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# IDEAL CLASS GROUPS OF WITT RINGS

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Let $F$ be a formally real field with only finitely many orderings. Let $R$ denote the Witt Ring of $F$. In [**1**] we gave necessary and sufficient conditions for every ideal of $R$ containing an odd dimensional form to be principal. The wish to place this result in the more natural context of multiplicative ideal theory led to the problem of computing the ideal class group $C(R)$ of $R$. Details will appear elsewhere.

DEFINITION. An element $a \in R$ is *regular* if it is not a zero-divisor and *strongly regular* if it is odd dimensional. An ideal $I \subset R$ is (*strongly*) *regular* if it contains an element which is (strongly) regular.

PROPOSITION 1. *Let $I \subset R$ be a strongly regular ideal. Then:*

(1) *$I$ is a unique (finite) product of prime ideals;*

(2) *$I$ has a unique primary decomposition.*

SKETCH OF PROOF. If $R$ is not reduced, then $R$ is a Prüfer Ring [**4**], and so (1) follows. (2) follows from (1) by the identity $(I+J)(I \cap J) = IJ$ (for ideals $I, J$ one of which is regular) which holds in Prüfer Rings [**3**].

If $R$ is reduced, then (2) follows from previous work on primary decomposition [**1**]. And (1) is deduced from (2) by a standard primary decomposition argument. □

PROPOSITION 2. *Let $[I]$ denote the class of a regular ideal $I$ in the ideal class group $C(R)$. Then there exists a strongly regular ideal $J$ such that $[I] = [J]$.*

PROOF. We may assume $I \subset IF$. Then $I_{IF}$ is principal, generated by a regular element $a \in I$. Now $I = (a, b)$ for some $b \in R$ by [**2**]. Since

$b \in I_{IF} = (a)_{IF}$ we have $by = ax$, for some $x, y \in R$ and $y$ strongly regular. Thus

$$(y)(a,b) = (ya, yb) = (ya, ax) = (a)(x,y).$$

Then $J = (x,y)$ is the desired ideal. □

Let $X_F$ denote the (finite) set of orderings on $F$. If $\alpha \in X_F$ and $m \in \mathbf{N}$ let $P(\alpha, m) = \{x \in R | \operatorname{sgn}_\alpha x \equiv 0 (\operatorname{mod} m)\}$. Propositions 1 and 2 reduce the computation of $C(R)$ to finding all relations among the $[P(\alpha, p)]$, where $\alpha \in X_F$ and $p$ is an odd prime.

DEFINITION. $F$ is *weakly $n$-stable* if, for all disjoint clopen subsets $A, B \subset X_F$, there exists a form $x \in I^n F$ such that $\operatorname{sgn}_\alpha x = 0$ for $\alpha \in A$, and $\operatorname{sgn}_\beta x = 2^n$ for $\beta \in B$.

Note that $X_F$ finite implies $F$ is weakly $n$-stable for some $n$. The main technical result on weak $n$-stability is

LEMMA 3. *$F$ is weakly $n$-stable if and only if $P(\alpha, m)$ is principal for all $\alpha \in X_F$ and $m \equiv \pm 1 (\operatorname{mod} 2^n)$.*

Thus, if $F$ is weakly $n$-stable, then the subgroup $H(\alpha) = \{[P(\alpha, m)] | m \text{ odd}\} \subset C(R)$ is isomorphic to the group of units $U(\mathbf{Z}_2 n)$ modulo $\pm 1$. This is known to be cyclic of order $2^{n-2}$ with generator 5. We get the following results on $C(R)$:

THEOREM 4. *$C(R) = \{1\}$ if and only if $F$ is weakly 2-stable.*

THEOREM 5. *Suppose $F$ is weakly $n$-stable but not weakly $(n-1)$-stable $(n \geq 3)$. Let $r = |X_F|$. Then:*

(1) *$C(R)$ is generated by $\{[P(\alpha, 5)] | \alpha \in X_F\}$;*

(2) *$C(R)$ is a finite group of 2-power order;*

(3) *Every $[I] \in C(R)$ has order at most $2^{n-2}$;*

(4) $2^{n-2} \leq |C(R)| \leq 2^{(n-2)(r-1)}$.

We isolate two noteworthy consequences:

COROLLARY 6. *Let $F$ be weakly $n$-stable. Let $I \subset R$ be an ideal containing a regular element $a$. Then:*

(1) *If $I$ is invertible, $I = (a, b)$ for some $b \in R$;*

(2) *$I^{2^{n-2}}$ is principal.*

We close with one explicit computation (using the easy half of the representation theorem [**5**, 6.8]).

PROPOSITION 7. *Let $G_n$ be a group of exponent 2 and order $2^n$. Let $R = \mathbf{Z}[G_n]$ (e.g., $F = \mathbf{R}((t_1))\cdots((t_n))$). Then*

$$C(R) \approx \oplus_{i=1}^{n-2}(\mathbf{Z}_{2^i})^{\binom{n}{i+2}}. \ \square$$

## REFERENCES

**1.** R. Fitzgerald, *Primary Ideals in Witt Rings,* J. Alg. **96** (1985), 368-385.

**2.** R. Gilmer and W. Heinzer, *On the Number of Generators of an Invertible Ideal,* J. Alg. **14** (1970), 139-151.

**3.** M. Griffin, *Prüfer Rings with Zero Divisors,* J. Reine Angew. Main. **239** (1969), 55-67.

**4.** ———, *Valuations and Prüfer Rings,* Can. J. Math. **26** (1974), 412-429.

**5.** T.Y. Lam, *Orderings, Valuations and Quadratic Forms,* CBMS Regional Conference Series in Math., No. 52, 1983.

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