ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 17, Number 3, Summer 1987

AN α–APPROXIMATION THEOREM FOR R^∞–MANIFOLDS

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0. Introduction and preliminaries. Generalizing the CE-approximation theorem of Armentrout [**1,2**] and Siebenmann [**20**] for finite-dimensional manifolds, Ferry proved an α–approximation theorem for Q–manifolds in [**8**] and an α–approximation theorem for manifolds of dimensions ≥ 5 in a joint work with Chapman [**6**].

Recently, the author proved in [**16**] an α–approximation theorem for Q^∞–manifolds: "Given an open cover α of a Q^∞–manifold N, then there is an open cover β of N such that every β–equivalence from a Q^∞–manifold M to N is α–close to a homeomorphism".

It will be shown in this note that such an α–approximation theorem also holds true for R^∞–manifolds. So, the question (NLC 8) in [**9**] has an affirmative answer.

As in [**16**], in the process of proving the main theorem, some results similar to a few properties of Z–sets in Q and ℓ_2–manifold theory will be proved. These include:

(1) relative R^∞–deficient embedding approximation theorem (Theorem 2.3);

(2) unknotting theorem for R^∞–deficient embeddings (Theorem 3.3);

(3) collar theorem (Theorem 4.2); and

(4) R^∞–deficient subsets being strongly negligible (Theorem 5.3).

For standard concepts such as the cone(X) of a topological space X, the mapping cylinder $M(f)$ of a map f, the infinite mapping cylinder $M(f_1, f_2, ...)$ of a sequence of maps $f_i : X_{i-1} \to X_i$, the limitation of a homotopy $H : X \times I \to Y$ by an open cover α of Y, the n^{th}–star $\mathrm{St}^n(\alpha)$ of an open cover α, etc., we refer to [**8**] or [**16**] for more details. All topological spaces are separable.

Throughout this note, let R^∞ be the direct–limit space $\lim_{\to}\{R^n\}$

* This research is partially supported by Summer Research Grant 1045 of University of Alabama, 1980
Received by the editors on May 28, 1983.

endowed with the direct limit topology, where R^n is the n–euclidean space. It has been proved that $\mathrm{R}^\infty \approx \mathrm{R}^\infty \times \mathrm{R}^\infty$ [**13**, p.395] and that R^∞ is paracompact [**11**, p.298]. By an R^∞–manifold, we mean a separable paracompact space that is locally homeomorphic to R^∞. Let us denote $[0,1]$ by either I or J, and $\lim_{\rightarrow}\{[0,1]^n, i_n\}$ by I^∞ or J^∞, where $i_n(x) = (x,0)$. Then, from the corollary in [**14**], it follows that $\mathrm{R}^\infty \approx I^\infty$. A subset X of an R^∞–manifold M is said to be *R^∞– deficient* if X is closed in M and there is a homeomorphism $h : M \to M\times\mathrm{R}^\infty$–such that $h(X) \subset M\times 0$, where $D = (0,0,\cdots) \in \mathrm{R}^\infty$. Recall that M is R^∞ stable [**12**, p.48]. An embedding $f : X \to M$ is said to be R^∞–deficient if $f(X)$ is R^∞*–deficient* in M. A closed subset X of M is said to be *collared* in M if there is an open embedding $f : X \times [0,1) \to M$ such that $f(x,0) = x$ for all $x \in X$. A collar $f : X \times [0,1) \to M$ is *normal* if $f(X \times [0,s])$ is closed in M for each $s \in [0,1)$, and the restriction $f|(X \times [0,s])$ of a normal collar f is called a *closed collar* of X in M. Observe that, if M is a paracompact and X is a collared subset in M, then there is a normal collar of X in M. Therefore, throughout this note, we will use normal collars without further notice if the involved spaces are paracompact.

A map $f : X \to Y$ is said to be a *near homeomorphism* if f is α–close to a homeomorphism for any pre–chosen open cover α of Y. Given an open cover α of Y, a map $f : X \to Y$ is said to be an *α–equivalence* if there is a map $g : Y \to X$ such that $g\circ f \overset{f^{-1}(\alpha)}{\simeq} \mathrm{id}_X$ and $f\circ g \overset{\alpha}{\simeq} \mathrm{id}_Y$ (refer to [**8**] or [**16**]). A map $f : X \to Y$ is called a fine homotopy equivalence if f is an α–equivalence for each open cover α of Y.

Let q be a nonnegative integer and G an open subset of a metric space (M,d). G is said to be q–LC at $x \in M$ (rel. M) (refer to [**18**, p.45]) provided that, given an open set $U = N(x;\varepsilon)$, there is an open set $V = N(x;\delta)(\delta = \delta(x,\varepsilon) > 0)$ such that every map from the q–sphere into $V\cap G$ is null–homotopic in $U\cap G$, where $N(x;r)$ denotes $\{y : y \in M$ and $d(x,y) < r\}$ for $r > 0$. G is said to be $q - LC($ rel. $M)$ if it is q–LC at each $x \in X$ (rel. M). G is said to be *q–LC* (rel M) if it is q-LC at M (rel. M). If the choice of δ is independent to x for all $x \in M$, G is said to be q–ULC (rel. M). A closed subset X of M is said to be q-LCC in M if $M - X$ is $q - LC$ at X(rel. M). A closed embedding $f : \mathrm{Z} \to M$ is said to be 1–LCC if $f(\mathrm{Z})$ is 1–LCC in M. Observe that if M is a finite–dimensional manifold and if $f : \mathrm{Z} \to M$ is 1–LCC embedding, then $M - f(\mathrm{Z})$ is 1–LC (rel. M). By LC^p (ULC^p), we mean q–LC (q–ULC) for each $q = 0,1,...,p$. The proof of the following lemma is straightforward.

LEMMA 0. *Let M be a compact manifold and X a compact subset of its interior. Then $M - X$ is 1–ULC (rel.M) if and only if X is 1–LCC in M. If $\dim X \leq \dim M - 2$, then $M - X$ is 0–ULC.*

We now state some known results that we will use in the sequel. From the theorem in [**3**] restated in §1 below, a closed embedding of a compact PL–manifold N^k into a manifold M^m without boundary, with $2k + 2 \leq m$ and $m \geq 5$, is locally flat if and only if it is a 1–LCC embedding. Hence, the following lemma can be deduced from Theorem 0 in [**12**] and the proposition in Part C of [**14**].

LEMMA A. *A space M is an R^∞–manifold if and only if M is homeomorphic to $\lim_\rightarrow M_n$, where, for each n, M_n is a compact finite–dimensional manifold and it is a 1–LCC subset of the interior of M_{n+1} with $2 \dim M_n + 2 \leq \dim M_{n+1}$.*

The following is from Lemma 2.4 of [**10**].

