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LINEAR SPACES OF SEQUENCES

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ABSTRACT. If $A = (a_{nk})$ denotes a non-negative regular summation matrix and $1 < p < \infty$, then $L^p(A)$ denotes the space of sequences $s = \{s_n\}$ such that $||s||_p = \lim\sup(\sum a_{nk}|s_k|^p)^{1/p}$ is finite. The sequences in $L^P(A)$ are represented as bounded linear operators from $L^{p'}(A)$ to a space of continuous functions. The induced operator topologies are in part related to the strength of the matrix A. Conditions for a sequence to be an extreme point of the unit sphere of $L^p(A)$ are given.

0. Introduction. Let $A = (a_{nk})$ be a non-negative regular summation matrix, that is, the elements $a_{nk}(n = 0, 1, \ldots; k = 0, 1 \ldots)$ of A satisfy the conditions

(i) $a_{nk} \geq 0, \ n = 0, 1, \ldots; \ k = 0, 1, \ldots,$

(ii) $\lim_{n\to\infty} a_{nk} = 0, \ k = 0, 1, \ldots.$

(iii) $\lim_{n\to\infty} \sum_{k-1}^{\infty} a_{nk} = 1$

[**3**, p. 43]. By the space $L^p(A)$, where p is a number greater than or equal to 1, we understand the space of sequences $s = \{s_n\}$ such that $||s||_p = \lim\sup(\sum_{k=0}^{\infty} a_{nk}|s_k|^p)^{1/p} < \infty$; two sequence s and t are identified in $L^p(A)$ if $||s - t||_p = 0$, that is, if A evaluates the sequence $\{|s_n - t)n|^p\}$ to 0.

Lau [**4**] studies similar L^p spaces of functions defined on the real line rather than the set N of natural numbers. He obtains, among other things, the duals of the L^p spaces and a determination of the extreme points of the unit sphere in his L^p spaces. We will obtain analogous results for the space $L^p(A)$; also we regard the elements of $L^p(A)$ as bounded linear operators from $L^{p'}(A)$ (throughout, the symbol p' will denote the number $p/(p-1)$ for $p > 1$) to a space of continuous functions and we will relate the operator topologies to the strength of the summation matrix A. The matrix A will always

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be assumed to satisfy (i), (ii) and (iii). The number p will always be assumed to satisfy $1 < p < \infty$; often analogous results for the cases $p - 1$, thus $p' = \infty$ or $p - \infty$, can easily be obtained.

In most cases our results hold no matter whether real or complex sequences are involved. Therefore, we will assume the sequence to be real or complex, depending on which kind makes the proof simpler.

If E is a subset of N, then the quantity $\lim_n \sup \sum_{k \in E} a_{nk}$ will be denoted by $A(E)$; $\lim_{n \to \infty} \sum_{k \in E} a_{nk}$ (if it exists) will be denoted by $A_0(E)$.

We will be dealing frequently with the Stone-Čeck compactification βN of the discrete space N of natural numbers; therefore we make a few remarks about the Stone-Čech compactification. If X is a completely regular space, then there exists a compact space βX in which X is densely imbedded such that every bounded continuous function on X can be continuously extended to βX. For a description of the Stone-Čech compactification we refer the reader to [**2**, pp. 82-93]. If f is a bounded continuous function on the space X we will always denote its continuous extension to βX by f^β; if v is a point of βX, the symbol F_v^β will always express the fact that the function f^β has been evaluated at the point v. The symbol N will always denote the discrete space of natural numbers. If $E \subseteq N$, the closure of E in $\beta N - N$ will be denoted by E^*; in particular $N^* = \beta N - N$.

If t is in a sequence space $L^{p'}(A)$, then, for each point ν in N^* the functional $L(t, \nu)$ on $L^p(A)$ given by

$$(1) \qquad L(t,\nu)(s) = \left(\sum a_{nk} s_k t_k\right)_\nu^\beta, \quad s \in L^p(A), \quad p > 1,$$

is bounded and $||L|| = ||t||_{p'}$. We have

THEOREM 0.1. *The functionals $L(t,\nu)(s)$ given by (1), as t ranges over $L^{p'}(A)$ and ν ranges over N^* are weak * dense in the dual of $L^p(A)$.*

PROOF. Suppose that s is in $L^p(A), s \neq 0$. Let

$$t_k = |s_k|^{p-2} \bar{s}_k \text{ if } s_k \neq 0,$$

$$t_k = 0 \text{ if } s_k = 0.$$

Then $t \in L^{p'}(A)$ and $||t||_{p'} = ||s||_p^{p'}$. We have

$$L(t,\nu)s = \Big(\sum_{k=0}^{\infty} a_{nk}|s_k|^p\Big)_\nu^\beta$$

There is a point $\nu \in B^*$ such that $L(t,\nu)s = ||s||_p^p \neq 0$. In other works if all functionals of the form (1) annihilate s, then $s = 0$. The result follows. □

