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REGULARITY AND UNIQUENESS OF CERTAIN SYSTEMS OF FUNCTIONS ANNIHILATED BY A FORMALLY INTEGRABLE SYSTEM OF VECTOR FIELDS

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1. Introduction and the statement of the main theorems. It is well known that a local CR diffeomorphism of a smooth CR manifold of CR codimension 1 with a nondegenerate Levi form is determined by a finite number of constants. Moreover, if M and M' are real analytic (C^∞, respectively) CR manifolds as above and $F : M \to M'$ is a CR diffeomorphism of class C^7 then F is real analytic (C^∞, respectively). These are consequences of the existence of the invariant Cartan connection on the bundle of pseudo conformal frames over M ([**3**], cf. also [**9**]). If M' is a real hypersurface in $\mathbf{C}^{n+1}$ the above two facts are easier to see: Let r be a local defining function of M' and let $\{L_1, \dots, L_n\}$ be an independent set of C^ω tangential Cauchy-Riemann vector fields on M. Then the components of $F = (f_1, \dots, f_{n+1})$ satisfy an equation

$$(1) \qquad r \cdot F = 0$$

and a system of partial differential equations

$$(2) \qquad L_i f_j = 0, \quad i = 1, \dots, n, \quad j = 1, \dots, n+1.$$

Through a process of repeated differentiation of (1), reduction of order of derivatives using (2) and introducing new variables, we can construct a C^ω pfaffian system whose integral manifolds correspond to CR diffeomorphisms of M onto M'. The regularity and the uniqueness of F follow from the Frobenius theorem (cf. [**6**]). This method is a variant of the so-called 'prolangation' originated by E. Cartan, which he used as a basic tool for the equivalence problem (cf. [**2**]).

The purpose of this paper is to generalize the above properties of CR diffeomorphisms to certain systems of functions annihilated by a

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formally integrable system of complex vector fields. We restrict our interest to the C^ω category. Our viewpoint is purely local so, for instance, a "function" must be understood as a germ of a function at the origin. The following definitions are adopted from Treves [**8**]:

Let Ω be an open subset of $\mathbf{R}^N$ containing the origin and $\mathcal{V}$ be a C^ω subbundle of the complexified tangent bundle $\mathbf{C}T\Omega$ which satisfies the formal integrability:

If L_1 and L_2 are sections of $\mathcal{V}$, then their commutator $[L_1, L_2]$ is again a section of $\mathcal{V}$.

$\mathcal{V}$ is called a complex structure if $\mathbf{C}T\Omega = \mathcal{V} \oplus \overline{\mathcal{V}}$ and a CR structure if $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$. A function (or distribution) f is said to be annihilated by $\mathcal{V}$ if

$$Lf = 0 \text{ for any section } L \text{ of } \mathcal{V}.$$

To state our assumption on $\mathcal{V}$ we define a module B_k, $k = 1, 2, \ldots,$ over $C^\omega(\Omega)$ of linear partial differential operators as follows:

Let $n =$ complex dimension of $\mathcal{V}$ and let $L_1, \ldots, L_n$ be independent C^ω sections of $\mathcal{V}$. Let $\beta = (b_1, \ldots, b_n)$ be a sequence of nonnegative integers. We denote, by $\mathbf{L}^\beta$, a linear partial differential operator

$$L_n^{b_n} \cdots L_2^{b_2} L_1^{b_1}.$$

A block of length j is a differential operator of the form

$$\overline{\mathbf{L}}^{\beta_j} \cdots \overline{\mathbf{L}}^{\beta_3} \mathbf{L}^{\beta_2} \overline{\mathbf{L}}^{\beta_1} \quad \text{if } j \text{ is odd},$$

$$\mathbf{L}^{\beta_j} \cdots \overline{\mathbf{L}}^{\beta_3} \mathbf{L}^{\beta_2} \overline{\mathbf{L}}^{\beta_1} \quad \text{if } j \text{ is even},$$

where β's are multiindices as above. Then define B_k as a module generated by all the blocks of length $\leq k$ and $B_0 = C^\omega(\Omega)$. B_k is well defined due to the formal integrability of $\mathcal{V}$. Let B be the algebra of all linear partial differential operators on Ω with C^ω coefficients. Then

$$B_k \subseteq B_{k+1}, \quad k = 0, 1, 2, \ldots,$$

and

$$\cup_{k=1}^\infty B_k \subseteq B.$$

First, we consider

CONDITION 1. There exists a positive integer ν such that $B_\nu = B$ when restricted to the distributions annihilated by $\mathcal{V}$.

REMARKS. If $\mathcal{V}$ is a complex structure, Condition 1 holds with $\nu = 1$. If $\mathcal{V}$ is a CR structure of CR codimension 1 with a nondegenerate Levi form, Condition 1 holds with $\nu = 2$. If the Levi form has a nonzero eigenvalue, Condition 1 holds with $\nu = 3$ (Prop. 1 of §3). Some CR manifolds with degenerate Levi forms satisfy Condition 1. See 3.1 of §3 for an example. The author does not know yet how Condition 1 is related to the notions of 'finite type' as in [**5**].

THEOREM 1. *Let $\mathcal{V}$ be a C^ω formally integrable subbundle of the complexified tangent bundle of an open set Ω of $\mathbf{R}^N$. Let $F = (f_1, \ldots, f_l)$ be a system of complex valued functions on Ω annihilated by $\mathcal{V}$. Suppose that $\mathcal{V}$ satisfies Condition 1 and that each f_j can be expressed as*

$$f_j = F_j(x, D^\alpha \overline{f}_i : |\alpha| \leq m, i = 1, \ldots, l),$$

where F_j is an analytic function, namely a convergent power series of the variables in the parenthesis. Let

$$\lambda = \begin{cases} (\nu+1)m, & \text{if } \nu \text{ is odd} \\ \nu m, & \text{if } \nu \text{ is even, where } \nu \text{ is as in Condition 1.} \end{cases}$$

Then $F = (f_1, \ldots, f_l)$ is determined by their partial derivatives at the origin of order $\leq \lambda$. Furthermore, if $F \in C^{\lambda+1}$ then $F \in C^\omega$.