LEMMA B. *Let $X = \lim_\rightarrow\{X_n\}$, where X_n is a metric subspace of X_{n+1} for each n. If K is a compact subset of X, then there is an integer n_0 such that K is contained in X_{n_0}.*

Throughout this note, let $\operatorname{Int} M$ and ∂M denote the interior and the boundary of a finite–dimensional manifold M respectively. For convenience, we will use the same notations R^n for $\mathrm{R}^n \times 0 \subset \mathrm{R}^\infty$, M for $M \times 0 \subset M \times \mathrm{R}^\infty$, $M \times \mathrm{R}^k$ for $M \times \mathrm{R}^k \times 0 \subset M \times \mathrm{R}^\infty$, etc.

1. Relative unknotting theorem in $\mathbf{R}^n$. Bryant [**3**] has shown that "*if X is a metric compact space and if $f, g : X \to R^n$ are two ε–homotopic 1–LCC embeddings, with $n \geq 5$ and $2 \dim X + 2 \leq n$, then there is an ε–isotopy F_t of R^n, $t \in I$, such that $F_0 = id$ and $F_1 \circ f = g$*". In this section, we will prove a relative version of Bryant's theorem that, according to our knowledge, has not appeared elsewhere. We need some notations and observations for the proof.

Given a subset Z of a metric space (X, d) and a $\delta > 0$, let $N(Z; \delta)$ denote the δ–neighborhood of Z in X, $\{x \in X | d(x, Z) < \delta\}$. It follows from Lemma 0 that if X is a compact 1–LCC subset of R^n, then $\mathrm{R}^n - X$ is 1–ULC, i.e., given an $\varepsilon > 0$, there is a $\delta > 0$ such that every δ–loop in

$R^n - X$ is ε–null homotopic in R^n–X. By use of the PL–approximation theorem [**19**], Theorem 5.3 and the 0–ULC property of $R^{n-1} - X$ from Lemma 0, we can show that if X is a compact subset of $R^{n-1} \subset R^n$ with $\dim X \leq n-3$, then X is 1–LCC in R^n. Observe that if X is a 1–LCC subset of R^n and if h is a homeomorphism of R^n, then $h(X)$ is also 1–LCC in R^n.

THEOREM 1.1. *Let (X, X_0) be a pair of compact metric spaces and let $H : X \times I \to R^n (n \geq 5)$ be a homotopy* (rel. X_0) *from a* 1–LCC *embedding f to another g. Let $\delta : X \to I$ be a map such that $\delta^{-1}(0) = X_0$. If $2\dim X + 4 \leq n$, then there is an isotopy F_t of R^n, $t \in I$, such that*

$$F_0 = \mathrm{id}, \tag{1}$$

$$F_t = \mathrm{id} \text{ on } f(X_0) \cup (R^n - U), \qquad \text{for each } t \in I, \tag{2}$$

where U is the union of all members of $\mathcal{U} = \{N(H(x \times I); 2\delta(x)) | x \in X - X_0\}$,

(3) *$F(x \times I)$ is either constant or limited by $\mathcal{U}$, for each $x \in R^n$, and*

(4) *$F_1 f = g$.*

We need a few preliminary lemmas for its proof.

LEMMA 1.2. *If X and Y are two 1–LCC compact subsets of M^n, a manifold without boundary, and if $\dim(X \cup Y) \leq n-3$, then $X \cup Y$ is 1–LCC in M.*

PROOF. Let $x \in X \cup Y$. If $x \in X \cap Y$, the 1–LCC property at x (rel.M) is easily verified. For $x \in X \cap Y$, given a neighborhood U of x in M^n, let V be a neighborhood of x in U obtained from the 1–LCC property of X at x. We will show that every map $f : \partial\Delta^2 \to V - (X \cup Y)$ is null–homotopic in $U - (X \cup Y)$, where Δ^2 denotes the standard 2–simplex.

Let $\phi : \Delta^2 \to U - X$ be an extension of f over Δ^2. Given an $\varepsilon > 0$ such that $\varepsilon < \min\{\mathrm{dist}(\phi(\Delta^2), X), \mathrm{dist}(\phi(\Delta^2), M^n - U\}$, there is a $\delta (0 < \delta < \varepsilon/3)$ such that every δ–loop in $M^n - Y$ is $(\varepsilon/3)$–null homotopic in $M^n - Y$. Moreover, since $\dim Y \leq n-3$, $M^n - Y$ is 0–ULC. Hence, there is a positive number η such that if x and y are in $M^n - Y$ with $d(x, y) < \eta$, then there is a $(\delta/2)$-path in $M^n - Y$

joining x and y. Now, let K be a PL-subdivision of Δ^2 such that $\operatorname{diam}\phi(\sigma) < \eta/3$ for each 2–simplex σ of K. Then, we can construct, by induction , a map $\psi : K^{(0)} \cup K^{(1)} \to U - Y$ such that

(1) $\psi|\partial\Delta^2 = \phi|\partial\Delta^2$,

(2) $d(\psi(v), \phi(v)) < \eta/3$ for each $v \in K^{(0)}$, and

(3) $\operatorname{diam}\phi(\tau) < \eta/2$ for each 1–simplex τ of K.

Finally, we extend ψ over each 2–simplex σ of $K^{(2)}$ into $M^n - Y$ such that $\operatorname{diam}\psi(\sigma) < \varepsilon/3$. Then, we can show that $d(\psi(x), \phi(x)) < \delta$ for all $x \in K^{(0)} \bigcup K^{(1)}$ and it follows that $d(\psi(x), \phi(x)) < \varepsilon$ for all $x \in K$. Hence $\psi(\Delta^2) \subset U - (X \cup Y)$ by the choice of ε, and the proof of Lemma 1.2 is complete.

Let Z be a closed subset of a PL–manifold M without boundary. If Z is 1–LCC, then $M - Z$ is 1–LC (rel.M). Therefore, if $2 \le p \le n - k - 2$ where $k = \dim Z$, it follows from Theorem 4 in [**18**] that $M - Z$ is LC^p (rel.M) since $M - Z$ is ℓc^p (rel.M) [**18**, p.45]. So, for $k \geq 2$ and $2k + 1 \leq n$, by use of the compactness of I^k, Lemma 0 and Theorem 6 in [**18**], it is tedious to show that a closed subset Z of Int$M$ has the property Z^k as in [**14**] if and only if Z is 1–LCC in M. Therefore, we can obtain the following lemma which is a special case of the Corollary 5 in [**14**].

LEMMA 1.3. *Let n be an integer ≥ 5 and let (X, X_0) be a pair of locally compact metric spaces with $2\dim X + 1 \leq n$. Let $f : X \to M^n$ be a proper map such that $f|X_0$ is a 1–LCC embedding, where M is a* PL*–manifold without boundary. Then, given a map $\varepsilon : X \to [0,1]$ such that $\varepsilon^{-1}(0) = X_0$, there is a proper $1 - LCC$ embedding $g : X \to M^n$ such that*

(1) $g(x) = f(x)$ if $x \in X_0$,

(2) $d(f(x), g(x)) < \varepsilon(x)$, all $x \in X - X_0$.