1. The sequences in $\mathbf{L^p}$ as bounded operators on $\mathbf{L^{p'}}$. Operator topologies. Throughout, the operator from $L^{p'}(A)$ to $C(N^*)$ corresponding to a sequence in $L^p(A)$ will be denoted by the corresponding capital letter, that is, if s is a sequence in $L^p(A)$, then the operator S is defined by

$$S(t)(\nu) = \Big(\sum_{k=0}^{\infty} a_{nk}s_k t_k\Big)_\nu^\beta, \quad t \in L^{p'}(A), \quad \nu \in N^*.$$

In addition to the norm topology and the weak topology on $L^p(A)$ we have

(a) The strong operator topology: A net of operators $S^{(\alpha)}$ corresponding to sequences $\{s^{(\alpha)}\}$ in $L^p(A)$ converges to 0 in the strong operator topology if and only if

$$\lim_\alpha \limsup_n \Big|\sum_{k=0}^{\infty} a_{nk}s_k^{(\alpha)}t_k\Big| = 0$$

for all $t \in L^{p'}(A)$.

(b) The weak operator topology: A net $\{s^{(\alpha)}\}$ in $L^p(A)$ converges to 0 in the weak operator topology if and only if the quantities $\{\sum_{k=0}^{\infty} a)nks_k^{(\alpha)}t_k\}$ are uniformly bounded and

$$\lim_\alpha \Big(\sum_{k=0}^{\infty} a_{nk}s_k^{(\alpha)}t_k\Big)_\nu^\beta = 0$$

for each sequence $t \in L^{p'}(A)$ and each point $\nu \in B^*$. (cf. [**1**, p. 265].)

The summation matrix B is said to include the summation matrix A if every sequence evaluated by A is evaluated by B to the same value. Then we have

THEOREM 1.1. *If A and B are non-negative regular summation matrices, A is invertible and B includes A, then $L^p(A) \subseteq L^p(B)$. Also the identity mapping injecting $L^p(A)$ into $L^p(B)$ is continuous relative to the norm topology.*

PROOF. Let (s_n) be a sequence in $L^p(A)$. Let $\{u_n\}$ and $\{\nu_n\}$ be the sequences $\{\sum_{k=0}^{\infty} a_{nk}|s_k|^p\}$ and $\{\sum_{k=0}^{\infty} b_{nk}|s_k|^p\}$ respectively, thus $\nu = BA^{-1}u$ (clearly v exists). If we let $c = (c_{nk})$ denote BA^{-1}, then

$$\begin{aligned}\text{LUB}|\nu_n| &\leq \text{LUB}\sum_{k=0}^{\infty} |c_{nk}| \\ &\leq \text{LUB}\sum_{k=0}^{\infty} |c_{nk}|\ \text{LUB}|u_n|.\end{aligned}$$

The fact that B includes A and thus BA^{-1} is regular guarantees that $\text{LUB}\sum_{k=0}^{\infty}|c_{nk}| < \infty$. Hence S is in $L^p(B)$. Also, the norm of S in $L^p(B)$ is at most $\text{LUB}\sum_{k=0}^{\infty}|c_{nk}|$ times the norm of s in $L^p(A)$. This completes the proof. □

However it is not true that if the regular matrix B strictly includes the regular matrix A, then the space $L^p(B)$ strictly includes the space $L^p(A)$. For example if A is the identity matrix and B is the Nörlund matrix defined by the equations

$$b_{00} = 1,$$

$$b_{n,n} = b_{n,n-1} = 1/2,\ b_{n,k} = 0 \quad k \neq n, k \neq n \geq 1,$$

then B strictly includes A, yet $L^p(A)$ coincides with $L^p(B)$ (and the $L(A)$ norm is equivalent to the $L^p(B)$ for all p). However we have a result in the opposite direction.

THEOREM 1.2. *Suppose that the matrices A and B are non-negative and regular, and that there exist disjoint subsets E, F of N such that*

(2) $A_0(E)$ *does not exist,*

(3) $B_0(E) = 0$, and

(4) $A(F) > 0,\ B(F) > 0,\ \lim_n \max_{k\in F} b_{n.k} = 0$.