If $\mathcal{V}$ is a CR structure, Condition 1 may be replaced by a weaker condition to get the analyticity of F: Let $\mathcal{V}$ be a C^ω CR structure of the complex dimension n and CR codimension d on an open set $\Omega \in \mathbf{R}^{2n+d}$. Choose C^ω real vector fields $T_i, i = 1, \ldots, d$, so that $\{L_1, \ldots, L_n, \overline{L}_1, \ldots, \overline{L}_n, T_1, \ldots, T_d\}$ generates $\mathbf{C}T\Omega$. Let $\alpha = (a_1, \ldots, a_d)$ and $\beta = (b_1, \ldots, b_n)$ be multiindices. We denote by $\langle \mathbf{T}^\alpha \overline{\mathbf{L}}^\beta, \ldots \rangle$ the set of all linear combinations with C^ω coefficients of $\mathbf{T}^\alpha \overline{\mathbf{L}}^\beta, \ldots$.

CONDITION 2. For some positive integer ν, B_ν has the following property: For each $i = 1, \ldots, d$ and each positive integer p there exists a linear operator

$$A_{i,p} \in \langle \mathbf{T}^\alpha \overline{\mathbf{L}}^\beta : |\alpha| \leq p-1, |\alpha| + |\beta| \leq p \rangle \text{ such that}$$
$$(T_i)^p f + A_{i,p} f \in B_\nu f \text{ for any CR distribution } f,$$

where $B_\nu f$ is the set $\{Lf : L \in B_\nu\}$.

THEOREM 2. *Let $\mathcal{V}$ be a C^ω CR structure of the complex dimension n and CR codimension d on an open set $\Omega \subseteq \mathbf{R}^{2n+d}$. Let $F = (f_1, \cdots, f_l)$ be a system of CR functions. Suppose that $\mathcal{V}$ satisfies Condition 2 and that each f_j can be expressed as*

$$f_j = F_j(x, D^\alpha \overline{f}_i : |\alpha| \leq m, i = 1, \cdots, l), \text{ where } F_j \text{ is analytic.}$$

$$\text{Let } \lambda = \begin{cases} (\nu+1)m, & \text{if } \nu \text{ is odd,} \\ \nu m, & \text{if } \nu \text{ is even, where } \nu \text{ is as in Condition 2,} \end{cases}$$

and let $\lambda' \geq \lambda + 1$ be an even number. Then $F \in C^{\lambda'}$ implies that $F \in C^\omega$.

In §3, we give a list of examples of Theorem 1 and 2 in CR structures. We find that much of §3 is covered by recent results [**1**] by Baouendi-Jacobowitz-Treves. However, we give proofs from our viewpoint. The author thanks S. Webster for answering many questions. The author also thanks the referee for pointing out several mistakes and for helpful suggestions.

2. Proof of Theorems 1 and 2. Let n be the complex dimension of $\mathcal{V}$. Let $L_1, \ldots, L_n$ be C^ω linearly independent sections of $\mathcal{V}$. Choose C^ω independent real vector fields $T_i, i = 1, \ldots d$, so that $\{L_1, \ldots, L_n, \overline{L}_1, \ldots, \overline{L}_n, T_1, \ldots, T_d\}$ spans the complexified tangent space of Ω at each point of Ω. Note that these vectors may not be independent. Let $\alpha = (a_1, \ldots, a_d)$, $\beta = (b_1, \ldots, b_n)$ and $\gamma = (c_1, \ldots, c_n)$ be multiindices. Observe that any linear partial differential operator $p(x, D)$ of order λ with C^ω coefficients on Ω can be expressed as a linear combination with C^ω coefficients of

$$\{\mathbf{T}^\alpha \overline{\mathbf{L}}^\beta \mathbf{L}^\gamma : |\alpha| + |\beta| + |\gamma| \leq \lambda\},$$

and if f is a distribution annihilated by $\mathcal{V}$, then $p(x,D)f$ can be expressed as a linear combination with C^ω coefficients of $\{\mathbf{T}^\alpha\overline{\mathbf{L}}^\beta f : |\alpha|+|\beta| \leq \lambda\}$. Let $F = (f_1,\ldots,f_l)$ be a system of functions annihilated by $\mathcal{V}$. For each pair of integers k and k' $(k \geq k')$ let C_k be the set of all analytic functions of the local coordinates x and $\{\mathbf{T}^\alpha\overline{\mathbf{L}}^\beta f_j : |\alpha|+|\beta| \leq k, j=1,\ldots,l\}$ and $C_{k,k'}$ be the subset of C_k which consists of all the analytic functions of x, let C_k be the set of all analytic functions of the local coordinates x and $\{\mathbf{T}^\alpha\overline{\mathbf{L}}^\beta f_j : |\alpha|+|\beta| \leq k, |\alpha| \leq k', j=1,\ldots,l\}$ and let $\overline{C}_k$ and $\overline{C}_{k,k'}$ be their complex conjugates, respectively. Namely, $\overline{C}_k$ is the set of all the analytic functions of x and $\{\mathbf{T}^\alpha\mathbf{L}^\beta\overline{f}_j : |\alpha|+|\beta| \leq k, j=1,\ldots,l\}$, etc. We will denote, by $\overline{\mathbf{L}}^\beta C_k$, the set $\{\overline{\mathbf{L}}^\beta u : u \in C_k\}$, etc.