PROOF OF THEOREM 1.1. Let $\tilde{X}$ denote the quotient space $(X \times I)/\sim$, where $(x, 1) \sim (x', 0)$ if $f(x') = g(x)$, or $(x, t) \sim (x, 0)$ if $x \in X_0$. Then, the given homotopy $H : f \simeq g$ induces a map $\tilde{H} : \tilde{X} \to \mathrm{R}^n$. Observe that $\dim \tilde{X} \leq \dim X + 1$ as follows. Let $X_1 = (X \times [0, 1/2])/\sim_1$ and $X_2 = (X \times [1/2, 1])/\sim_2$, where $(x, t) \sim_1 (x, 0)$ if $x \in X_0$ and $0 \leq t \leq 1/2$, and where $(x, t) \sim_2 (x, 1)$ if $x \in X_0$ and $1/2 \leq t \leq 1$. Then, X_1 and X_2 are homeomorphic to the subspace $\cup\{x \times [0, \lambda(x)] | x \in X\}$ of $X \times I$ for some map $\lambda : X \to [0,1]$ with $\lambda^{-1}(0) = X_0$. Hence, $\dim X_1 = \dim X_2 \leq \dim(X \times I) = \dim X + 1$. The latter follows from the Remark on p. 34 in [**15**] since X is compact. Now, set $A =$

$f^{-1}(f(X)\cap g(X))$ and $B=g^{-1}(f(X)\cap g(X))$ and think of $X\times\{1/2\}$ as compact subsets of X_1 and X_2. Then, $\tilde{X}=X_1\bigcup_\phi X_2$, where $\phi=g^{-1}f\cup\mathrm{id}:((A\times\{0\})\cup(X\times\{1/2\}))/\sim_1\to((B\times\{1\})\cup(X\times\{1/2\}))/\sim_2$. In other words, X is the union of two compact subspaces whose intersection is the compact subspace $((A\times\{0\})\cup(X\times\{1/2\}))/\sim_1\equiv((B\times\{1\})\bigcup(X\times\{1/2\})))/\sim_2$. Hence, $\dim\tilde{X}\le\dim X+1$, by Theorem III 2[**15**]. Let $\overline{X_0}$ denote$((X_0\times I)\cup(X\times\{0,1\}))/\sim$. Then, $\tilde{H}|\overline{X_0}$ is a 1–LCC embedding by Lemma 1.2.

Now, from Lemma 1.3, there is a 1–LCC embedding $\overline{H}:\tilde{X}\to\mathrm{R}^n$ such that: (1) $d(\overline{H}(z),\tilde{H}(z))<\delta(x)$ if $z\in\tilde{X}-\overline{X_0}$ is represented by a point (x,t) in $X\times I$; and (2) $\overline{H}(z)=\tilde{H}(z)$ if $z\in\overline{X_0}$. First, by considering the homotopy $\overline{H}$ from $\overline{H_0}$ to $\overline{H_{1/2}}$, we can assume that $f(X-X_0)\cap g(X-X_0)=\emptyset$. Observe that if Z is a codimension 3 1–LCC compact subset of R^{n-1}, then every closed subset of $Z\times R$ is 1–LCC in R^n. Therefore, if $e:X\to\mathrm{R}^{n-1}$ is a 1–LCC embedding, then the map $\overline{e}:\tilde{X}\to\mathrm{R}^n$ defined by $\overline{e}(z)=(e(x),t\cdot\lambda(x))$, where (x,t) is a representative of $z\in\tilde{X}$, is a 1–LCC embedding. Hence, again by [**3**], we can assume that $f(X)\subset\mathrm{R}^{n-1}\times\{0\}$ and $\overline{H}(x,t)=(f(x),t\lambda(x))$. (Recall that $2\dim\tilde{X}+2\le n$.)

As in the proof of Theorem 9.1 in [**5**], we need only to construct an isotopy corresponding to such a homotopy $\overline{H}$. Let $\mu,\eta:\mathrm{R}^{n-1}\to[0,1]$ denote the extensions of $\lambda f^{-1}$and δf^{-1} respectively over R^{n-1} such that the set $Y=\{x|\mu(x)>0\}=\{|\eta(x)>0\}$ is contained in the neighborhood $U\cap\mathrm{R}^{n-1}$ of $f(X-X_0)$ in R^{n-1}. Then, an isotopy $F_t^1:\mathrm{R}^n\to\mathrm{R}^n$ such that F_t^1f is equal to $\overline{H}_{t/2}$ for each $t, 0\le t\le 1$, can be defined as follows:

$$F_t^1(x,s)=\begin{cases}(x,s) & \text{if } x\in(\mathrm{R}^{n-1}-Y)\text{ or}(x\in Y\text{ and } s \text{ is not in}\\ & \text{the open interval } (-\eta(x)\mu(x),(1+\eta(x))\mu(x))),\\ (\frac{s+\eta(x)\mu(x)}{\eta(x)})(\eta(x)+t)-\eta(x)\mu(x), & \text{if } x\in Y \text{ and}\\ & -\eta(x)\mu(x)\le s\le 0,\\ (\frac{s}{1+\eta(x)})(1+\eta(x)-t)+t\mu(x), & \text{if } x\in Y\text{ and} 0\le s\le\\ & (1+\eta(x))\mu(x).\end{cases}$$

Similarly, we can define an isotopy F_t^2 of R^n such that $F_t^2F_1^1f$ is equal to $\overline{H}_{t/2}$ for each $1\le t\le 2$. Now, we define F_t to be: (1) F_{2t}^1 if $0\le t\le 1/2$, and (2) $F_{2t-1}^2F_1^1$ if $1/2\le t\le 1$. Since F_0^1 and F_0^2 are the identity, F_t is a well–defined isotopy that we want to establish. Therefore, the proof is complete.

ADDENDA TO THEOREM 1.1. (1) *For each* t, F_t *is an extension of* $H_t f^{-1}$ *if the induced map* $\tilde{H} : \tilde{X} \to R^n$ *is a* 1*–LCC embedding.*

Such notions as proper homotopy, proper homotopy equivalence, etc., are defined in analogy with the corresponding notions from the ordinary homotopy category. Observe that it is straightforward to show that every proper map is a closed map.

(2) *If* X *is a locally compact,* f *and* g *are proper maps and* X_0 *is a closed subset of* X *such that* $\overline{X - X_0}$ *is compact, then Theorem 1.1 also holds true.*

(3) *Theorem* 1.1 *and the above Addenda* (1) *and* (2) *also hold true if* R^n *is replaced by a manifold which is homeomorphic to an open subset of* R^n.

