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JACOBI THETA SERIES

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Dedicated to Professor Bruno Schoeneberg on his 80^{th} birthday

Let V be a quadratic space over $\mathbf{Q}$ of dimension N with quadratic form Q and bilinear form B such that $Q(x) = \frac{1}{2}B(x,x)$. Let Λ be an even unimodular positive definite lattice of rank N (i.e., a $\mathbf{Z}$-module such that $B(x,x) \equiv 0 \bmod 2$ for all $x \in \Lambda$ and $\Lambda^* = \{x \in V | B(x,\Lambda) \subset \mathbf{Z}\} = \Lambda$). Then N is a multiple of 8. We fix a $y \in \Lambda$ and consider the Jacobi theta series for Λ with respect to $y \in \Lambda$:

$$\theta_{Q,y}(\tau, z) := \sum_{x \in \Lambda} q^{Q(x)} \zeta^{B(x,y)}$$

(where $q = \exp(2\pi i\tau)$, $\zeta = \exp(2\pi iz)$), which is a Jacobi form of weight $k = N/2$ and index $m = Q(y)$ on $\mathrm{SL}(\mathbf{Z})$ [**2**, Theorem 7.1]. We consider the

QUESTION. If $y_1, y_2 \in \Lambda$ with $Q(y_1) = Q(y_2) = m$, in which cases is $\theta_{Q,y_1} = \theta_{Q,y_2}$?

This is interesting, e.g., because of the following evident fact: If the automorphism group of the lattice $\mathrm{Aut}(\Lambda)$ acts transitively on $\Lambda_{2m} := \{x \in \Lambda | B(x,x) = 2m\}$, then $\theta_{Q,y_1} = \theta_{Q,y_2}$ for $y_1, y_2 \in \Lambda$ with $Q(y_1) = Q(y_2) = m$.

Considering now only extremal lattices (defined by $\min_{x\in\Lambda, x\neq 0} Q(x) = [N/24] + 1$), it is well known that $\mathrm{Aut}(\Lambda)$ acts transitively on Λ_2 in dimensions 8 and 16 and $\mathrm{Aut}(\Lambda)$ acts transitively on Λ_4 in dimension 24 (the Leech lattice being the only extremal lattice in this dimension); in dimension 32 it is in general open whether $\mathrm{Aut}(\Lambda)$ acts transitively on Λ_4. However, we have the result that $\theta_{Q,y}$ is independent of y with

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$Q(y) = 2$ for any extremal lattice in dimension 32. The situation is analogous in dimensions 48 and 56. (see §1). Consequences are certain divisibility properties of Fourier coefficients of Siegel theta series of degree 2. Furthermore we consider lattices with non-empty root systems (i.e., $\Lambda_2 \neq \emptyset$). (see §2). Finally we remark the coincidence of the "singular series" for the simultaneous representation of two numbers by a quadratic and a linear form with an Eisenstein series in the theory of Jacobi forms in certain cases (see §3).

The terminology follows, throughout, the book of Eichler-Zagier [**3**]. I am grateful to M. Ozeki and R. Schulze-Pillot for critical remarks.

1. Extremal lattices.

PROPOSITION. *$\theta_{Q,y}$ is independent of y with $Q(y) = 2$ for any extremal (even unimodular positive definite) lattice in dimension 32.$\theta_{Q,y}$ is independent of y with $Q(y) = 3$ for any extremal lattice in dimensions 48 and 56. Furthermore, for extremal lattices in dimension 32 (respectively 48 and 56), there exists a unique Jacobi theta series.*

PROOF. We use an explicit basis of $J_{k,m}$ (the $\mathbf{C}$-vector space of Jacobi forms of weight k and index m). In [**3**], Theorem 9.2 implies $\dim J_{16,2} = 4$, $\dim J_{24,3} = 8$, $\dim J_{28,3} = 9$. We use the Fourier expansion

$$\theta_{Q,y}(\tau, z) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathbf{Z} \\ r^2 \leqq 4nm}} c(n,r) q^n \zeta^r,$$

where we abbreviate $c(n,r) =: c(4nm - r^2)$, since, for $k \in 2\mathbf{Z}, m = 1$ or prime, $c(n,r)$ depends only on $4nm - r^2$ [**3**, Theorem 2.2]. Thus cusp forms in our situation are characterized by $c(0) = 0$.