LEMMA 1. *If there exists an integer $m > 0$ such that each $f_j \in \overline{C}_{m,j} = 1,\ldots,l$, then we have:*

1) *for any multiindex β, $\overline{\mathbf{L}}^\beta f_j \in \overline{C}_m, j=1,\ldots,l$, and*

2) *for any integers k and k' with $k \geq k' \geq 0$,*

$$C_{k,k'} \subseteq \overline{C}_{m+k'}, \text{ or equivalently}, \overline{C}_{k,k'} \subseteq C_{m+k'}.$$

PROOF. First, we show that if L is a C^ω section of $\mathcal{V}$ and $\mu > 0$ is any integer, then

$$LC_\mu \subset C_\mu.$$

Let $u \in C_\mu$. Since u is an analytic function of x and $\{\mathbf{T}^\alpha\overline{\mathbf{L}}^\beta f_j : |\alpha|+|\beta| \leq \mu, j=1,\ldots,l\}$, by the chain rule Lu is an analytic function of x, $\{\mathbf{T}^\alpha\overline{\mathbf{L}}^\beta f_j\}$ and $\{L\mathbf{T}^\alpha\overline{\mathbf{L}}^\beta f_j\}$. But $L(\mathbf{T}^\alpha\overline{\mathbf{L}}^\beta)f_j = (\mathbf{T}^\alpha\overline{\mathbf{L}}^\beta)Lf_j$ + terms arising from commutating L with $\mathbf{T}^\alpha\overline{\mathbf{L}}^\beta$. In the right side $Lf_j = 0$, and the sum of the terms arising from commutating is a linear combination with C^ω coefficients of $\{\mathbf{T}^\alpha\overline{\mathbf{L}}^\beta f_j : |\alpha|+|\beta| \leq \mu\}$. Therefore, $Lu \in C_\mu$. A repeated application of the above argument shows that for any multiindex β

$$\mathbf{L}^\beta C_m \subseteq C_m, \text{ or equivalently, } \overline{\mathbf{L}}^\beta\overline{\mathbf{C}}_m \subseteq \overline{C}_m.$$

Now, since $f_j \in \overline{C}_m$, we have

$$\overline{\mathbf{L}}^\beta f_j \in \overline{\mathbf{L}}^\beta\overline{C}_m \subset \overline{C}_m, \text{which proves 1).}$$

Let α be any multiindex with $|\alpha| \leq k'$. Apply $\mathbf{T}^\alpha$ to the conclusion part of 1) to get $\mathbf{T}^\alpha \overline{\mathbf{L}}^\beta f_j \in \overline{C}_{m+|\alpha|}$.

Since $C_{k,k'}$ is the set of analytic functions of $\{x, \mathbf{T}^\alpha \overline{\mathbf{L}}^\beta f_j : |\alpha| \leq k'\}$ and each $\mathbf{T}^\alpha \overline{\mathbf{L}}^\beta f_j \in \overline{C}_{m+|\alpha|} \subseteq \overline{C}_{m+k'}$, we have $C_{k,k'} \subseteq \overline{C}_{m+k'}$. This proves 2). □

LEMMA 2. *Let B_k be as defined in* §1. *If there exists an integer $m > 0$ such that each $f_j \in \overline{C}_m, j = 1, \dots, l$, then we have*

$$B_k f_j \subset C_{(k+1)m} \text{ if } k \text{ is odd}$$

and

$$B_k f_j \subset C_{km} \text{ if } k \text{ is even.}$$

PROOF. For any multiindices $\beta_i, i = 1, 2, \dots,$

$$\overline{\mathbf{L}}^{\beta_1} f_j \in \overline{C}_m \text{ by 1) of Lemma 1.}$$

Apply $\mathbf{L}^{\beta_2}$ to the above to get

$$\mathbf{L}^{\beta_2} \mathbf{L}^{\beta_1} f_j \in \mathbf{L}^{\beta_2} \overline{C}_m \subseteq \overline{C}_{|\beta_2|+m,m}.$$

But $\overline{C}_{|\beta_2|+m,m} \subset C_{m+m}$ by 2) of Lemma 1. Therefore, $\mathbf{L}^{\beta_2} \overline{\mathbf{L}}^{\beta_1} f_j \in C_{m+m}$. Thus the lemma is proved for $k = 1$ and $k = 2$. Then use induction on k. □

PROOF OF THEOREM 1. Let ν be as in Condition 1 and λ be as in the statement of Theorem 1. Then, by Lemma 2, $B_\nu f_j \subseteq C_\lambda, j = 1, \dots, l$. But Condition 1 implies that $B_\nu f_j$ contains all the partial derivatives of f_j of order $\lambda + 1$. Thus we have

$$D^\alpha f_j = F_j^\alpha(x, D^\beta f_i : |\beta| \leq \lambda, i = 1, \dots, l)$$

for each α with $|\alpha| = \lambda + 1$ and each $j = 1, \dots, l$, where F_j^α is an analytic function.

Therefore, F is determined by its partial derivatives at the origin up to order λ and $F \in C^\omega$ if $F \in C^{\lambda+1}$ (see the following Remark). □

REMARK. Let $\mathbf{u} = (u_1, \dots, u_m)$ be a system of real valued functions on an open set of $\mathbf{R}^n$. Suppose that all the partial derivatives of $u_j, j = 1, \dots, m$, of order $\lambda + 1$ can be expressed as an analytic function of the local coordinates $\mathbf{x}$ and the derivatives of $\mathbf{u}$ of order $\leq \lambda$. For example, let $m = 1, n = 2$ and $\lambda = 1$. Then we have

$$\begin{aligned} u_{xx} &= a(x, y, u, u_x, u_y) \\ u_{xy} &= b(x, y, u, u_x, u_y) \\ u_{yy} &= c(x, y, u, u_x, u_y), \end{aligned}$$

where a, b and c are analytic functions. Then we have

$$\begin{aligned} du &= u_x dx + u_y dy \\ du_x &= a\, dx + b\, dy \\ du_y &= b\, dx + c\, dy. \end{aligned}$$