THEOREM 1.4. *Let* X_0 *be a closed subset of a locally compact metric space* X *with* $2\dim X + 4 \leq n (n \geq 5)$ *and* $H : X \times I \to R^n$ *a proper homotopy* (rel. X_0) *from a proper* 1*–LCC embedding* $f : X \to R^n$ *to another* g. *Then, there is an isotopy* F_t *of* $R^n (t \in I)$ *such that:*
(1) $F_0 = \mathrm{id}$;
(2) $F_1 f = g$;
(3) *If* $\delta : X \to [0, \infty)$ *is a map that* $\delta^{-1}(0) = X_0$, *let* $\mathcal{U} = \{N(H(x \times I); \delta(x)) : x \in X - X_0$ *and* U *the union of all members of* $\mathcal{U}$, *then the isotopy* F *can be chosen such that* $F(x \times I) = \{x\}$ *if* $x \in f(X_0), \cup (R^n - U)$ *or* $F(x \times I)$ *is limited by* $\mathcal{U}$ *if* $x \in U$; *and*
(4) *If the induced map* $\tilde{H}$ *is a proper 1-*LCC *embedding, then* F *can be chosen such that* F_t *is an extension of* $H_t f^{-1} (t \in I)$.

PROOF. Consider the one-point compactification $\mathrm{R}^n \bigcup \{\infty\}$ of R^n as the n-sphere S^n. Let $X_\infty = X \bigcup \{\infty\}$ and $X_{0,\infty} = X_0 \bigcup \{\infty\}$ denote the one-compactification of X and X_0, respectively. Since H is proper, we can extend H to $H' : X_\infty \times I \to S^n$ by defining $H'(\infty \times I) = \infty \in S^n$. If H is a 1-LCC proper embedding, it is straightforward to show that $\tilde{H}' : \tilde{X}_\infty \to S^n$ is 1-LCC at $\infty \in S^n$; hence, $\tilde{H}'$ is a 1-LCC embedding. The rest of the proof is the same as that of Theorem 1.1 for the pair of compact spaces $(X_\infty, X_{0,\infty})$ by noting that S^{n-1} is bicollared in S^n as R^{n-1} is in R^n and that Bryant's unknotting theorem also holds true for 1-LCC embeddings of compacta into S^n. Moreover, since the isotopy F of S^n keeps ∞ fixed,

$F|(S^n - \{\infty\})$ is a desired isotopy of R^n and the proof is complete.

THEOREM 1.5. *Let M be a manifold of dimension $n \geq 5$. Let (X, X_0) be a pair of compact metric spaces, and let $H : X \times I \to \mathrm{Int}\, M$ be a homotopy (rel. X_0) from a $1-LCC$ embedding $f : X \to \mathrm{Int}\, M$ to another g. If $2 \dim X + 4 \leq n$, then there is an isotopy $F_t (t \in I)$ of M such that*

(1) $F_0 = \mathrm{id}_M$;

(2) $F_1 f = g$;

(3) *If $\delta : X \to [0,1]$ is a continuous map such that $\delta^{-1}(0) = X_0$, let $\mathcal{U} = \{N(H(x \times I); \delta(x)) : x \in X - X_0\}$ and U the union of all members of $\mathcal{U}$, then the isotopy F can be chosen such that $F(X \times I) = \{x\}$ if $x \in f(X_0) \bigcup (M - U)$ or $F(X \times I)$ is limited by $\mathcal{U}$ if $x \in U$; and*

(4) *If H induces a 1–LCC embedding $\tilde{H} \to \mathrm{Int} M$, then F can be chosen such that $F(f(x) \times [0,s]) \subset F(f(x) \times [0,t]) \subset H(x \times I)$ for all $x \in X$ and $0 \leq s \leq t \leq 1$ and $F(f(x) \times I) = H(x \times I)$.*

PROOF. Consider a PL–triangulation of $\mathrm{int}\, M$. Since X is compact, without loss of generality, we can assume that M is a compact PL–manifold and we will work with a handle decomposition of M by using $\mathrm{st}(\hat{\sigma}, K'')$, where $\hat{\sigma}$ is the barycenter of a simplex σ of a triangulation K of M and where K" is the second barycentric subdivision of K[**19**, p.81]. Let $M = \mathcal{H}_0 \cup \mathcal{H}_1 \cup \ldots \cup \mathcal{H}_n$, where $\mathcal{H}_i$ is the union of the i^{th}-handles. The proof will be inductive on the indices of handles.

Following the proof of Theorem 1.1, we can assume that $f(X) \cap g(X) = f(X_0)$ and $\tilde{H} : \tilde{X} \to M^n$ is a 1–LCC embedding. Then, since X is compact, we can break the given homotopy into small pieces and assume that H is an ε–homotopy where ε is so small that the $(n + 4)\varepsilon$–neighborhood of each $\mathcal{H}_i$ is the union of a finite family of pairwise–disjoint open balls.

SUBLEMMA. *Let $\alpha : X \to [0, \infty)$ be a map such that $\alpha^{-1}(0) = X_0$. Then, there is a map $\beta : X \to [0, \infty)$ such that*

(a) $\beta^{-1}(0) = X_0$, *and*

(b) *if $x, y \in X - X_0$ and $d(H(x \times I), H(y \times I)) < \beta(x)$, then $d_{\mathcal{H}}(H(x \times I),\ H(y \times I)) < \alpha(x)/4$, where $d_{\mathcal{H}}$ denotes the Hausdorff metric* [**7**, p.205].

PROOF. The proof is straightforward by use of the fact that $\tilde{H}$ is an embedding and that $X - X_0$ is locally compact and σ–compact.

Using the sublemma, we can define inductively a decreasing sequence of functions $\eta_k : X \to [0,1)$, $k = 0, 1, ..., n$, such that

(i) $\eta_k^{-1}(0) = X_0$;

(ii) for each $x \in X - X_0$, $\delta(x) = \eta_{n+1}(x) > \eta_{n+1}(x)/4 > \eta_n(x) > \eta_n(x)/4 >...> n_1(x) > \eta_1(x)/4 > \eta_0(x)$; and

(iii) if $x, y \in X - X_0$ and $d(H(x \times I), H(y \times I)) < \eta_i(x)$, then $d_{\mathcal{H}}(H(x \times I), H(y \times I)) < \eta_{i+1}(x)/4$ for each i.