Then, for each $p > 0$, there exists a sequence of elements $\{s^{(m)}\}$ in $L^p(A)\cap L^p(B)$ which converges to 0 *in the norm topology of $L^p(B)$ but not in the weak operator topology of $L^p(A)$.*

Conditions (2) and (3) insure that B evaluates the sequence 1_E (i.e., the sequence $s = \{s_n\}$, where $s_n = 1\ k \in E,\quad s_n = 0,\ k \notin E$) to 0 while A does not evaluate 1_E.

PROOF. Let $s^{(m)} = 1_{E\cup F_m}(m = 1, 2, \dots)$ where, for each m, F_m is a subset of F such that $B(F_m) > 0, \lim_{m\to\infty} B(F_m) = 0$. The sets F_m with the stated properties exist because of (4). Certainly $\{s^{(m)}\}$ tends to 0 in the norm topology of $L^p(B)$. On the other hand let $t = 1_E$. Then

$$\limsup \sum_{k=0}^{\infty} a_{nk} s_k^{(m)} t_k = A(E) > 0.$$

Hence there is a point $\nu \in N^*$ such that

$$L(s^{(m)}, t)(\nu) = A(E) > 0$$

for all m. The sequence $\{s^{(m)}(t)\}$ does not tend to 0 pointwise on N^* and hence is not weakly convergent to 0 in $C(N^*)$. Thus the sequence $\{s^{(m)}\}$ does not converge to 0 in the weak operator topology in $L^p(A)$.

THEOREM 1.3A. *Suppose that A and B are non-negative regular summation matrices, A is invertible and that B includes A. If, for each index $\alpha, s^{(\alpha)}$ is in $L^p(A)\cap L^p(B)$ and $s^{(\alpha)}$ tends to* 0 *in the strong operator topology of $L^p(A)$, then $\{s^{(\alpha)}\}$ tends to* 0 *in the strong operator topology of $L^p(B)$.*

PROOF. Let t be a sequence in $L^p(A) \cup L^p(B)$. For each index α let

$$u_n^{(\alpha)} = \sum_{k=0}^{\infty} a_{nk} s_k^{(\alpha)} t_k,$$

$$\nu_n^{(\alpha)} = \sum_{k=0}^{\infty} b_{nk} s_k^{(\alpha)} t_k.$$

We will show that if $\lim_\alpha \lim_n \sup|u_n^{(\alpha)}| = 0$, then $\lim_\alpha \lim_n \sup|\nu_n^{(\alpha)}| = 0$ Let $C = (c_{nk}) = BA^{-1}$. Since C is a regular summation matrix, $\{\sum_{k=0}^{\infty} |c_{nk}|\}$ is bounded (cf. [**3**, p. 43]). We have

$$\limsup |\nu_n^{\alpha}| \leq \limsup \sum_{k=0}^{\infty} |c_{nk}| \limsup_n |u_n^{(\alpha)}|$$

for all α. Hence if $\lim_\alpha \lim_n \sup|u_n^{(\alpha)}| = 0$, then $\lim_\alpha \lim_n \sup|\nu_n^{(\alpha)}| = 0$.

Now suppose that $\lim_\alpha \lim_n \sup|\sum_{k=0}^{\infty} a_{nk} s_k^{(\alpha)} t_k| = 0$ for all sequences t in $L^{p'}(A)$, but $\lim_\alpha \text{suplim}_n \text{sup} \Sigma b_{nk} s_k^{(\alpha)} w)k > 0$ for some sequence w in $L^{p'}(A)$ not in $L^{p'}(A)$. By the uniform boundedness theorem the norms of the sequences $\{s^{(\alpha)}\}$ are bounded in $L^p(A)$. By Theorem 1.1 the norms of the sequences $\{s^{(\alpha)}\}$ are bounded in $L^p(B)$. Hence without loss of generality we may assume that $\sum_{k=0}^{\infty} b_{nk}|s_k^{(\alpha)}|^p \leq 1$ for all n and all α. Since $w \notin L^{p'}(A)$ there is a set $E \subseteq N$ such that, for each positive integer j, there is an index n_j such that

$$\sum_{k \in E} a_{n_j,k} |w_k|^{p'} > j,$$

and w_k tends to infinity as k tends to infinity through E, while $\sum_{k \in N-E} a_{nk}|w_k|^{p'}$ is bounded by some constant M. Let $w = w' + w''$ where

$$w'_k = w_k \quad \text{if } k \in E,$$
$$w'_k = 0 \text{ if } k \notin E.$$

We note that $w'' \in L^{p'}(A)$ and hence $\lim_\alpha \lim_n \sup\left|\sum_{k=0}^{\infty} a_{nk} s_k^{(\alpha)} w''_\alpha\right| = 0$. By what was shown above,

$$\lim_\alpha \lim_n \sup |\sum_{k=0}^{\infty} b_{nk} s_k^{(\alpha)} w''_\alpha| = 0.$$