Introduce the new variables $p = u_x, q = u_y$ and let

$$\begin{aligned} \omega_1 &= du - p\, dx - q\, dy \\ \omega_2 &= dp - a\, dx - b\, dy \\ \omega_3 &= dq - b\, dx - c\, dy. \end{aligned}$$

Then the mapping $\mathbf{x} \longmapsto (\mathbf{x}, u(\mathbf{x}), u_x(\mathbf{x}), u_y(\mathbf{x}))$ is an integral manifold of the Pfaffian system $\omega_j = 0, \quad j = 1, 2, 3$ in $\mathbf{R}^5$. Thus u is determined on a neighborhood of the origin by a finite number of constants $u(0), u_x(0)$ and $u_y(0)$ and $u \in C^\omega$ if $u \in C^2$. □

PROOF OF THEOREM 2. Let λ and λ' be as in the statement of Theorem 2. Condition 2 implies that, for each $i = 1, \dots, d$, there exists a differential operator $A_i \in < \mathbf{T}^\alpha \overline{\mathbf{L}}^\beta : |\alpha| \langle \lambda', |\alpha| + |\beta| \leq \lambda' \rangle$ so that

$$(T_i)^{\lambda'} f_j + A_i f_j \in B_\nu f_j.$$

But, by Lemma 2, $B_\nu f_j \subset C_\lambda$. Therefore

$$(T_i)^{\lambda'} f_j + A_i f_j \in C_\lambda, \; j = 1, \dots, l, i = 1, \dots, d. \tag{2.1}$$

On the other hand, for each $j = 1, \ldots, l$,

$$(\overline{L}_k L_k)^{\frac{\lambda'}{2}} f_j = 0, \quad k = 1, \ldots, n. \tag{2.2}$$

We choose a coordinate system $(x_1, y_1, \ldots, x_n, y_n, t_1, \ldots, t_d)$ of Ω such that

$$L_k(0) = \frac{1}{2}\Big(\frac{\partial}{\partial x_k} + \sqrt{-1}\frac{\partial}{\partial y_k}\Big) \text{ and } T_j(0) = \frac{\partial}{\partial t_j},$$

and express the associated convectors in the variable $(\xi, \tau) = (\xi_1, \ldots, \xi_{2n}, \tau_1, \ldots, \tau_d)$.

Consider the equation

$$G\sum_{k=1}^{n}(\overline{L}_k L_k)^{\frac{\lambda'}{2}} f_j = 0, G >> 0. \tag{2.2$'$}$$

The principal symbol at the origin of the system (2.1) with $i = 1, \ldots, d$ and (2.2)$'$ is

$$G[(\xi_1^2+\xi_2^2)^{\frac{\lambda'}{2}}+\cdots+(\xi_{2n-1}^2+\xi_{2n}^2)^{\frac{\lambda'}{2}}]+\tau_1^{\lambda'}+\cdots+\tau_d^{\lambda'}+\sum a_{\alpha\beta}\tau^\alpha\xi^\beta, \tag{2.3}$$

where $|\alpha| + |\beta| = \lambda'$ and $|\alpha| < \lambda'$.

If we take G sufficiently large, (2.3) ≥ 0, with equality only when $(\xi, \tau) = 0$. By the theory of elliptic partial differential equations (cf. [**7**]) $F \in C^\omega$ if $F \in C^{\lambda'}$. □

3. Applications. This section deals with applications of Theorems 1 and 2 to the cases of embedding of abstract CR manifolds into $\mathbf{C}^N$. Let $\mathcal{V}$ be a C^ω CR structure on an open set $\Omega \subseteq \mathbf{R}^{2n+d}$ of the complex dimension n and CR codimension d. It is well known that there exists a C^ω CR embedding of $(\Omega, \mathcal{V})$ into $\mathbf{C}^{n+d}$ as a generic submanifold. This is a consequence of the analytic version of the Frobenius theorem (cf. [**9**]). However, not every CR embedding is C^ω even if its image is a C^ω submanifold of $\mathbf{C}^{n+d}$. The following example shows that if the Levi form is identically equal to zero there is much freedom in the choice of CR embeddings.

EXAMPLE. Let $\Omega = \mathbf{R}^3 = \{(x, y, t)\}$ and be the CR structure generated by the vector field $\frac{1}{2}(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y})$. Let $\phi : \mathbf{R}^1 \to \mathbf{R}^1$ be any C^1 diffeomorphism. Then $F : \Omega \to \mathbf{C}^2$, defined by

$$F(x, y, t) = (x + \sqrt{-1}y, \phi(t))$$

is a *CR embedding.* However, under certain nondegeneracy assumptions of the Levi form, Theorem 1 applies to the component functions of a CR embedding $F = (f_1, \ldots, f_{n+d})$ to conclude that F is determined by a finite number of constants and $F \in C^\omega$ if $F \in C^k$ for a sufficiently large k. In this section a tangential Cauchy-Riemann vector field will be denoted by either $\overline{Z}$ or $\overline{V}$ (instead of L).

PROPOSITION 1. *Let* $(\Omega, \mathcal{V})$ *be a* C^ω *CR manifold of* CR *codimension* 1. *If the Levi form has a nonzero eigenvalue, then Condition* 1 *holds with* $\nu = 3$.