Let $\mathcal{B}_k = \mathcal{H}_0 \cup \mathcal{H}_1 \cup ... \cup \mathcal{H}_k$. We will inductively construct isotopies $F^0, F^1, ..., F^n$ of M that have the following properties: for each $k = 0, 1, ..., n$:

(A) $F_0^k = \mathrm{id}$

(B) F_t^k fixes $f(X_0) \cup \mathcal{B}_{k-1} \cup (M - N(\mathcal{H}_k; (k+4)\varepsilon))(t \in I)$,

(C) if $\mathcal{U}_k = \{N(H^k(x \times I); \eta_k'(x)/2) | x \in X - X_0^k\}$, where a homotopy H^k, a closed subset X_0^k and a map η_k' with $0 \leq \eta_k' < \eta_k$ (which will be defined later). Let U_k denote the union of all members of $\mathcal{U}_k$; then F_t^k fixes $M - U_k$ and $F^k(x \times I)$ is limited by $\mathcal{U}_k$ if $x \in U_k$; and

(D) for each $t \in I$, $F_t^0 f = H_t^0$ and $F_t^k H_1^{k-1} = H_t^k$ if $k \geq 1$. Then, a wanted isotopy F of M will be defined by

$$F(x,t) = F^i(x, (n+1)t - i) \text{ if } \frac{i}{n} \leq t \leq \frac{i+1}{n+1} \quad \text{where } i = 0, ..., n.$$

First let us define F^0. Let $Z_0 = H^{-1}(\overline{N(\mathcal{H}_0; \varepsilon)}), \tilde{X}_0 = \mathrm{pr}_X(Z_0)$, $X_{0,\lambda} = f^{-1}(N(\mathcal{H}_0; \lambda))$ and $X'_{0,\lambda} = f^{-1}(\overline{N(\mathcal{H}_0; \lambda)})$, for $\lambda > 0$. Observe that $\tilde{X}_0 \cup X_{0,2\varepsilon}$. Define $X_0^0 = (X - X_{0,3\varepsilon}) \cup X_0$ and let $\phi_0 : X \to [0,1]$ be a map such that

(i) ${}_0\phi_0^{-1}(1) = X'_{0,2\varepsilon}$, and

(ii) ${}_0\phi_0^{-1}(0) = X - X_{0,3\varepsilon}$.

Define $\eta_0'(x) = \phi_0(x)\eta_0(x)$, for each $x \in X$ and $H^0 : X \times I \to M$ by $H^0(x,t) = H(x, t\phi_0(x))$. Then $0 \leq \eta_0' \leq \eta_0$, and $\tilde{H}^0$ is a 1–LCC embedding. Now, applying Theorem 1.1 and its addenda to each component of $N(\mathcal{H}_0; 5\varepsilon)$, we can obtain a desired isotopy F^0 (rel. $f(X_0^0) \cup (M - N(\mathcal{H}_0; 4\varepsilon)))$ of M such that $F_t^0 f = H_t^0 (t \in I)$.

Now, let us define F^1. Let $Z_1 = H^{-1}(\overline{N(\mathcal{H}_0 \cup \mathcal{H}_1; \varepsilon)})$, $\tilde{X}_1 = \mathrm{pr}_X(Z_1)$, $X_{1,\lambda} = f^{-1}(N(\mathcal{H}_0 \cup \mathcal{H}_1; \lambda))$ and $X'_{1,\lambda} = f^{-1}(\overline{N(\mathcal{H}_0 \cup \mathcal{H}_1; \lambda)})$, for $\lambda > 0$. Observe that $\tilde{X}_1 \subset X_{1,2\varepsilon}$. Define $X_0^1 = (X - X_{1,4\varepsilon}) \cup X_0 \cup X'_{0,2\varepsilon}$. Let $\phi_1 : X \to [0,1]$ be a map such that

(i) ${}_1\phi_1^{-1}(1) = X'_{1,3\varepsilon} \supset X_{0,3\varepsilon} \supset \phi_0^{-1}((0,1])$ by (ii)${}_0$,

(ii) ${}_2\phi_1^{-1}(0) = X - X_{1,4\varepsilon}$.

Define $\eta_1'(x) = \phi_1(x)\eta_1(x)$ for each $x \in X$ and $H^1 : X \times I \to M$ by $H^1(x,t) = H(x,(1-t)\phi_0(x) + t\phi_1(x))$. Then $0 \leq \eta_1' \leq \eta_1$, and H^1 is a homotopy (rel. X_0^1) from H_1^0 to H_1^1 such that $\tilde{H}^1$ is a 1–LCC embedding. Again, applying Theorem 1.1 and its addenda to each component of $N(\mathcal{H}_1;6\varepsilon)$, we can obtain a desired isotopy F^1 (rel. $F(X_0^1) \cup \mathcal{H}_0 \cup (M - N(\mathcal{H}_1;5\varepsilon)))$ of M such that $F_t^1 H_1^0 = H_t^1 (t \in I)$.

Similarly, we can obtain inductively the isotopies $F^2, F^3, ..., F^n$ satisfying (A)-(D) as we wanted ($\phi_n(x) = 1$ for all $x \in X$.)

Now, we outline the proof to show that F is an ε–isotopy. Given a point $y_0 \in M$ for each $i = 1,2,...,n$, let $y_i = F_1^{i-1}...F_1^1F_1^0(y_0)$. For each $i = 0,1,2,...,n$, it follows from (C) that

(*) if $y_0 \in U$, $F^i(y_i \times I) \subset N(H^i(x_i \times I); 1/2\eta_i(x_i))$, for some $x_i \in X - X_0$ or $F^i(y_i \times I) = \{y_i\}$;

(**) if $y_0 \notin U$, then $F^i(y_i \times I) = \{y_0\}$, for all $i = 0,...,n$.

Let $T_i = F^0(y_0 \times I) \cup ... \cup F^i(y_i \times I)$. We will prove by induction that $T_i \subset N(H(\tilde{x_i} \times I); 2/\eta_{i+1}(\tilde{x}_i),)$, for some $\tilde{x_i} \in X - X_0$, if $y_0 \in U$. By (*), it is clear that the statement holds true for $i = 0$. Now, let us assume that $y_0 \in U$ and that $T_{k-1} \subset N(H(\tilde{x}_{k-1}I); \times\eta_k(\tilde{X_{k-1}})/2)$ for some $\tilde{x}_{k-1} \in X - X_0$. Then, $T_k = T_{k-1}F^k(y_k \times I)$ is contained in $N(H(\eta_{k-1} \times I); \eta_k(\tilde{x_{k-1}})/2) \cup N(H^k(x_k \times I); \eta_k(x_k)/2)$ by (*). Let $\tilde{x}_k \in \{\tilde{x_{k-1}}, x_k\}$ such that $\eta_k(\tilde{x}_k) = \max\{\eta_k(\tilde{x}_{k-1}), \eta_k(x_k)\}$. Since $F^{k-1}(y_{k-1},1) = y_k = F^k(y_k,0)$ belongs to both $N(H(\tilde{x}_{k-1} \times I); \eta_k(\tilde{x}_{k-1})/2)$ and $N(H^k(x_k \times I); \eta_k(x_k)/2)$, we have $d(H(\tilde{x}_{k-1} \times I), H(x_k \times I)) < d(H(\tilde{x}_{k-1} \times I), H^k(x_k \times I)), \eta)k(\tilde{x_k})$; so, $d_{\mathcal{H}}(H(\tilde{x}_{k-1} \times I), H(x_k \times I)) < \eta_{k+1}(\tilde{x_k})/4$ by (iii). Therefore, both $N(H(\tilde{X}_{k-1} \times I); \eta_k(\tilde{x}_{k-1})/4)$ and $N(H(\tilde{x}_k \times I); \eta_k(\tilde{x_k} \times I)/2; \eta_k(\tilde{x_k})/2)$ are contained in $N(H(\tilde{x_k} \times I); \eta_k(\tilde{x_k})/2 + \eta_{k+1}(\tilde{x_k})/4)$. Consequently, by (ii), $T_k \subset N(H(\tilde{x_k} \times I); \eta_{k+1}(\tilde{x_k})/4 + \eta_k(\tilde{x_k})/2) \subset N(H(\tilde{x_k} \times I); \eta_{k+1}(\tilde{x_k})/2)$.