Now let Λ be extremal of dimension 32, i.e., there are no $z \in \Lambda$ with $B(z,z) = 2$. Thus, since $B(x+y, x+y) = B(x,x) + B(y,y) + 2B(x,y)$, if $B(x,x) = 6$ and $B(y,y) = 4$, it follows that $B(x,y) \neq \pm 4$. Consequently $c(3,4) = c(4 \cdot 2 \cdot 3 - 4^2) = c(8) = 0$. Similarly one gets $c(7) = 0$, $c(4) = 0$. Now $ZE_8/12$, $X\Delta$, $(XE_{12} + YE_{10})/24$ (terminology of [**3**, Chapter III]) have the Fourier coefficients

	$c(0)$	$c(4)$	$c(7)$	$c(8)$
$S_1 := \frac{1}{12} Z E_8$	0	1	-4	6
$S_2 := X\Delta$	0	0	0	1
$S_3 := \frac{1}{24}(XE_{12} + YE_{10})$	0	1	8	27, ...

,

so they are linearly independent cusp forms and we can write

$$\theta_{Q,y} = E_{16,2} + a_y^{(1)} S_1 + a_y^{(2)} S_2 + a_y^{(3)} S_3.$$

Using $c(4) = c(7) = c(8) = 0$ we get linear equations for $a_y^{(i)} \in \mathbf{C}$ which fix them uniquely independent of y. By the way, Venkov [**8**, Theorem 3], showed (in other terminology) that $c(12), c(15), c(16)$ are independent of the lattice and of y and computed the explicit values.

For extremal lattices in dimensions 48 and 56 we have, analogously,

$$c(3) = c(8) = c(11) = c(12) = c(15) = c(20) = c(23) = c(24) = 0.$$

Here we have $\theta_{Q,y} = E_{24,3} + \sum_{i=1}^{7} a_y^{(i)} S_i$ with following cusp forms S_i and their Fourier coefficients

	$c(3)$	$c(8)$	$c(11)$	$c(12)$	$c(15)$	$c(20)$	$c(23)$	$c(24)$
$J_{6,3}^{+} E_{18}$	1	-6	15	20	*	*	*	*
$\frac{1}{12}(E_{8,1}^3 - E_{6,2}E_{4,1}E_{14})$	0	1	20	102	*	*	*	*
$E_{6,1}^2 \phi_{12,1}$	0	0	1	10	2	-146	-2506	-7084
$E_{4,1}^3 \Delta$	0	0	0	1	0	3	168	378
$J_{8,2}^{+} S_{16,1}^{(1)}$	0	0	0	0	1	-6	15	-20
$\frac{1}{12}(J_{8,2}^{+}(S_{16,1}^{(2)} - S_{16,1}^{(1)}))$	0	0	0	0	0	1	-4	6
$E_{4,1}^2 S_{16,1}^{(2)} - E_{6,1}^2 \phi_{12,1}$	0	0	0	0	0	278	3880	9828

where $S_{16,1}^{(1)}, S_{16,1}^{(2)}$ denote the two forms with $k = 16$ in [**3**, Table 2]; $J_{8,2}^{+}, J_{6,3}^{+}$ are from [**3**, Table 3].

Now we see directly the claim of the proposition for dimension 48. By the way, Venkov [**10**, §3] gives without proof the result that $c(27), c(32), c(35), c(36)$ are independent of the extremal lattice and independent of y, and he gives explicit values for these Fourier coefficients.