By our choice of w there exists a positive constant η and arbitrarily large values of α such that

$$\limsup \left| \sum_{k=0}^{\infty} b_{nk} s_k^{(\alpha)} w_k \right| \geq \eta.$$

Hence, for arbitrarily large α, there are arbitrarily large integers n such that

$$\sum_{k=0}^{\infty} b_{nk} s_k^{(\alpha)} w_k' \geq \eta/2 \tag{5}$$

or

$$\sum_{k=0}^{\infty} b_{nk} s_k^{(\alpha)} w_k' \geq \eta/2 \tag{6}$$

and for such n, $\sum_{k=0}^{\infty} b_{nk} |w_k'|^{p'}$ exceeds a positive constant. But (5) or (6) can hold only if

$$s_k^{(\alpha)} \geq (\eta/8\tau)|w_k'|^{p'-1} \quad \text{if } k \in E_1$$

or

$$s_k^{(\alpha)} \leq (\eta/8\tau)|w_k'|^{p'-1} \quad \text{if } k \in E_1$$

for some subset E_1 of E such that

$$\sum_{k \in E_1} b_{nk} |w_k'|^{p'} \geq (\eta/8)^{p'},$$

where τ denotes the quantity $\sum_{k \in E_1} b_{nk} |w_k'|^{p'}$ which is bounded away from 0 - recall that $\sum_{k=0}^{\infty} b_{nk} |s_k^{(\alpha)}|^p \leq 1$ for all n and α. Finally, we let the sequence w''' be defined by the equations

$$w_k''' = w_k' \qquad k \in E_1,$$

$$w_k''' = 0 \quad k \notin E_1.$$

There are arbitrarily large n such that

$$\left| \sum_{k=0}^{\infty} b_{nk} s_k^{(\alpha)} w_k''' \right| \geq (\eta/8\tau)^{p'+1} \tau > 0.$$

Also, for all n,

$$\sum_{k=0}^{\infty} a_{nk}|w_k'''|^{p'} \leq (8\tau/\eta) \sum_{k=0}^{\infty} a_{nk}|s_k^{(\alpha)}|^p$$

which is bounded. Hence $w''' \in L^p(A)$. But since the net $\{\lim_n \sup|\sum_{k=0}^{\infty} b_{nk} s_k^{(\alpha)} w_k'''\}$ does not tend to 0, neither does the net $\{\lim_n \sup \sum_{k=0}^{\infty} a_{nk} s_k^{(\alpha)} w_k'''\}$, by the first part of the proof. Hence $\{s^{(\alpha)}\}$ does not tend to zero in the strong operator topology of $L^{p'}(A)$. □

THEOREM 1.3B. *Under the hypotheses of Theorem* 1.3A, *if a net* $\{s^{(\alpha)}\}$ *in* $L^p(A) \cap L^p(B), p > 1$, *tends to* 0 *in the weak operator topology of* $L^p(A)$, *then it tends to* 0 *in the weak operator topology* $L^p(B)$.

PROOF. We first note that if the norms of $s^{(\alpha)}$ are bounded in $L^p(A)$, then these norms are bounded in $L^p(B)$. We next show that if $t \in L^{p'}(A) \cap L^{p'}(B)$ and the net

$$(u^{(\alpha)})_r^\beta = \Big(\sum_{k=0}^{\infty} a_{nk} s_k^{(\alpha)} t_k\Big)_r^\beta$$

tends to 0 for each $\nu \in N^*$, then so does the net $(\nu^{(\alpha)})_r^\beta = (\sum_{k=0}^{\infty} b_{nk} s_k^{(\alpha)} t_k)_r^\beta$. To do this we note that $\nu = T(u)$ where T is the linear operator represented by the matrix $C = (c_{nk}) = BA^{-1}$. The operator T may be regarded as a linear operator on $C(N^*)$ which is continuous in the norm topology and consequently in the weak topology of $C(N^*)$ (cf. [**1**, pp. 422-423].) Hence if the net $\{u^{(\alpha)}\}^\beta$ tends to 0 pointwise on N^*, so does the net $\{(\nu^{(\alpha)})^\beta\}$. To rule out the possibility that the net $(\Sigma b_{nk} s_k^{(\alpha)} t_k)_\nu^\beta$ does not tend to 0 pointwise on N^* for some sequence w in $L^{p'}(B)$ not in $L^{p'}(A)$ we use the method of Theorem 1.3A. This completes our sketch of the proof. □