Therefore, $F(y_0 \times I) = T_n \subset N(H(\tilde{x_n} \times I); 1/2\eta_{n+1}(\tilde{x_k}))$ for some $\tilde{x_n} \in X - X_0$. Hence, $F(y_0 \times I)$ is limited by $\mathcal{U}$ and F satisfies (3). Moreover, (1) and (2) follow since $F_0^0 = \text{id}$ (by (A)) and $F_1^n...F_1^0 f = H_1^n = g$, and (4) follows from (**D**). So the proof is complete.

COROLLARY 1.6. *Let X_0 be a closed subset of a locally compact metric space X with $2\dim X + 4 \leq n, n \geq 5$, and M a piecewise–linear n–manifold without boundary endowed with a complete metric d, Let $H : X \times I \to M$ be a proper homotopy (rel,X_0) from a 1–LCC embedding $f : X \to M$ to another g. Then there is an isotopy $F_t (t \in I)$ of M enjoying the properties similar to* (1)-(4) *in Theorem* 1.5.

PROOF. The proof is similar to that of Theorem 1.5. We assume that $\tilde{H}$ is a proper 1–LCC embedding. Fix an $x_0 \in M$ and let $\overline{\lambda} : M \to [0, \infty)$ be a map defined by $\overline{\lambda}(x) = d(x, x_0)$. By use of Theorem 3.5 on p. 298 in [7], it can be shown that $\overline{\lambda}$ is proper. Let λ be a PL–approximation of $\overline{\lambda}$ such that, for each $r > 0$, $M_r = \lambda^{-1}([0, r])$ is a compact PL–submanifold of M whose boundary ∂M_r is PL–bicollared in M. For $0 \leq p \leq q$, define $M_{p,q} = \lambda^{-1}([p, q]), Z_{p.q} = H^{-1}(M_{p,q})$ and $X_{p,q} = \text{pr}_X(Z_{p,q})$; all $M_{p,q}$, $Z_{p,q}$ and $X_{p,q}$ are compact.

Choose an increasing sequence $0 < r_1 < r_2 <...$ such that $\cup\{M_{r_k} | k = 1, 2, ...\} = M$ and that $H(X_{r_k, r_{k+1}} \times I)$ misses both $\partial M_{r_{k-1}}$ and $\partial M_{r_{k+2}}$. Consequently, if $A_s = X_{r_{4s}, r_{4s+2}} (s = 1, 2, ...)$, then $\{H(A_s \times I) | s = 1, 2, ...\}$ is a pairwise–disjoint family. Let $\phi_1 : X \to [0, 1]$ be a map such that $\phi_1^{-1}(1) = \cup\{A_s | s = 1, 2, ...\}$ and $\phi_1^{-1}(0) = X - \cup\{N(A_s; \varepsilon_s) | s = 1, 2, ...\}$, where $\varepsilon_s > 0$ is chosen such that $\{N(A_s; 2\varepsilon_s) \colon s = 1, 2, \cdots\}$ is a pairwise–disjoint family. Define $H_t^1(x) = H(x, t\phi_1(x))$, a homotopy from f to H_1^1. Since $\{N(A_s; \varepsilon) : s = 1, 2, ...\}$ is pairwise–disjoint, from Theorem 1.5, we can obtain an isotopy F^1 of M corresponding to the homotopy H^1 such that $F_1^1 f = H_1^1$.

As in the proof of Theorem 1.5, define $H^2 : X \times I \to M$ by $H^2(x, t) = H(x, (1 - t)\phi_1(x) + t)(\phi_2(x) = 1$ for all $x \in X)$. Then H^2 is a homotopy (rel. $\phi_1^{-1}(1) \cup X_0$) from $H_1^1 = H_0^2$ to $H_1^2 = H_1 = g$ and $H^2((X - \phi_1^{-1}(1)) \times I)$ is a family of relative compact sets of pairwise–disjoint closures. Therefore, from Theorem 1.5 again, we can obtain an isotopy F^2 of such m that $F_1^2 H_1^1 = H_1^2 = g$.

Finally, define $F_t = F_{2t-i+1}^i$ if $\frac{(i-1)}{2} \leq t \leq \frac{i}{2}$, $i = 1, 2$. Then F is a desired isotopy of M if we choose η_1 and η_2 carefully to control the tracks of the isotopies F^1 and F^2 as in the proof of Theorem 1.5. Also, it should be pointed out that the property (4) in Theorem 1.5 is strong enough to carry out a similar property in Corollary 1.6.

2. Approximation theorem. In this section, B^n denotes the PL n-cell $\prod_{i=1}^n [-n, n]_i$. By Lemma A, we can think of R^∞ as $\lim_\to B^n$. If (X, X_0) is a pair of closed subsets of R^∞, let us set $X_k = (X \cap B^k) \cup X_0 (k > 0)$. Let $P_Z : Y \times Z \to Z$ denote the projection. For convenience, we also use M to denote the subspace $M \times 0$ of $M \times \mathrm{R}^\infty$, etc.

LEMMA 2.1. *Let (X, X_0) be a pair of closed subsets of R^∞, and let M be an R^∞-manifold. Given a map $f : (X, X_0) \to (M \times R^\infty, M)$ such that $f|X_0$ is a closed embedding and given an open cover α of $M \times R^\infty$,* *there is a closed embedding* $g : X \to M \times R^\infty$ *such that*

(i) $g|X_0 = f|X_0$,

(ii) g *is* α*-close to* f,

(iii) $p_{\mathrm{R}^\infty} g$ *induces an embedding* $\overline{g} : X/X_0 \to R^\infty$ *such that* $\overline{g}|(X_k/X_0)$ *is a* $1-LCC$ *embedding into* R^{d_k}, *with* d_k*'s subject to the relations* $2d_{k-1}+4 \le d_k$, $d_0 = 6$*; consequently,* $2\dim(X_k/X_0)+4 \le d_{k-1}$.

PROOF. Let $M = \lim_\to M_n$ as in Lemma A. By Lemma B, we can take a subsequence of $\{M_n\}$ and assume that $f(X \cap B^k) \subset M_k \times \mathrm{R}^{d_k}$. For a map $\phi : X \to M \times \mathrm{R}^\infty$, we set $\phi_0 = P_M\phi, \phi_1 = P_{\mathrm{R}_1}\phi, \phi_2 = P_{\mathrm{R}_2}\phi, \cdots, \phi_{m,\cdots,n} = (P_{\mathrm{R}_m \times \cdots \times \mathrm{R}_n})\phi$ and $\phi_{k,\cdots} = (P_{\mathrm{R}_k^\infty})\phi$, where $\mathrm{R}_k^\infty = \lim_\to\{0 \times \mathrm{R}_k \to 0 \times \mathrm{R}_k \times \mathrm{R}_{k+1} \to \cdots\} \subset \mathrm{R}^\infty$; and we will use similar notations for a map $\phi : X \to M \times \mathrm{R}^P \times I^\infty_{p+1}$.