For extremal lattices in dimension 56 - which exist according to Ozeki [4] (cf. [2, p. 156]) - we have $\theta_{Q,y} = E_{28,3} + \sum_{i=1}^{8} a_7^{(i)} S_i^*$. We take $S_i^* = S_i \cdot E_4$ with the S_i defined above (in the case of dimension 48) and $S_8' = J_{10,2}^+ \cdot S_{18,1}^{(2)}$, where $J_{10,2}^+$ is from [2, Table 3] and $S_{18,1}^{(2)}$ is the second form with $k = 18$ in [2, Table 2]. Then

	$c(3)$	$c(8)$	$c(11)$	$c(12)$	$c(15)$	$c(20)$	$c(23)$	$c(24)$
S_1^*	1	-6	15	20	*	*	*	*
S_2^*	0	1	20	102	*	*	*	*
S_3^*	0	0	1	10	2	-146	-2266	-4684
S_4^*	0	0	0	1	0	3	168	378
S_5^*	0	0	0	0	1	-6	15	-20
S_6^*	0	0	0	0	0	1	-4	6
S_7^*	0	0	0	0	0	278	3880	9828
S_8'	0	0	0	0	1	18	63	-164
$\frac{1}{24}(S_8' - S_5^*)$	0	0	0	0	0	1	2	6

Let $S_8^* := (S_8' - S_5^*)/24$ and one easily sees the claim of the proposition in dimension 56. □

An application of the proposition to Fourier coefficients of Siegel theta series of degree 2 is as follows. Let $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$, $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ and consider the Siegel theta series of degree 2,

$$\theta_Q^{(2)}(Z) = \sum_{T \geqq 0} r_Q(T) \exp(2\pi i \operatorname{tr}(TZ)),$$

(where

$$\begin{aligned} &r_Q\left(\begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix}\right) \\ &= \#\{(x,y) \in \Lambda \times \Lambda \| Q(x) = n, \quad Q(y) = m, \quad B(x,y) = r\} \\ &= \sum_{m=0}^{\infty} \phi_m(\tau, z)\exp(2\pi i m\tau') \end{aligned}$$

(Fourier-Jacobi expansion, where $\phi_m \in J_{k,m}$)).

Then

$$\phi_m(\tau, z) = \sum_{\substack{y\in\Lambda \\ Q(y)=m}} \theta_{Q,y}(\tau, z)$$

[**3**, Theorem 7.3]. The proposition then implies the

COROLLARY. *For* Λ *extremal, we have*

$$\textit{in dimension } 32 : |\Lambda_4| = 146880 | r_Q\left(\begin{pmatrix} n & r/2 \\ r/2 & 2 \end{pmatrix}\right)$$

$$\textit{in dimension } 48 : |\Lambda_6| = 52416000 | r_Q\left(\begin{pmatrix} n & r/2 \\ r/2 & 3 \end{pmatrix}\right)$$

$$\textit{in dimension } 56 : |\Lambda_6| = 15590400 | r_Q\left(\begin{pmatrix} n & r/2 \\ r/2 & 3 \end{pmatrix}\right).$$

REMARK. This was checked numerically in dimension 32 in many cases in [**5**].

The corresponding result is not true in dimension 40, as was pointed out to me by Ozeki, cf. [**6**]. Thus our proposition is not true in dimension 40.

2. Lattices with non-empty root systems. Let Λ be an even unimodular positive definite lattice of rank N with non-empty root system, i.e., $\Lambda_2 \neq \emptyset$. Take $y \in \Lambda$ with $Q(y) = 1$ so that $\theta_{Q,y} \in J_{k,1}, k = N/2$. Let $\Lambda_2 = R_l \oplus \cdots \oplus R_1$ with R_i an irreducible root system. If $y \in R_i$, then

$$\sum_{\substack{B(r_i,y)=\pm 1\\ 1 r_i \in R_i}} 1 = 4h(R_i) - 8 \quad [\mathbf{1}; \text{Chapter VI, §1, 11, Proposition 32}]$$

$$= 2c_{Q,y}(1,1) = 2c_{Q,y}(3).$$

where $h(R_i)$ denotes the Coxeter number of R_i and $c_{Q,y}$ denote the Fourier coefficients of $\theta_{Q,y}$. Thus, $\theta_{Q,y}$ independent of y with $Q(y) = 1$ implies $h(R_1) = \cdots = h(R_l)$. Conversely, if $h(R_1) = \cdots = h(R_l)$, then $c_{Q,y}(3) =: c(3)$ is independent of y.