THEOREM 1.4. *If* $\{s^{(\alpha)}\}$ *is a net of sequences in* $L^p(A), p > 1$ *and there exists a subset* E *of* N *and an infinite collection of indices* $\{\alpha'\} \subseteq \{\alpha\}$ *such that*

$$A(E) = \limsup_n \sum_{k \in E} a_{nk} \geq \eta > 0$$

and, for each k in E, either $s_k^{(\alpha')} \geq z$ for all α' or $s_k^{(\alpha')} \leq -z$ for all α', where z is a positive constant, then $\{s_k^{(\alpha)}\}$ does not tend to 0 in the weak operator topology.

PROOF. Let the sequence t be defined by the equations

$$t_k = 1 \quad \text{if } k \in E \text{ and } s_k^{(\alpha')} \geq z \text{ for all } \alpha'$$

$$t_k = -1 \quad \text{if } k \in E \text{ and } s_k^{(\alpha')} \leq -z \text{ for all } \alpha'$$

$$t_k = 0 \text{ if } k \neq E.$$

Then

$$\limsup \sum a_{nk} s_k^{(\alpha')} t_k \geq \eta z$$

for infinitely many indices α', that is, the net $\{s_k^{(\alpha')}$ does not tend to 0 in the weak operator topology. □

COROLLARY. *If, for each integer k such that the k^{th} column of A contains infinitely many non-zero elements, there are arbitrarily large values n_k such that $a_{n_k,k}$ exceeds a positive constant μ, then, for each $p > 1$, the weak operator topology coincides with the norm topology.*

PROOF. Let E denote the set of integers k such that the k-th column contains infinitely many non-zero elements. By the preceding theorem, if $s^{(\alpha)}$ tends to zero in the weak operator topology, then $\lim_{\alpha} s_k^{(\alpha)} = 0$ for each k in E. But then $\lim ||s^{(\alpha)}||_p = 0$. □

The same proof actually works also for the case $p = 1$.

The preceding results indicates that for weak summation matrices, that is, matrices evaluating few divergent sequences the various topologies we are considering coincide. The next result indicates that if the summation matrix A represents a fairly strong method, there is a considerable gap between the norm topology and the weak operator topology.

THEOREM 1.5. *If, for each subset E of N such that*

$$A_0(E) = \lim \sum_{k \in E} a_{nk}$$

exists and is positive, there exist disjoint subsets $E_1 E_2$ of E such that $E_1 \cup E_2 = E, A_0(E_1)$ and $A_0(E_2)$ exist and are both equal to $A_0(E)/2$, then there is a sequence $\{s^{(m)}\}$ from $L^p(A)$ for each $p > 1$, which tends to 0 in the weak operator topology but not in the norm topology of $L^p(A)$.

PROOF. Let E be a subset of N such that $A_0(E) > 0$. There exist sets E_{11}, E_{12} such that

$$E = E_{11} \cup E_{12}$$

$$A_0(E_{11} = A_0(E_{12} = A_0(E)/2.$$

Also there exist $E_{21}, E_{22}, E_{23}, E_{24}$ such that $E_{11} = E_{21} \cup E_{22}, E_{12} = E_{23} \cup E_{24}$ and $A_0(E_{21}) = A_0(E_{22}) = A_0(E_{23}) = A_0(E_{24}) = A_0(E)/4$. In general, we have, for each i, sets ${}_{i+1.2j-1}E_{i+1.2j}$ such that

$$E_{ij} = E_{i+1.2j-1} \cup E_{i+1.2j}$$

and

$$A_0(E_{i+1.2j-1}) = A_0(E_{i+1.2j}) = A_0(E_{ij})/2 = A_0(E)/2^{i+4}$$

for $j = 1, 2, \ldots, 2^i$. For the construction of the sequence $\{s^{(m)}\}$ with the stated properties, we use a technique similar to the one used in constructing the Rademacher functions. Let $s^{(1)} = 1_E$, that is,

$$s_k^{(1)} = 1 \quad k \in E$$

$$s_k^{(1)} = 0 \quad k \notin E,$$

And for $m > 1$, let

$$s_k^{(m)} = 1, \quad k \in E_{m1} \cup E_{m3} \cdots E_{m,2^m-1},$$

$$s_k^{(m)} = -1, \quad k \in E_{m2} \cup E_{m4} \cdots E_{m,2^m},$$

$$s_k^{(m)} = 0, \quad k \notin \cup_{j=1}^{2m} E_{mj}.$$

Clearly $||s^{(m)}||$ does not tend to 0 since $||s^{(m)}||_p = (A(E))^{1/p}$ for all m and all p. Now let t be a sequence in $L^{p'}(A)$. If t is unbounded, then we may write $N = F \cup G$ where t is bounded on F and $|t_k|$ tends to infinity as k tends to infinity through values of k in G. We have, for each m,

$$\Big|\sum_{k \in G} a_{nk} s_k^{(m)} t_k\Big| \leq \sum_{k \in G} a_{nk} |t_k|.$$