PROOF. Since the Levi form has a nonzero eigenvalue, there exists a C^ω section $\overline{Z}$ of $\mathcal{V}$ such that

$$[Z, \overline{Z}] \neq 0, \operatorname{mod} \mathcal{V} \oplus \overline{\mathcal{V}}.$$

Let $T = \sqrt{-1}[Z, \overline{Z}]$, then T is a C^ω real vector field on Ω. Choose a set of generators $\{Z_1 = Z, Z_2, \ldots, Z_n\}$ of $\overline{\mathcal{V}}$. Let f be any CR distribution. Any partial derivative $D^\alpha f(|\alpha| = q)$ is a linear combination with C^ω coefficients of $\{\mathbf{Z}^\beta T^t f : |\beta| + t \le q\}$. We shall show that, for any multiindex $\beta = (b_1, \ldots, b_n)$ and any nonnegative integer t,

$$\mathbf{Z}^\beta T^t f \in B_3 f. \tag{3.1}$$

Since

$$\overline{Z}Zf = (Z\overline{Z} - [Z, \overline{Z}])f \tag{3.2}$$

$$Z\overline{Z}f + \sqrt{-1}Tf = \sqrt{-1}Tf,$$

we have

$$Tf \in B_2 f.$$

By induction on t, it is easy to see that

$$\overline{Z}^t Z^t f = t!(\sqrt{-1}T)^t f + \sum A_{i.\gamma}\mathbf{Z}^\gamma T^i f, \tag{3.3}$$

where $i + |\gamma| \le t$ and $i < t$. We show (3.1) by induction on t. If $t = 1$, apply $\mathbf{Z}^\beta$ to (3.2) to get $\mathbf{Z}^\beta T f \in \mathbf{Z}^\beta B_2 f \subset B_3 f$. Now apply Z^β to (3.3), to get

$$\mathbf{Z}^\beta \overline{Z}^t Z^t f = t! \quad \mathbf{Z}^\beta(\sqrt{-1}T)^t f + \sum b_{i.\alpha}\mathbf{Z}^\alpha T^i f$$

where

$$|\alpha| + i \le |\beta| + t \text{ and } i < t.$$

By induction hypothesis $\mathbf{Z}^\alpha T^i f \in B_3 f$. But $\mathbf{Z}^\beta \overline{Z}^t Z^t f \in B_3 f$, thereafter (3.1) follows. □

Now we present several examples of applications of the Theorems 1 and 2.

3.1. Hypersurfaces of $\mathbf{C}^{n+1}$ with degenerate Levi forms Let M be a C^ω real hypersurface in $\mathbf{C}^{n+1}$. Let F be a local defining function of M. By a holomorphic change of coordinates we can get local coordinates $(z_1, \ldots, z_{n+1})$ so that

$$r(z_1, \overline{z}_1, \ldots, z_{n+1}, \overline{z}_{n+1}) = z_{n+1} + \overline{z}_{n+1} - \phi(z_1, \overline{z}_1, \ldots, z_{n+1}, \overline{z}_{n+1}),$$

where $\phi(0) = 0, d\phi(0) = 0$ and the Taylor series expansion of ϕ has no pluriharmonic part. Suppose that for each $j = 1, \ldots, n$ there is an n-tuple of nonnegative integers $\beta_j = (b_j^i, \ldots, b_j^n)$ such that

$$\det \left[\frac{\partial^{|\beta_j|+1} r}{\partial \overline{z}^{\beta_j}\, \partial z_i}\right]_{i,j=1,\ldots,n} \ne 0,$$

where $(\frac{\partial}{\partial \overline{z}})^{\beta_j} = (\frac{\partial}{\partial \overline{z}_n})^{b_j^n} \cdots \left(\frac{\partial}{\partial \overline{z}_1}\right)^{b_j^1}$.

If $(\Omega, \mathcal{V})$ is an abstract C^ω CR manifold and $F = (f_1, \ldots, f_{n+1}) : \Omega \to M$ is a CR diffeomorphism, it is proved in **[6]** that $(\Omega, \mathcal{V})$ satisfies Condition 1 with $\nu = 2$ and we can express each f_j as

$$f_j = F_j(x, D^\alpha \overline{f}_i : |\alpha| \le m, i = 1, \ldots, n+1),$$

where $m = \max|\beta_j|, j = 1, \ldots, n$, and F_j is an analytic function. Thus by Theorem 1, F is determined by its partial derivatives at 0 up to order $\lambda = 2m$ and $F \in C^{2m+1}$ implies that $F \in C^\omega$.

3.2. Tube hypersurfaces A tube hypersurface in $\mathbf{C^n} = \mathbf{R^n} + \sqrt{-1}\mathbf{R^n}$ is a hypersurface M of the form $M = S + \sqrt{-1}\mathbf{R^n}$, where S is a hypersurface of codimension 1 in $\mathbf{R^n}$.

We can prove

PROPOSITION 2. *Let $M = S + \sqrt{-1}\mathbf{R^n}$ be a tube hypersurface in $\mathbf{C^n}$ where S does not contain a real line. Let $(\Omega, \mathcal{V})$ be a C^ω CR manifold and $F = (f_1, \ldots, f_n) : \Omega \to M$ be a CR diffeomorphism. Then, for each point $P \in \Omega$, there exists an integer k_p so that, on a neighborhood of P, F is determined by its partial derivaties of order $< k_p$ and $F \in C^\omega$ whenever $F \in C^{k_p}$.*

SKETCH OF THE PROOF. Assume that P and $F(P)$ are the origins of $\mathbf{R}^{2n-1}$ and $\mathbf{C^n}$, respectively. Let $x + \sqrt{-1}y$ be the standard coordinates of $\mathbf{C}^n$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. By a repeated linear change of the coordinates x and using the condition that S contains no real line, we get coordinates $(t_1, \ldots, t_n)$ of $\mathbf{R^n}$ so that S is locally the graph $t_n = \phi(t_1, \ldots, t_{n-1})$ and, for each $j = 1, \ldots, n-1$, there exists an integer $m_j > 0$ such that the matrix

$$\left[\left(\frac{\partial}{\partial t_i}\right)\left(\frac{\partial}{\partial t_j}\right)^{m_j} \phi(0)\right]_{i,j=1,\ldots,n-1}$$

forms an upper triangular matrix with nonzero diagonal entries.