Evidently we have the identity

$$(*) \qquad 2c_{Q,y}(3) + c_{Q,y}(4) + 2 = |\Lambda_2|.$$

Now let us consider the cases of low dimensions:

$$\textit{Dimension } 8 : \theta_{Q,y} = E_{4,1}$$

$$\textit{Dimension } 16 : \theta_{Q,y} = E_{8,1}$$

$$\textit{Dimension } 24 : \theta_{Q,y} = E_{12,1} + a_y \phi_{12,1}.$$

So the first Fourier coefficients are (using [**2**, Table 1])

$$c_y(3) = e(3) + a_y \cdot 1$$

$$c_y(4) = e(4) + a_y \cdot 10.$$

Using $(*)$ we get $2e(3) + e(4) + 12a_y + 2 = |\Lambda_2|$; thus, a_y is independent of y, therefore $h(R_1) = \cdot = h(R_l)$ (see [**6**] for a different proof). Thus $\theta_{Q,y}$ is independent of y.

Dimension 32. There are lattices with different Coxeter numbers in their irreducible components of the root system, e.g., [**9**] mentions a lattice with root system $D_{20} + D_{12}$. But if we *restrict* to the case of lattices with $h(R_1) = \cdots = h(R_l)$, then one can show that $\theta_{Q,y}$ is independent of y with $Q(y) = 1$, using an explicit basis of $J_{16,1}$ with dim $J_{16,1} = 3$.

Dimension 40. dim $J_{20,1} = 3$ and the same result is true as in Dimension 32.

Dimension 48. dim $J_{24,1} = 4$ and the same result is true as in dimension 32 and 40.

The proof of these claims in dimension 32, 40 and 48 is similar, let us only describe the case of dimension 48:

$$\theta_{Q,y} = E_{24,1} + a^{(1)}_{24,1} \int_{24,1}^{(1)} + a_y^{(2)} S^{(2)}_{24,1} + a_y^{(3)} S^{(3)}_{24,1},$$

where $S^{(i)}_{24,1}$ are the 3 forms with $k = 24$ in [**3**, Table 2].

Inserting the first Fourier coefficients, we have

$$c(3) = e(3) + a_y^{(1)} 1 + a_y^{(2)} 1 + a_y^{(3)} 1$$

$$c(4) = e(4) + a_y^{(1)}(-2) + a_y^{(2)} \cdot 10 + a_y^{(3)} \cdot 10$$

$$c(7) = e(7) + a_y^{(1)}(-40) + a_y^{(2)} \cdot 632 + a_y^{(3)}(-1096),$$

$c(3)$ is independent of y by hypothesis, $c(4)$ by $(*)$ and $c(7)$, since

$$c(7) = -24(8 + 2c(3)) - 4c(4) + 2|\Lambda_2| + \frac{1}{12}|\Lambda_4|$$

which follows from Venkov's "fundamental equations" [**8**, §2]. Our claim follows then by solution of the above linear system for $a_y^{(1)}, a_y^{(2)}, a_y^{(3)}$.

3. Singular series. The computation of the number of simultaneous representations of two numbers by a quadratic and a linear form is equivalent to the computation of a Jacobi theta series. The former question was handled earlier by computation of the so-called "singular series", a product of all local representation densities, and estimation of the remainder term (see [**11**] and the literature quoted there).

PROPOSITION. *For even unimodular positive definite quadratic forms of rank N which represent* 1, *the singular series coincides with the Eisenstein series $E_{k,1}$ ($k = N/2$). The same result is true with $E_{k,m}$ if m is square free, if one takes y with $Q(y) = m$.*

The *proof* follows by explicit computation using [**11**, **3**]. As a consequence we have the asymptotic result

$$c_{Q,y}(4n-r^2) = e_{k,1}(4n-r^2) + O\Big((4n-r^2)^{\frac{k}{2}-\frac{1}{2}+\epsilon}\Big)$$

for $(4n - r^2) \to \infty$, where $c_{Q,y}$, respectively $e_{k,1}$, are the Fourier coefficients of $\theta_{Q,y}$, respectively $E_{k,1}$.

The proposition is analogous to the situation in Siegel's theorem in the analytic theory of quadratic forms.

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