But since $\sum_{k \in G} a_{nk} |t_k|^{p'}$ is bounded and $p' > 1, \sum_{k \in G} a_{nk}|t_k|$ tends to 0. Hence we need only show that

$$\lim_{m \to \infty} \Big(\sum_{k \in F} a_{nk} s_k^{(m)} t_k\Big)_\nu^\beta = 0$$

for all points $\nu \in N^*$ and for all $t \in L^{p'}(A)$. This is shown in the same way that the Rademacher functions on the unit interval are shown to converge weakly to 0 in all Lebesque spaces $L^p[0,1], p > 1$ (cf. [**1**, p. 342]). (In fact the Boolean algebra generated by the diadic subintervals of the interval [**0**,**1**]. Hence the sequence $\{s^{(m)}\}$ tends to 0 in the weak operator topology. □

THEOREM 1.6. *Suppose that in addition to satisfying* (i), (ii), (iii) *the matrix A is triangular and there exist pairwise disjoint subsets $\mathcal{S}^{(m)} (m = 1, 2, \dots)$ of N such that*

$$\limsup \sum_{k \in \mathcal{S}}^{(m)} a_{nk} |s_k^{(m)}|^p \geq \eta,$$

for some positive constant η. Then the sequence $\{s^{(m)}\}$ fails to converge to 0 in the weak operator topology of $L^p(A)$.

PROOF. We will construct a sequence t in $L^{p'}(A)$ and a double sequence of integers $\{n_{ij}\}$, $i = 1, 2, \dots$, $j = 1, 2, \dots i + 1$, tending to infinity such that

$$\sum a_{n_{ij}k} s_k^{(j)} t_k$$

exceeds a positive constant for each value of i and j. To begin with, there is a positive constant η and an integer $n_1 = n_{11}$ such that

$$\sum_{k \in \mathcal{S}^{(1)}} a_{n_{11}k} |s_k^{(1)}|^p \geq \eta.$$

We let

$$t_k = |s_k^{(1)}|^{p-2} \bar{s}_k^{(1)} \quad \text{if } k \in \mathcal{S}^{(1)}, \quad k \leq n_1,$$

$$t_k = 0 \quad \text{if } s_k^{(1)} = 0 \text{ or } k \notin \mathcal{S}^{(1)}, \quad k \leq n_1.$$

We note that t_k is now defined for $k \leq n_1$.

There exists an integer $n_{12} > n_1$ such that

$$\sum_{k \in \mathcal{S}^{(2)}} a_{n_{12},k} |s_k^{(2)}|^p \geq \eta/2;$$

the integer n_{12} may also be chosen so large that

$$\sum_{k \leq n_1} a_{nk} |s_k^{(1)}|^p \leq \eta/8, \quad n \geq n_{12}.$$

Now let the integer n_{21} be chosen so large that $n_{21} > n_{12}$,

$$\sum_{k > n_{12}} a_{n_{21}k} |s_k^{(1)}|^p \geq \eta/2, \; k \in \mathcal{S}^{(1)}$$

and

$$\sum_{k \leq n_{12}} a_{nk} |s_k^{(2)}|^p \leq \eta/8$$

for $n \geq n_{21}$. We define, for $n_{12} < k \leq n_{21}$:

$$t_k = |s_k^{(1)}|^{p-2} \; \bar{s}_k^{(1)} \text{ if } k \in \mathcal{S}^{(1)} \text{ and } s_k^{(1)} \neq 0,$$
$$t_k = 0 \quad \text{if } k \notin \mathcal{S}^{(1)} \text{ or } \mathcal{S}_k^{(1)} = 0.$$

The sequence $\{t_k\}$ has now been defined for $k \leq n_{21}$.

