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LINEAR COMBINATIONS OF HYPONORMAL OPERATORS

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1. Introduction. It is well known that if N_1 and N_2 are two normal operators such that their linear span consists of normal operators, then N_1 and N_2 commute [**8**]. This paper addresses the question whether this is true for hyponormal and subnormal operators. The answer is no. Two noncommuting hyponormal operators are given in this note such that their linear span consists entirely of hyponormal operators. What is true (Proposition 2.3)is that if A and B are two hyponormal operators and if $AB^* = B^*A$, then the linear span of A and B consists of hyponormal operators and both AB and BA are hyponormal.

The linear span of two normal operators consists entirely of normal operators if and only if the operators commute. There are, however, examples of subnormal operators A and B such that $AB = BA$ but neither $A + B$ nor AB is hyponormal [**7**] (also see pages 23-24 of [**5**]). Since half of this result for normal operators fails to generalize to the hyponormal case it is not too surprising that the other half also fails to generalize. The counterexample demonstrating this (Example 2.4) arises by constructing hyponormal operators A and B such that $AB^* = B^*A$, but $AB \neq BA$. In light of the result mentioned above, this shows that span$\{A, B\} \equiv \{aA + bB : a, b \in \mathbf{C}\}$ consists of hyponormal operators even though A and B do not commute.

This leads to a consideration of the question, "If A and B are hyponormal operators and $AB^* = B^*A$, when does $AB = BA$"? That is, when is the converse of Fuglede's Theorem valid for hyponormal operators? It is well known that Fuglede's Theorem does not extend to hyponormal operators or even subnormal operators. Moreover, as mentioned above, the possible generalization of the converse of

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the Fuglede Theorem to hyponormal operators, though previously overlooked, is also false. In §2 of this paper, conditions are given under which the linear span of two hyponormal operators consists of only hyponormal operators. Example 2.4 illustrates that the question raised at the beginning of this paper has a negative answer. Included in §2 is one necessary and sufficient condition (Proposition 2.5) which is used to show that if A is a hyponormal operator and N is a normal operator, then the linear span of A and N consists entirely of hyponormal operators if and only if $AN = NA$. A similar result is established for a subnormal operator A (Proposition 2.9). These results are far from the ultimate answer to this question and indicate the need for further work. In fact, this question is one more indication that our understanding of hyponormal and subnormal operators has much further to go.

In §3 a number of situations will be given in which the converse of Fuglede's Theorem does hold. It will be shown that it often happens that if A and B are hyponormal and $AB = BA$ and $AB^* = B^*A$, then either A or B is normal. In particular, this is the case when A is a finitely multicylic hyponormal operator.

The remainder of the present section is devoted to setting the notation and some preliminaries. The notation and terminology used here will usually conform to that of [**3**] and [**4**]. For the convenience of the reader some terminology is recalled here. An operator A on a Hilbert space $\mathcal{H}$ is said to be *hyponormal* if $A^*A - AA^*$ is a positive operator; A is *cohyponormal* if A^* is hyponormal.

In this paper $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded linear operators on the Hilbert space $\mathcal{H}$. The basic facts concerning hyponormal and subnormal operators can be found in [**3**]. If X and Y belong to $\mathcal{B}(\mathcal{H})$, the *commutator* of X and Y is defined as $[X, Y] = XY - YX$. The *self-commutator* of an operator A is $[A^*, A]$.

Recall that, for $1 \leq n \leq \infty, \mathcal{H}^{(n)}$ denotes the Hilbert space formed by taking the direct sum of $\mathcal{H}$ with itself n times (a countable number of times if $n = \infty$). Say that a set of operators is *hyponormal* (respectively, *subnormal*) if each operator in the set is hyponormal (respectively, subnormal).

2. The linear span of two hyponormal operators. This section will explore conditions under which span $\{A, B\}$ is hyponormal. The proof of the next lemma is straightforward.