Let $t_j = \sum_{k=1}^n C_j^k x_k, j = 1, \ldots, n$. We make the corresponding change of the complex coordinates $(z_1, \ldots, z_n)$, where $z_j = x_j + \sqrt{-1}y_j$, by

$$\zeta_j = \sum_{k=1}^n C_j^k z_k.$$

Then $\operatorname{Re}\zeta_j = t_j$. Let

$$r = 2t_n - 2\phi(t_1, \ldots, t_{n-1}) \equiv \zeta_n + \overline{\zeta}_n + \psi(\zeta_1, \overline{\zeta}_1, \ldots, \zeta_{n-1}, \overline{\zeta}_{n-1}).$$

Then

$$\left(\frac{\partial}{\partial\zeta_i}\right)\left(\frac{\partial}{\partial\overline{\zeta}_j}\right)^{m_j} r(0) = \left(\frac{\partial}{\partial\zeta_i}\right)\left(\frac{\partial}{\partial\overline{\zeta}_j}\right)^{m_j} \psi(0)$$
$$= \left(\frac{1}{2}\right)^{m_j+1}(-2)\left(\frac{\partial}{\partial t_i}\right)\left(\frac{\partial}{\partial t_j}\right)^{m_j} \phi(0).$$

So,

$$\left[\frac{\partial^{m_j+1} r}{\partial S_i(\partial\overline{S}_j)m_j}(0)\right]_{i,j=1,\dots,n-1}$$

is an upper triangular matrix with nonzero diagonal entries (therefore, nonsingular). Thus this reduces to a case of 3.1. □

3.3. Holomorphic decomposition of a defining function (cf. [4]) Let M be a hypersurface in $\mathbf{C}^{n+1} = \{(z_1, \dots, z_n, w)\}$ with a defining function of the form

$$r(z, \overline{z}, w, \overline{w}) = w + \overline{w} + \sum_{j=1}^{N} \varepsilon_j u_j(z, w)\overline{u_j(z, w)}, \tag{3.4}$$

where each ε_j is either 1 or -1 and each u_j is a holomorphic function vanishing at the origin. We can prove

PROPOSITION 3. *Let $(\Omega, \mathcal{V})$ be a C^ω CR manifold satisfying Condition 1 and let $F : \Omega \to M$ be a CR diffeomorphism, where M is defined by a local defining function (3.4). Suppose that $\{u_j(z, 0), j = 1, \dots, N\}$ are linearly independent functions of $z = (z_1, \dots, z_n)$. Let $h_j = u_j \circ F, j = 1, \dots, N$ and $h_{N+1} = w \circ F$. Then there exists an integer λ such that each $h_j \in C^\omega$, whenever $F \in C^{\lambda+1}$, and $(h_1, \dots, h_{N+1})$ is determined by their partial derivatives at the origin of order $\leq \lambda$.*

PROOF. Let $u_j(z, 0) = \sum_{|\alpha|=0}^{\infty} a_j^\alpha z^\alpha$, α: multi-index, $j = 1, \dots, N$. Since $\{u_j(z, 0)\}$ is linearly independent, we can choose N multi-indices $\alpha_1, \dots, \alpha_N$ so that

$$\det\left[a_j^{\alpha^i}\right]_{i,j=1,\dots,N} \neq 0.$$

Let $m = \max\{|\alpha_i| : i = 1, \ldots, N\}$. Now we will express each h_j as

$$h_j = H_j(x, D^\beta \overline{h}_i : |\beta| \le m, i = 1, \ldots, N+1),$$

where each H_j is an analytic function of the variables x and $D^\beta \overline{h}_i$. Let $V_j = \frac{\partial}{\partial z_j} - \left(\frac{\partial r}{\partial z_j} \Big/ \frac{\partial r}{\partial w}\right)\frac{\partial}{\partial w}, j = 1, \ldots, n$. Since $V_j r = 0, V_j$ is tangent to M and $\overline{V}_j$ is a tangential Cauchy-Riemann vector field. Let $\overline{Z}_j$ be a C^ω section of $\mathcal{V}$ belonging to the same class of m-jets as $F_*^{-1}(\overline{V}_j)$, (two vector fields $X = \Sigma a_i \frac{\partial}{\partial x_i}$ and $Y = \Sigma b_i \frac{\partial}{\partial x_i}$ are said to belong to the same class of m-jets if all the partial derivatives of a_i at the origin up to the order m are equal to those of b_i). We apply $\overline{\mathbf{Z}}^{\alpha_i} (i = 1, \ldots, N)$ to

$$r \cdot F = h_{N+1} + \overline{h}_{N+1} + \sum_{j=1}^{N} \varepsilon_j h_j \overline{h}_j = 0. \tag{3.5}$$

Since h_j's are CR functions, we have

$$\overline{\mathbf{Z}}^{\alpha_i} \overline{h}_{N+1} + \sum_{j=1}^{N} \varepsilon_j h_j (\overline{\mathbf{Z}}^{\alpha_i} \overline{h}_j) = 0. \tag{3.6}$$

But

$$\overline{\mathbf{Z}}^{\alpha}_{\alpha i} \overline{h}_j(0) = (F_*^{-1} \overline{\mathbf{V}}_i^{\alpha}) \overline{h}_j(0) = \overline{\mathbf{V}}^{\alpha_i} \overline{u}_j(0) = c_i \overline{a}_j^{\alpha_i},$$

where c_i is a positive integer. Since $\det\left[\overline{a}_j^{\alpha_i}\right]_{i,j=1,\ldots,N} \ne 0$, we can solve (3.5) and (3.6) with $i = 1, \ldots, N$ for $h_1, \ldots, h_{N+1}$ in terms of $\overline{Z}^{\alpha_i} \overline{h}_j, j = 1, \ldots, N+1, i = 1, \ldots, N$, which gives

$$h_j = H_j(x, D^\alpha \overline{h}_i : |\alpha| \le m, i = 1, \ldots, N+1),$$

where H_j is an analytic function of those variables. The conclusion follows from applying Theorem 1 to $(h_1, \ldots, h_{N+1})$.