We continue in this manner. Suppose that a sequence of integers $n_{11}, n_{12}, n_{21}, n_{22}, n_{23}, \ldots, n_{ij}$ has been chosen and that the sequence t_k

has been defined for $k \leq n_{ij}$. If $j \leq i$ we choose $n_{i.j+1}$ so that, if $j \leq i$, we choose $n_{i.j+1} > n_{i.j}$ and so that

$$\sum_{k > n_{ij}} a_{nk_{i.j+1}} |s_k^{(j+1)}|^p \geq \eta/2, \; k \in \mathcal{S}^{(j+1)}$$

and

$$\sum_{k \leq n_{ij}} a_{n.k} \max_{0 \leq m} \leq j |s_k^{(m)}|^p \leq \eta/8$$

for $n \geq n_{i.j+1}$. We define t_k for $n_{ij} < k < n_{i.j+1}$ by the equation

$$\begin{aligned} t_k &= |s_k^{(j+1)}|^{p-2} \; \bar{s}_k^{(j+1)} \\ &\text{if } k \in \mathcal{S}^{(j+1)} \text{ and } s_k^{(j+1)} \neq 0, \\ t_k &= 0 \text{ if } k \in \mathcal{S}^{(j+1)} \text{ or } s_k^{(j+1)} = 0. \end{aligned}$$

Likewise, for $j = i + 1$, we first choose the integer $n_{i+1.j}$ so that

$$\sum_{k \in \mathcal{S}^{(1)}} a_{nk_{i+1.j}} |s_k^{(1)}|^p \geq \eta/2, \; k > n_{ij}$$

and

$$\sum_{k \leq n_{ij}} a_{nk} \max_{m \leq j} |s_k^{(m)}|^p \leq \eta/8$$

for $n \geq n_{i.j+1}$. In this case we define t_k for $n_{i.i+1} < k < n_{i+1.1}$ by the equations

$$t_k = |s_k^{(1)}|^{p-2} \bar{s}_k^{(1)} \qquad \text{if } k \in \mathcal{S}^{(1)} \text{ and } s_k^{(1)} \neq 0,$$

$$t_k = 0 \qquad \text{if } k \notin \mathcal{S}^{(1)} \text{ or } s_k^{(1)} = 0.$$

This completes the inductive definition of the sequence t. We note that, for each i and j, $j \leq i$,

$$\begin{aligned} \sum a_{nk_{ij}} s_k^{(j)} t_k \geq & \sum_{\substack{k \in \mathcal{S}^{(j)} \\ k \geq n_{i,j-1}}} k \in S^{(j)} a_{nk} |s_k^{(j)}|^p \\ & - \sum_{k \leq n_{i,j-1}} a_{nk} \max_{m \leq j-1} |s_k^{(m)}|^p > \eta/4. \end{aligned}$$

Hence, if ν is a point in N^* which is in the closure of $\{n_{ij}\}$, then

$$(7) \qquad \Big(\sum_{k=0}^{\infty} a_{nk} s_k^{(j)} t_k\Big)_{\nu}^{\beta} \geq \eta/4.$$

To show that $s(m)$ does not tend to 0 in the weak operator topology in $L^p(A)$ we need only show that $t \in L^{p'}(A)$. We may assume that the norms $||s^{(m)}||$ are uniformly bounded, for otherwise, by the uniform limitedness theorem, the sequence $\{s^{(j)}\}$ cannot converge in the weak operator topology. Hence there exists a universal constant M such that

$$\sum_{n=0}^{\infty} a_{nk} |s_k^{(j)}|^p \leq M.$$

For $n_{ij} \leq n < n_{i,j+1}$,

$$\begin{aligned}\sum_{k=0}^{n} a_{nk} |t_k|^{p'} &\leq \sum_{k<n_{i,j-1}} a_{nk} \max |s_k^{(m)}|^{p'} + \sum_{k=n_{i,j-1}}^{\infty} ank |t_k|^{p'} \\ &\leq \eta/8 + \sum_{k\in S}^{(j)} a_{nk} |s_k^{(j)}|^p + \sum_{k\in S^{(j+1)}} a_{nk} |s_k^{(j+1)}|^p \\ &\leq \eta/8 + 2M + 1,\end{aligned}$$

provided n is sufficiently large. Similar estimates can be given if $n_{i,i+1} \leq n < n_{i+1,1}$. Thus $t \in L^{p'}$, and (7) shows that the sequence $\{s^{(j)}\}$ does not tend to 0 in the weak operator topology. □