LEMMA 2.1. *If A and B belong to $\mathcal{B}(\mathcal{H})$, then span$\{A, B\}$ is hyponormal (respectively, subnormal) if and only if, for every a in $\mathbf{C}$ both A and $aA + B$ are hyponormal (respectively, subnormal).*

Note that if $w \in \mathbf{C}$ and $T = wA + B$, then

$$[T^*, T] = |w|^2[A^*, A] + [B^*, B] + 2\mathrm{Re}\,(w[B^*, A]). \tag{2.2}$$

PROPOSITION 2.3. *If A and B are hyponormal and $AB^* = B^*A$, then:*

(a) span $\{A, B\}$ *is hyponormal;*

(b) *AB and BA are hyponormal.*

PROOF. Since $AB^* = B^*A$, $[B^*, A] = 0$. The result (a) is now immediate from Lemma 2.1 and Equation 2.2. To prove (b) note that $AB(AB)^* = ABB^*A^* \leq AB^*BA^* = B^*AA^*B \leq B^*A^*AB = (AB)^*AB$; thus AB is hyponormal. Similarly, BA is also hyponormal. □

The condition of Proposition 2.3 is not necessary for span $\{A, B\}$ to be hyponormal Indeed, if $A = B =$ the uniliateral shift of multiplicity one, then span$\{A, B\}$ is hyponormal but $AB^* \neq B^*A$.

The next example shows that it is possible for span$\{A, B\}$ to be hyponormal without having $AB = BA$, thus answering the question posed at the beginning of this paper.

EXAMPLE 2.4. If A is the unilateral shift of infinite multiplicity, then there is a hyponormal operator B such that $AB^* = B^*A$, and so span $\{A, B\}$ is hyponormal, but $AB \neq BA$.

Let U be the unilateral shift of multiplicity one and let $\mathcal{K}$ be an infinite dimensional Hilbert space. Thus $A = I_{\mathcal{K}} \otimes U$ is a unilateral shift of infinite multiplicity. Let $B = X \otimes 1 + Y^* \otimes U^*$ for some X and Y in $B(\mathcal{K})$. Clearly $AB^* = B^*A$ but $AB \neq BA$. Computing shows that

$$B^*B = X^*X \otimes 1 + X^*Y^* \otimes U^* + YX \otimes U + YY^* \otimes UU^*,$$
$$BB^* = XX^* \otimes 1 + XY \otimes U + Y^*X^* \otimes U^* + Y^*Y^* \otimes 1$$

Assume for the moment that X and Y can be found such that $XY = YX$. Then

$$[B^*, B] = (X^*X - XX^* - Y^*Y) \otimes 1 + YY^* \otimes UU^*.$$

So in order to get B hyponormal and satisfying the other conditions stipulated in the example, it suffices to find operators X and Y such that $XY = YX, Y \neq 0$, and $X^*X - XX^* - Y^*Y \geq 0$.

To do this let $\mathcal{L}$ be any Hilbert space of dimension at least 2 and put $\mathcal{K} = \mathcal{L}^{(\infty)}$ (the direct sum of $\mathcal{L}$ with itself countably many times). Thus operators on $\mathcal{K}$ can be represented as infinite matrices with entries from $\mathcal{B}(\mathcal{L})$. Let P be a proper projection on $\mathcal{L}(P \neq 0$ or $1)$ and define X and Y by

$$X = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ P & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \cdots \end{bmatrix} \quad Y = \begin{bmatrix} 0 & 1-P & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots \end{bmatrix}.$$

If follows that

$$X^*X = \begin{bmatrix} P & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \cdots \end{bmatrix} \quad XX^* = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & P & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \cdots \end{bmatrix}$$

$$Y^*Y = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1-P & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots \end{bmatrix}.$$

Thus

$$X^*X - XX^* - Y^*Y = \begin{bmatrix} P & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots \end{bmatrix}$$

and $XY = YX = 0$. This concludes the example.

The next result is a necessary and sufficient condition for span $\{A, B\}$ to be hyponormal, though in practice it is difficult to verify. Its main virtue is that it is the natural approach to obtain Corollary 2.8.

PROPOSITION 2.5. *If A and B are hyponormal, then* span$\{A, B\}$ *is hyponormal if and only if, for every h in $\mathcal{H}$ such that $[B^*, A]h \neq 0$, the inequality*

$$| \langle [B^*, A]h, h \rangle |^2 \leq \langle [A^*, A]h, h \rangle \langle [B^*, B]h, h \rangle$$

holds.

Before proving this proposition, a lemma about complex numbers is necessary.