3.4. CR manifolds of codimension 1 with nondegenerate Levi forms. Let $\mathcal{V}$ be a CR structure on $\Omega \subseteq \mathbf{R}^{2n+d}$ of complex dimension n and CR codimension d. We fix definitions and notations: At each point $P \in \Omega$ let $W_p = \mathbf{C}T_p(\Omega)/\mathcal{V}_{P \oplus} \overline{\mathcal{V}}_p$, where $\mathcal{V}_p$ is the fibre of $\mathcal{V}$ over P. Let $W_p = W_p^r \otimes \mathbf{C}$ for a real subspace W_p^r. A Levi form of $\mathcal{V}$ is the vector-valued hermitian form

$$\mathcal{L}_p : \mathcal{V}_p \times \mathcal{V}_p \to W_p$$

defined by

$$\mathcal{L}_p(v_1, v_2) = i[V_1, \overline{V}_2], (\mathrm{mod} \mathcal{V}_p \oplus \overline{V}_p),$$

where $V_j (j = 1, 2)$ is any section of $\overline{\mathcal{V}}$ such that $V_j(P) = v_j$. The Levi form is nondegenerate if $\mathcal{L}_p(v_1, v_2) = 0$ for all v_2, implying that $v_1 = 0$. The image of $\mathcal{L}_p$ is the set $\{\mathcal{L}_P(v, v) \in W_p^r : v \in \mathcal{V}_p\}$.

Webster proved in [**9**], using an analytic disk method, that if M is a C^ω generic submanifold of $\mathbf{C}^{n+d}$ such that the Levi form is nondegenerate at each point and its image contains an open set of W_p^r at each point $P \in M$ and if F is a CR diffeomorphism of M onto another such submanifold M', then $F \in C^1$ implies that $F \in C^\omega$.

Assuming that $F \in C^4$, we can weaken Webster's hypothesis on the Levi form:

PROPOSITION 4. *Let $(\Omega, \mathcal{V})$ be a $C^\omega CR$ manifold of the complex dimension n and CR codimension d and let $F = (f_1, \ldots, f_{n+d}) : \Omega \to \mathbf{C}^{n+d}$ be a CR diffeomorphism onto a C^ω generic submanifold. Suppose that the Levi form of $\mathcal{V}$ is nondegenerate and the image of the Levi form at $P \in \Omega$ spans W_p^r. Then $F \in C^4$ implies that $F \in C^\omega$.*

PROOF. We will show that $\mathcal{V}$ and $(f_1, \ldots, f_{n+d})$ satisfy the hypotheses of Theorem 2 with $m = 1, \nu = 2$, and therefore $\lambda' = 4$, from which the above conclusion follows. Let P be the origin. Let $M = F(\Omega) \subseteq \mathbf{C}^{n+d}$ be locally defined as

$$\cap_{j=1}^d \{r_j = 0\},$$

where each $r_j \in C^\omega$ and $\partial r_1 \wedge \cdots \wedge \partial r_d \neq 0$ on M. By a homomorphic change of coordinates we get a coordinate system

$$\{z_1, \ldots, z_n, w_1, \ldots, w_d\},$$

with respect to which r_j is of the form $r_j = w_j + \overline{w}_j - \phi_j(z, \overline{z}, w, \overline{w})$, where $\phi_j(0) = 0, d\phi_j(0) = 0$ and the Taylor series expansion of ϕ_j has no pluriharmonic part.

Now, for each $j = 1, \ldots, n$, there exists a complex vector field V_j tangent to M of the form

$$V_j = \frac{\partial}{\partial z_j} - \sum_{i=1}^{d} a_j^i \frac{\partial}{\partial w_i}.$$

The coefficients a_j^i's are uniquely determined by

$$V_j r_k = \frac{\partial r_k}{\partial z_j} - \sum_{i=1}^{d} a_j^i \frac{\partial r_k}{\partial w_i} = 0, \ k = 1, \ldots, d.$$

Note that $\left[\frac{\partial r_k}{\partial w_i}(0)\right]_{i,k=1,\ldots,d}$ is an identity matrix. Furthermore, we see that

$$a_j^i(0) = 0,$$
$$\left(\frac{\partial}{\partial z}\right)^\alpha \left(\frac{\partial}{\partial w}\right)^\beta a_j^i(0) = 0,$$

for any multi-indices α and β, and

$$\frac{\partial a_j^k}{\partial \overline{z}_u}(0) = \frac{\partial^2 r_k}{\partial z_j \partial \overline{z}_u}(0).$$

Therefore, at the origin,

$$\begin{aligned}\mathcal{L}(V_j, V_l) &= \sqrt{-1}\left[\frac{\partial}{\partial z_j} - \sum_{i=1}^{d} a_j^i \frac{\partial}{\partial w_i}, \frac{\partial}{\partial \overline{z}_l} - \sum_{i=1}^{d} \overline{a}_l^i \frac{\partial}{\partial \overline{w}_i}\right](0) \\ &\sqrt{-1}\sum_{i=1}^{d}\left(\frac{\partial \overline{a}_l^i}{\partial z_j}\frac{\partial}{\partial \overline{w}_i} + \frac{\partial a_j^i}{\partial \overline{z}_l}\frac{\partial}{\partial w_i}\right)(0) \\ &= \sqrt{-1}\sum_{i=1}^{k} \frac{\partial^2 r_i}{\partial z_j \partial \overline{z}_l}(0)\left(\frac{\partial}{\partial w_i} - \frac{\partial}{\partial \overline{w}_i}\right).\end{aligned}$$

Now for each $i = 1, \ldots, d$, we make a linear change of coordinates

$$(z_1, \ldots, z_n) \to \zeta^i = (\zeta_1^i, \ldots, \zeta_n^i)$$

so that

$$r_i = w_i + \overline{w}_i + \sum_{k=1}^{n_i} b_i^k \zeta_k^i \overline{\zeta}_k^i + \psi_i(\zeta, \overline{\zeta}, w, \overline{w}), \text{ for some } n_i \leq n,$$

where

$$\frac{\partial^2 \psi_i}{\partial \zeta_k \partial \overline{\zeta}_j}(0) = 0,$$

for any $j, k = 1, \dots n$, and each b_i^k is a nonzero constant.

