\tilde{W}$, see [**6**, p. 162].

REMARK. The above facts imply that:

(a) $T =$ real part of the closure, in the Zariski topology, of $\tilde{\hat{\varphi}}(\tilde{\hat{V}})$.

(b) $d\tilde{\hat{\varphi}}$ has maximum rank on an open dense set of $\tilde{\hat{V}}$.

We deduce that, for any $x \in T$, we have three disjoint kinds of analytic irreducible germs of $T_x$:

(1) germs of the image of $\tilde{V}'$, of dimension $p$;

(2) germs of the image of $\tilde{V}''$, of dimension $p$;

(3) germs of the image of $\tilde{\hat{\varphi}}^{-1}(S)$, of dimension lower than $p$;

This proves that $T' = \hat{\varphi}(V') \cup S$ is an analytic subset of $W$ and, clearly,

$$\{\alpha\} = \{\text{fundamental class of } T'\}.$$

In fact, for any $y \in T$, the germ

$$Y_y = |\cup_i \tilde{\hat{\varphi}}(\tilde{V}'_{y_i})|_R$$

is real analytic, where

$$\cup y_i = \tilde{\hat{\varphi}}^{-1}(y) \cap \tilde{V}', \quad |Z|_R = \text{ real part of } Z.$$

So $Y_y \cup S_y$ is real analytic and, clearly, $Y_y \cup S_y = T'_y$. In fact, in any point, $T'$ is the union of a finite set of irreducible germs of $T$.

So the theorem is proved under the hypothesis that $W$ is compact. In the general case, we may take a representative element $\alpha$ of $\{\alpha\}$ contained in a relatively compact open set $U$ of $W$.

We can now realize $U$, up to analytic isomorphism, as an open set of a compact analytic manifold $Z$ (take the unique analytic structure on the double of $U$).

We can now prove the analyticity of $\{\alpha\}$ in $Z$ and this implies, clearly, the analyticity of $\{\alpha\}$ in $W$. □

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