LEMMA 2.6. *If a and b are non-negative real numbers and c is a non-zero complex number, then*

$$|z|^2 a + b + 2\mathrm{Re}\ (zc) \geq 0$$

for all complex numbers z if and only if $|c|^2 \leq ab$.

PROOF. Suppose $|z|^2 a + b + 2\mathrm{Re}\,(zc) \geq 0$ for all z in $\mathbf{C}$. For an arbitrary real number t, if $z = t\bar{c}$, then $t^2|c|^2 a + b + 2t|c|^2 \geq 0$. An examination of the discriminant of this quadratic expression shows that $|c|^4 \leq |c|^2 ab$. For the converse, if μ is any number in $\mathbf{C}$ with $|\mu| = 1$, then $\mathrm{Re}\,(\mu c)^2 \leq |c|^2 \leq ab$. Hence $t^2 a + b + 2t\mathrm{Re}\,(\mu c) \geq 0$ for every t in $\mathbf{R}$. Since every complex z can be written as $t\mu$ with t real and $|\mu| = 1$, this completes the proof. □

PROOF OF PROPOSITION 2.5. Let w be any complex number and put $T = wA + B$. It follows from Equation 2.2 that

$$\langle [T^*, T]h, h \rangle = a|w|^2 + b + 2\mathrm{Re}\,(cw),$$

where $a = \langle [A^*, A]h, h\rangle, b = \langle [B^*, B]h, h\rangle$, and $c = \langle [B^*, A]h, h\rangle$. The proposition is now immediate by Lemma 2.6. □

COROLLARY 2.7. *If A and B are hyponormal, $AB^* \neq B^*A$, and* span$\{A, B\}$ *is hyponormal, then neither A nor B is normal.*

PROOF. The preceding proposition implies that if either A or B is normal then $\langle [B^*, A]h, h\rangle = 0$ for all vectors h in $\mathcal{H}$. But this is equivalent to the condition that $[B^*, A] = 0$, contradicting the hypothesis. □

The next corollary completely answers the question raised at the beginning of this paper in the case that one of the operators is normal.

COROLLARY 2.8. *If A is a hyponormal operator and N is normal, then* span$\{A, N\}$ *is hyponormal if and only if $AN = NA$.*

PROOF. First assume that span$\{A, N\}$ is hyponormal. It is immediate from the preceding corollary that $AN^* = N^*A$. By Fuglede's Theorem, A and N commute. The converse is immediate by Fuglede's Theorem and Proposition 2.3. □

There is a corresponding result for subnormal operators. Part of this next result is in the folklore of subnormal operator theory and is presented here for completeness.

PROPOSITION 2.9. *If S is a subnormal operator and N is a normal operator on $\mathcal{H}$, then the following statements are equivalent.*

(a) *S and N commute.*

(b) span$\{S, N\}$ *is subnormal.*

(c) $\mathcal{A}(S, N)$, *the algebra generated by S and N, is subnormal.*

(d) *S and N have commuting normal extensions.*

PROOF. It is clear that (d) implies (c) and (c) implies (b). The fact that (b) implies (a) is an immediate consequence of Corollary 2.8. It remains to show that (a) implies (d). To do this, a result of Ito (Theorem 1 of [**6**]) that generalizes the Halmos-Bram criterion to families of operators will be used.

Let $\{h_{mn}\}$ be an arbitrary finite family of vectors in $\mathcal{H}$. By Fuglede's Theorem

$$\begin{aligned}\Omega \equiv & \sum_{m,n,j,k} \langle S^m N^n h_{jk}, S^j N^k h_{mn}\rangle \\ = & \sum_{m,n,j,k} \langle S^m N^{*k} h_{jk}, S^j N^{*n} h_{mn}\rangle.\end{aligned}$$

If $f_j \equiv \Sigma_k N^{*k} h_{jk}$, then

$$\Omega = \sum_{m,j} \langle S^m f_j, S^j f_m\rangle$$

and so, by the subnormality of S and the Halmos-Bram condition, $\Omega \geq 0$. If now follows by the result of Ito referred to above that S and N have commuting normal extensions.