Let $g_i = w_i \circ F, f_k = z_k \circ F$ and $f_k^i = \zeta_k^i \circ F$, $i = 1, \dots, d, k = 1, \dots, n$. Then

$$\text{(3.7)} \qquad r_i \circ F = g_i + \overline{g}_i + \sum_{k=1}^{n_i} b_i^k f_k^i \overline{f}_k^i + \psi_i \circ F = 0 \text{ on } \Omega.$$

Now, for each $i = 1, \dots, d$, and $j = 1, \dots, n$, let

$$V_j^i = \frac{\partial}{\partial \zeta_j^i} - \sum_{t=1}^{k} a_j^{i,t} \frac{\partial}{\partial \omega_t}.$$

Then the coefficients $a_j^{i,t}$ are uniquely determined by the condition $V_j^i r_k = 0, \quad k = 1, \dots, d$. Let $\overline{Z}_j^i$ be a C^ω section of $\mathcal{V}$ belonging to the same class of 4-jets as $F_*^{-1}(\overline{V}_j^i)$. Apply $\overline{Z}_j^i$ to (3.7) to get

$$\text{(3.8)} \qquad \overline{Z}_j^i \overline{g}_i + \sum_{k=1}^{n_i} b_i^k f_k^i (\overline{Z}_j^i \overline{f}_k^i) + \overline{Z}_j^i (\psi^i \circ F) = 0.$$

Since the Levi form is nondegenerate and the set $\{\mathcal{L}(v, v) | v \in \mathcal{V}_p\}$ generates W_p^r, we can choose n pairs (i, j), where $i \in \{1, \dots, d\}$ and $j \in \{1, \dots, n\}$ such that $\{\zeta_j^i\}$ for those chosen pairs (i, j) are linearly independent. Since

$$\begin{aligned} \overline{Z}_j^i \overline{f}_k^i(0) &= \overline{V}_j^i \overline{\zeta}_k^i(0) \\ &= \delta_{jk} \text{ (Kronecker delta)}, \end{aligned}$$

we can solve (3.7) with $i = 1, \dots, d$ and (3.8) with the chosen pairs (i, j) for $g_1, \dots, g_d$ and f_j^i. Since each f_j is a linear combination of f_j^i's, we get

$$\begin{aligned} g_i &= G_i(x, D^\alpha \overline{f}, D^\alpha \overline{g}, |\alpha| \le 1), \quad i = 1, \dots, d, \\ f_j &= F_j(x, D^\alpha \overline{f}, D^\alpha \overline{g}, |\alpha| \le 1), \quad j = 1, \dots, n, \end{aligned}$$

where G_i and F_j are analytic functions of the variables in the parentheses.

Now, to show that $\mathcal{V}$ satisfies Condition 2 with $\nu = 2$, choose C^ω sections $V_1, \dots, V_d$ of $\overline{\mathcal{V}}$ so that $\{T_i \equiv \sqrt{-1}[V_i, \overline{V}_i], i = 1, \dots, d\}$ generates W_P^r, where $P \in \Omega$ is the origin. Then, for a CR distribution f,

$$\begin{aligned}\overline{V}_i V_i f &= (V_i \overline{V}_i - [V_i, \overline{V}_i])f \\ &= \sqrt{-1} T_i f,\end{aligned}$$

so $T_i f \in B_2 f$. By induction, we see that, for each $p = 1, 2, \dots,$

$$(\overline{V}_i)^p (V_i)^p = p!(\sqrt{-1}T_i)^p f + \sum_{\substack{|\alpha| + |\beta| \leq p \\ |\alpha| < p}} a^{i,p}_{\alpha,\beta} \mathbf{T}^\alpha \mathbf{Z}^\beta f, \tag{3.9}$$

where the coefficients a's are $C^\omega, \mathbf{T} = (T_1, \dots, T_d)$ is as above and $\mathbf{Z} = (Z_1, \dots, Z_n)$ is any C^ω sections that generates $\mathcal{V}$ at each point of Ω. If we let $A_{i,p} = \frac{1}{p!(\sqrt{-1})^p} \sum a^{i,p}_{\alpha\beta} \mathbf{T}^\alpha \mathbf{Z}^\beta f$, from (3.9) we have $(T_i)^p f + A_{i,p} f \in B_2 f$. This completes the proof. □

REFERENCES

1. M.S. Baouendi, H. Jacobowitz, and F. Treves, *On the analyticity of CR mappings*, Ann. of Math. (2) **122** (1985), 365-400.

2. D. Burns and S. Shnider, *Real hypersurfaces in complex manifolds*, Proc. Symp. Pure Math. **30** (1976), 141-167.

3. S.S. Chern and J.K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219-271.

4. J. D'Angelo, *Real hypersurfaces, order of contact and applications*, Ann. of Math. **115** (1982), 615-637.

5. ———, *Finite-type conditions for real hypersurfaces in* $\mathbf{C}^n$, preprint.

6. C.K. Han, *Analyticity of CR equivalences of some real hypersurfaces in* $\mathbf{C}^n$ *with degenerate Levi forms*, Invent. Math. **73** (1983), 51-69.

7. L. Nirenberg, *Lectures on linear partial differential equations*, CBMS Reg. Conf. Series in Math. **17** (1973).

8. F. Treves, *Hypo-analytic structures*, Contemporary Math. **27** (1984), 23-44.

9. S. Webster, *Analytic discs and the regularity of CR mappings of real submanifolds in* $\mathbf{C}^n$, Proc. Symp. Pure Math. **41** (1984), 69-79.

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