2. Extreme points of the unit ball of $L^p(A)$.

THEOREM 2.1. *Suppose that the matrix A is triangular. Suppose also that s is a bounded sequence in $L^p(A)$ of norm one such that the set of integers n for which*

$$\sum_{k=0}^{n} a_{nk} |s_k|^p \leq 1 - \delta$$

for a fixed positive constant δ, can be written as a disjoint union of intervals I_m arranged in increasing order, and that the intervals I'_m complementary to I_m are also arranged in increasing order in such a *way that I'_m lies between I_m and I_{m+1}. Moreover, if I_m is the interval $[n_m, m'_m]$ (so that I'_m is the interval (n'_m, n_{m+1}), then suppose that there exists a positive constant $\delta' < \delta$ and an integer n_δ, such that $n_m > n_\delta$, and*

$$n_{nk}/a_{n_{m+1},k} < (1-\delta')^{-1} \tag{8}$$

if $0 \leq k \leq n, n'_m \leq n < n_{m+1}$. Finally, suppose that

$$A(\cup_m I_m) > 0. \tag{9}$$

Then s is not an extreme point of the unit ball Ω of $L^p(A)$.

PROOF. We will show that the sequences $s \pm \alpha t$, where

$$t_n = 1 \quad \text{if } n \in \cup_m I_m$$

$$t_n = 0 \quad \text{if } n \notin \cup_m I_m$$

and α is a suitable constant, are in Ω. (We note by (9) that $||t||_p > 0$). If $n \in I_m$ for some n, then, since

$$\sum_{k=0}^{n} a_{nk}|s_k|^p \leq 1-\delta$$

we have

$$\sum_{k=0}^{n} a_{nk}|s_k \pm \alpha t_k|^p \leq 1-\delta' \tag{10}$$

if α is sufficiently small. If $n \notin \cup I_m$, then n must lie in some interval $(n'_m, n_{m+1}$. In this case

$$\sum_{k=0}^{n} a_{nk}|s_k \pm \alpha t_k|^p = \sum_{k=0}^{n} (a_{n,k}/a_{n_{m+1},k}) a_{n_{m+1},k}|s_k \pm \alpha t_k|^p.$$

By (8) and (10) the limit superior of this quantity is at most 1. It follows that s is not an extreme point of Ω. □

THEOREM 2.2. *Suppose that A is as in Theorem 2.1 and that $s \in L^p(A), ||s||_p = 1$ and moreover that the quantity*

$$\sum_{k=0}^{\infty} a_{nk}|s_k|^p$$

tends to 1 as n tends to infinity along a set $E = n_j$ such that, for some positive constant M,

$$a_{n,k} \leq M_{a_{n_m},k}, \quad 0 \leq k \leq n, n_{m-1} < n \leq n_m.$$

Then s is an extreme point of the unit sphere of $L^p(A)$.

PROOF. Suppose that there exists an element $t \neq 0$ in $L^{p'}(A)$ such that $||s+t||_p = ||s-t||_p = 1$. Let F be a subset of E; we first show that, for all such $F, \sum_{k=0}^{\infty} a_{nk}|t_k|^p$ tends to 0 as n tends to infinity through values in F. We use a method of Lau [**4**, p. 160]. Let $L_p(a_n)$ denote the Lebesque space with the measure a_n of a set $Q \subseteq N$ given by

$$a_n(Q) = \sum_{k \in Q} a_{nk}$$

so that the norm in $L_p(a_n)$ of a sequence s is

$$||s||_n = \Big(\sum_{k \in Q} a_{nk}|s_k|^p\Big)^{1/p}.$$

The space $L_p(a_n)$ is uniformly convex. If the quantity $(\sum a_{nk}|t_k|^p)^{1/p}$ does not tend to 0 as n tends to infinity through F, then there exists a positive constant η and arbitrarily large n such that $||t||_n \geq \eta$. Given a positive number ε, the values $||s+t||_n||s-t||_n$ are greater than $1-\varepsilon$ if n is a sufficiently large integer in F. Hence there exists a positive constant δ such that $||s||_n = ||((s+t)+(s-t))/2||_n \leq 1-\delta$ for arbitrarily large n in F. We have a contradiction. It follows that $\sum_{k=0}^{\infty} a_{nk}|t_k|^p$ tends to 0 as n tends to infinity through values of n in F; this holds for each subset F of E. In other words $\sum_{k=0}^{\infty} a_{nk}|t_k|^p$ tends to 0 as n tends to infinity through values in E. It remains to show that $\sum_{k=0}^{\infty} a_{nk}|t_k|^p$

tends to infinity through values not in E. If n is such an integer, say $n_{m-1} < n \leq n_m$, where n_{m-1} and n_m are consecutive integers in E,

$$\sum_{k=0}^{n} a_{nk}|t_k|^p \leq M \sum_{k=0}^{n_m} a_{n_m,k}|t_k|^p,$$

and this quantity tends to 0 as was shown above. Hence $||t||_p = 0$ and s is an extreme point of the sphere.

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