3. The converse of Fuglede's Theorem. As seen in the preceding section, there is a close relationship between the questions "When is span $\{A, B\}$ hyponormal"? and "When does $AB^* = B^*A$"? Attention will now be shifted to the latter question. To do this, some initial preparation is necessary. The proof of the first lemma in this process is left to the reader.

LEMMA 3.1. *Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, assume that $P \in \mathcal{B}(\mathcal{H}), Q \in \mathcal{B}(\mathcal{K})$, and $C : \mathcal{K} \to \mathcal{H}$ is a bounded operator. If an operator X in $\mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ is defined by the matrix*

$$X = \begin{bmatrix} P & C \\ C^* & Q \end{bmatrix},$$

then the following statements hold:

(a) *The operator X is positive if and only if for all h in $\mathcal{H}$ and k in $\mathcal{K}$,*

$$\langle Ph, h\rangle + \langle Qk, k\rangle + 2\mathrm{Re}\ \langle Ck, h\rangle \geq 0.$$

(b) *If* $X \geq 0$, *then* $P \geq 0$ *and* $Q \geq 0$.

(c) *If either* P *or* Q *is* 0 *and* X *is either positive or negative, then* $C = 0$.

LEMMA 3.2. *For* $1 \leq n \leq \infty$ *and* $A_{ij} \in \mathcal{B}(\mathcal{H})$, *for* $1 \leq i, j \leq n$, *let* $X = (A_{ij}) \in \mathcal{B}(\mathcal{H}^{(n)})$.

(a) *If* $X \geq 0$, *then* $A_{ii} \geq 0$ *for all* i.

(b) *If* X *is either positive or negative and* $A_{ii} = 0$ *for all* i, *then* $X = 0$.

PROOF. This follows by an application of the preceding lemma. □

PROPOSITION 3.3. *For* $1 \leq n < \infty$ *and* $1 \leq i, j \leq n$, *let* $T_{ij} \in \mathcal{B}(\mathcal{H})$ *and assume that* $T = (T_{ij})$ *is a bounded operator on* $\mathcal{H}^{(n)}$. *If each* T_{ij} *is hyponormal and* T *is cohyponormal, then* T *is normal.*

PROOF. Let $X = T^*T - TT^* = (A_{ij})$. It must be shown that $X = 0$. By assumption, $X \leq 0$. According to the preceding lemma, to show X is 0 it suffices to show that the diagonal entries of X are all 0.

A computation shows that

$$A_{kk} = \sum_{j=1}^{n} (T_{kj}^* T_{kj} - T_{jk} T_{jk}^*).$$

Since T is cohyponormal, $A_{kk} \leq 0$ for all k. Thus, by rearranging the order of the terms in the sum, it follows that

$$0 \leq \sum_{k=1}^{n} A_{kk} = \sum_{k=1}^{n} \sum_{j=1}^{n} (T_{kj}^* T_{kj} - T_{jk} T_{jk}^*) = \sum_{k=1}^{n} \sum_{j=1}^{n} (T_{kj}^* T_{kj} - T_{kj} T_{kj}^*).$$

But, by hypothesis, each T_{kj} is hyponormal and so $T_{kj}^* T_{kj} - T_{kj} T_{kj}^* \geq 0$. Hence $\sum_{k=1}^{n} A_{kk} \geq 0$. This implies that $\sum_{k=1}^{n} A_{kk} = 0$. But $A_{kk} \geq 0$ for each k and so $A_{kk} = 0$ for each k. By the preceding lemma, $X = 0$ and T is normal. □

Note that the proof of the preceding proposition also shows that each T_{ij} is normal since $A_{kk} = 0$ for all k and each T_{jk} is hyponormal. Also the preceding proposition is false for infinite matrices of operators. In fact, for $T =$ the backward unilateral shift of multiplicity 1, a cohyponormal operator that is not normal, the standard matrix representation of T has as its entries non-negative operators on the one dimensional Hilbert space $\mathbf{C}$.

COROLLARY 3.4. *Let $A \in \mathcal{B}(\mathcal{H})$ and assume its commutant, $\{A\}'$, is hypernormal. If $1 \leq n < \infty$, I_n is the identity on an n dimensional Hilbert space $\mathcal{H}_n$, and B is a hypernormal operator in $\mathcal{B}(\mathcal{H} \otimes \mathcal{H}_n)$ satisfying $B^*(A \otimes I_n) = (A \otimes I_n)B^*$, then $B(A \otimes I_n) = (A \otimes I_n)B$ and B is normal.*

PROOF. Because $\{A \otimes I_n\}'$ consists of the n by n matrices with entries from $\{A\}'$, it is an immediate consequence of the preceding proposition that B^* and hence B, is normal. The remainder of the corollary follows from Fuglede's Theorem. □

The preceding corollary shows that the construction in Example 2.4 is not possible for a shift of finite multiplicity. The next result is another instance where $AB^* = B^*A$ implies that $AB = BA$.

PROPOSITION 3.5. *If $A, B \in \mathcal{B}(\mathcal{H})$, A commutes with both B^* and B^*B, and B commutes with A^*A, then $AB = BA$. In particular, if A and B are isometries and $AB^* = B^*A$, then $AB = BA$.*

PROOF. Under the hypothesis, $(AB - BA)^*(AB - BA) = B^*A^*AB - B^*A^*BA - A^*B^*AB + A^*B^*BA = 0$. Thus $AB - BA = 0$. The statement about isometries is immediate, since in this case A^*A and B^*B are the identity operator. □

One might wonder whether the infinite multiplicity was essential in Example 2.4. If A is a hyponormal operator which is finitely multicyclic and B is a hyponormal operator such that $AB^* = B^*A$, must it be that $AB = BA$? Note that if this is true, then B is a hyponormal operator that commutes with both A and A^*. Thus $B \in W^*(A)'$, the commutant of $W^*(A)$, the von Neumann algebra generated by A. As the next result shows, this implies that B would have had to be normal so that Fuglede's Theorem would have sufficed.

PROPOSITION 3.6. *If A is a pure essentially normal operator and A_0 is a normal operator of finite multiplicity, then the only hyponormal operators in $W^*(A \oplus A_0)$ are the normal ones.*

PROOF. It is known ([**1**], also see [**3**; p. 296]) that an essentially normal operator can be written as the direct sum of irreducible ones and a normal operator. Therefore, because A is pure,

$$A = A_1^{(n_1)} \oplus A_2^{(n_2)} \oplus \cdots ,$$

where each A_j is irreducible and A_i and A_j are not unitarily equivalent for $i \neq j$. Note that $[A^*, A] = [A_1^*, A_1]^{(n_1)} \oplus [A_2^*, A_2]^{(n_2)} \oplus \cdots$. Since A is essentially normal, it follows that each A_j is essentially normal and that each $n_j < \infty$.

Suppose that A_j acts on the Hilbert space $\mathcal{H}_j$ and A_0 acts on $\mathcal{H}_{\bar{0}}$. Since A_j is irreducible, $W^*(A_j) = \mathcal{B}(\mathcal{H}_j)$. It is a standard consequence of Schur's Lemma and the fact that each n_j is finite that the commutant of $W^*(A_0 \oplus A)$ is spatially isomorphic to

$$W^*(A_0)' \oplus [M_{n_1} \otimes 1_{\mathcal{H}_1}] \oplus [M_{n_2} \otimes 1_{\mathcal{H}_2}] \oplus \cdots ,$$

where M_n denotes the n by n matrices. If B is a hyponormal operator in the commutant of $W^*(A_0 \oplus A)$, then $B = B_0 \oplus B_1 \oplus B_2 \oplus \cdots$, where each B_j is hyponormal, $B_0 \in W^*(A_0)'$, and, for $j \geq 1, B_j$ corresponds to a hyponormal element of M_{n_j}. Thus B_j is normal for all $j \geq 1$. Because of the finite multiplicity of A_0, multiplicity theory for normal operators (see, for example, [**4**, p. 307]) implies that B_0 is the direct sum of finite matrices with entries from the L^∞ space of some measure supported on a compact subset of the plane. Thus the entries in these matrices are normal operators and so, by Proposition 3.3, B_0 is a normal operator. Therefore B must be normal. □

COROLLARY 3.7. *If A is a finitely multicyclic hyponormal operator and B is a hyponormal operator such that $AB = BA$ and $A^*B = BA^*$, then B is normal.*

PROOF. This is an immediate consequence of the preceding result and the Berger-Shaw Theorem [**2**] which implies that A is essentially normal. □

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