The proof will be similar to that of Lemma 3.1 in [**16**]. We will only outline it. Let $\{\alpha_s | s = 0, 1, 2, \cdots\}$ be a sequence of open covers of $M \times \mathrm{R}^\infty$ such that

(i) $\alpha_0 = \alpha$,

(ii) $\mathrm{St}(\alpha_i, \alpha_i) < \alpha_{i-1}$, $(i \ge 1)$.

We will construct inductively a sequence of maps $\{g^{(n)} | n = 0, 1, 2, \cdots\}$, where $g^{(n)} : X \to M \times \mathrm{R}^\infty$, such that

(1) $g^{(n)}|X_{n-1} = g^{(n-1)}|X_{n-1}$;

(2) $g^{(n)}$ and $g^{(n-1)}$ are $\mathrm{St}(\alpha_{2n}, \alpha_{2n})$-close;

(3) $g^{(n)}(X_n - X_{n-1}) \cap (M_{n-1} \times \mathrm{R}^{d_{n-1}}) = \emptyset$; and

(4) $g^{(n)}|X_n$ is an embedding with $g^{(n)}(X_n) \subset M_n \times \mathrm{R}^{d_n}$.

Then, $g = \lim g^{(n)}$ will be a desired approximation; therefore the proof will be complete.

A. DEFINITION OF $g^{(0)}$. Let $g^{(0)} = f$.

B. DEFINITION OF $g^{(1)}$. First, we modify $g^{(0)}$ as follows to obtain a map $\check{g}^{(1)} : X \to M \times \mathrm{R}^\infty$ with the following properties:

(a) $\check{g}^{(1)}$ is α_2-close to $g^{(0)}$;

(b) $\check{g}^{(1)}(X_1 - X_0) \cap M = \emptyset$; and

(c) $\check{g}^{(1)}|X_0 = g^{(0)}|X_0$.

Let $\theta_1 : \mathrm{R}^\infty \to I^\infty$ be a homeomorphism with $\theta_1(0) = 0$. Define $\sigma_1 = \mathrm{id}_M \times \theta_1$ and $h^{(1)} = \sigma_1 g^{(0)}$. Since M is paracompact, by imitating the proof of Lemma 3.1 in [**16**], we can construct a map $\hat{h}_1^{(1)}$ approximating $h_1^{(1)}$ (rel. X_0) such that if we define $\check{h}^{(1)}$ to be $(h_0^{(1)}, \hat{h}_1^{(1)}, h_{2,\cdots}^{(1)}) : X \to M \times I^\infty$, then

(a') $\check{h}^{(1)}$ is $\sigma_1(\alpha_2)$-close to $h^{(1)}$, where $\sigma_1(\alpha_2)$ is the open cover $\{\sigma_1(V) : V \in \alpha_2\}$ of $M \times I^\infty$;
(b') $\check{h}^{(1)}(X_1 - X_0) \cap M = \emptyset$; and
(c') $\check{h}^{(1)}|X_0 = h^{(1)}|X_0$.
Define $\check{g}^{(1)} = \sigma_1^{-1}\check{h}^{(1)}$. Then, it is straightforward to verify that $\check{g}^{(1)}$ satisfies (a), (b), and (c), since $\sigma_1(x, 0) = (x, 0)$ for all $x \in M$

Next, since $\check{g}^{(1)}(X_0) \subset M$, we can assume that $\check{g}^{(1)}_{1,\cdots}(X_1/X_0)$ is a compact subset of R^{d_1} for some $d_1 \geq 2d_0 + 4$ with $d_0 = 6$ (we will use the same notations to denote the "induced" maps defined from the quotient spaces into R^∞.) Now, since M_1 is compact, for each $z \in \check{g}^{(1)}_{1,\cdots,d_1}(X_1/X_0)$, there is an open neighborhood V_z of z in R^{d_1} and an open cover $\mathcal{W}_{1,z}$ of M_1 such that the family $\{W \times V_z | W \in \mathcal{W}_{1,z}\} < \alpha_2$. Then, it follows from Lemma 1.3 that there is a 1-LCC embedding $\overline{g}^{(1)} : X_1/X_0 \to \mathrm{R}^{d_1}$ that is $\{V_z\}$-homotopic (rel.$\{X_0\}$) to $\check{g}^{(1)}_{1,\cdots,d_1}|(X_1/X_0)$. Consequently, $\check{g}^{(1)}|X_1$ is α_2–homotopic (rel.X_0) to $(\check{g}^{(1)}_0, \overline{g}^{(1)}, \check{g}^{(1)}_{d_1+1,\cdots})X_1$ since $\check{g}^{(1)}_{d_1+1}(X_1) = \{0\} \subset \mathrm{R}^\infty_{d_1+1}$.

Finally, by use of a homotopy extension and the paracompactness of X, we can show that $\check{g}^{(1)}$ is α_2-homotopic (rel.X_0) to a map $g^{(1)}$ that we wanted. Moreover, observe that $g^{(1)}|X_1 = (\check{g}^{(1)}_0, \overline{g}^{(1)}, \check{g}^{(1)}_{d_1+1,\cdots})|X_1$ is an embedding of X_1 into $M_1 \times \mathrm{R}^{d_1}$ and $g^{(0)}$ and $g^{(1)}$ are $\mathrm{St}(\alpha_2, \alpha_2)$-close.

C. DEFINITION OF $g^{(k)}$. Similarly, we will construct $g^{(k)}$ after a satisfactory $g^{(k-1)}$ has been defined as follows. Recall that $g^{(k-1)}|X_{k-1}$ is an embedding whose image is contained in $M_{k-1} \times \mathrm{R}^{d_{k-1}}$. Let $\theta_k : \mathrm{R}^\infty_{d_{k-1}+1} \to I^\infty_{d_{k-1}+1}$ be a homeomorphism with $\theta_k(0) = 0$. Define $\sigma_k = \mathrm{id}_{M \times \mathrm{R}^{d_{k-1}}} \times \theta_k$ and $h^{(k)} = \sigma_k g^{(k-1)}$. Again, we modify $h^{(k)}_{d_{k-1}+1}$ to obtain $\hat{h}^{(k)}_{d_{k-1}+1}$ such that if we define $\hat{h}^{(k)} = (h^{(k)}_{0,\cdots,d_{k-1}}, \hat{h}^{(k)}_{d_{k-1}+1}, h^{(k)}_{d_{k-1}+2,})$, then:
(a') $\check{h}^{(k)}$ is $\sigma_k(\alpha_{2k})$-close to $h^{(k)}$ where $\sigma_k(\alpha_{2k}) = \{\sigma_k(V) : V \in \alpha_{2k}\}$;
(b') $\check{h}^{(k)}(X_k - X_{k-1}) \cap (M \times \mathrm{R}^{d_{k-1}}) = \emptyset$; and
(c') $\check{h}^{(k)}|X_{k-1} = h^{(k)}|X_{k-1}$.
Now, define $\check{g}^{(k)} = \sigma_k^{-1}\check{h}^{(k)}$. Since $\sigma_k(x, 0) = (x, 0)$, for all $x \in M \times \mathrm{R}^{d_{k-1}}$, it follows that
(a) $\check{g}^{(k)}$ is σ_{2k}-close to $g^{(k-1)}$;
(b) $\check{g}^{(k)}(X_k - X_{k-1}) \cap (M \times \mathrm{R}^{d_{k-1}}) = \emptyset$; and
(c) $\check{g}^{(k)}|X_{k-1} = g^{(k-1)}|X_{k-1}$.
Then, as in step B, we can assume that the "induced" map $\check{g}^{(k)}_{1,\cdots}$:

$X/X_0 \to \mathrm{R}^\infty$ carries X_k/X_0 into R^{d_k} for some $d_k \geq 2d_{k-1}+4$. Again, from Lemma 1.3, it follows that $\check{g}^{(k)}_{1,\cdots,d_k}|(X_k/X_0)$ is homotopic (rel.X_{k-1}/X_0) to a 1-LCC embedding $\overline{g}^{(k)} : X_k/X_0 \to \mathrm{R}^{d_k}$ such that

(*) $\overline{g}^{(k)}((X_k/X_0)-(X_{k-1}/X_0))\cap\mathrm{R}^{d_{k-1}} = \emptyset$ (Recall that $\dim(X_k/X_0)$ $+d_{k-1} \leq k+d_{k-1} < 2d_{k-1} < d_k - 2$ since $\dim(X_k/X_0) \leq k$ and since $k < d_{k-1}$ by induction);

(**) $\check{g}^{(k)}|X_k$ is α_{2k}-homotopic (rel.X_{k-1}) to the embedding $(\check{g}^{(k)}_0, \overline{g}^{(k)}, \check{g}^{(k)}_{d_k+1,\cdots}) : X_k \to M_k \times \mathrm{R}^{d_k}$.

Now, we can extend the homotopy in (**) to an α_{2k}-homotopy from $\check{g}^{(k)}$ to a desired map $g^{(k)}$ which satisfies all properties (1)-(4).

Finally, the function $g = \lim g^{(k)}$ will be a desired approximation. In fact, g is well-defined by (1). Moreover, since each $x \in X$ belongs to X_k for some k, it follows that $g^{(p)}(x) = g^{(k)}(x)$ for all $p > k$ by (1). Therefore, by use of (2) and the construction of $\{\alpha_s | s = 0,1,2,\cdots\}$, we can prove by induction that $g^{(k)}$ is $\alpha_{2(k-j)}$-close to $g^{(k-j)}$ for each $j (0 \leq j \leq k)$. Hence, $g(x) = g^{(k)}(x)$ is α-close to $g^{(0}(x) = f(x)$. Also, by use of (3) and (4), we can show that g is a closed embedding as in Lemma 3.1 [**16**]. Finally, we can inductively define the sequence $\{d_k\}$ subject to (iii) as required.

Let us introduce some notations used in the following lemma. For a finite set A of positive integers, let R_A denote the product $\prod\{Z_i : i = 1,2,\cdots\}$, where $Z_i = \mathrm{R}$ if $i \in A$ and $Z_i = \{0\}$ if $i \notin A$. R_A is a subspace of R^∞. If A and B are two disjoint finite sets of positive integers, let $\mathrm{R}_A \times \mathrm{R}_B$ denote $\mathrm{R}_{A\cup B}$. Define $\mathrm{R}^{d_k} = \mathrm{R}_{\{1,2,\cdots,d_k\}}$ where d_k is a positive integer.

LEMMA 2.2. *Let $\phi : X \to R^\infty$ be a closed embedding such that* (use the notation in Lemma 2.1 and think of X_n as X_n/X_0 in Lemma 2.1):

(1) $\phi(X_{n+1} - X_n) \cap \mathrm{R}^{d_n} = \emptyset$;

(2) $\phi|X_n : X_n \to \mathrm{R}^{d_n}$ *is a* 1-LCC *embedding*;

(3) $2\dim X_n + 4 \leq d_{n-1}$, *and* $d_0 \geq 6$; *and*

(4) $2d_n + 4 \leq d_{n+1} (n \geq 0)$.

Then, ϕ is an R^∞-deficient embedding.

PROOF. Set ${}^1\mathrm{R}^{d_0} = \mathrm{R}^{d_0} = \mathrm{R}_{\{1,\cdots,d_0\}}$ and ${}^2\mathrm{R}^{d_1-d_0} = \mathrm{R}_{\{d_0+1,\cdots,d_1\}}$. For $k \geq 1$, define inductively,

$$\begin{cases} {}^1\mathrm{R}^{d_k} = {}^1\mathrm{R}^{d_{k-1}} \times \mathrm{R}_{\{d_k+1,\cdots,2d_k-d_{k-1}\}} \\ {}^2\mathrm{R}^{d_{k+1}-d_k} = {}^2\mathrm{R}^{d_k-d_{k-1}} \times \mathrm{R}_{\{2d_k-d_{k-1}+1,\cdots,d_{k+1}\}} \end{cases} .$$

It follows that, for each $k = 1, 2, \cdots$,

(A) $\mathrm{R}^{d_{k+1}} = {}^1\mathrm{R}^{d_k} \times {}^2\mathrm{R}^{d_{k+1}-d_k}$,

(B) $i^1_{k-1} : {}^1\mathrm{R}^{d_{k-1}} \subset {}^1\mathrm{R}^{d_k}$, and $\dim {}^1\mathrm{R}^{d_k} = d_k$,

(C) $i^2_{k-1} : {}^2\mathrm{R}^{d_k-d_{k-1}} \subset {}^2\mathrm{R}^{d_{k+1}-d_k}$, and $\dim {}^2\mathrm{R}^{d_{k+1}-d_k} = d_{k+1} - d_k$,

(D) $\mathrm{R}^\infty = \lim_\rightarrow \mathrm{R}^{d_k} \approx {}^{12}\mathrm{R}^\infty$ where ${}^1\mathrm{R}^\infty = \lim \rightarrow^1 \mathrm{R}^{d_k}$ and ${}^2\mathrm{R}^\infty = \lim^2 \mathrm{R}^{d_{k+1}-d_k}$.

For convenience, we will identify R^∞ with ${}^1\mathrm{R}^\infty \times {}^2\mathrm{R}^\infty$, ${}^1\mathrm{R}^\infty = \lim_\rightarrow {}^1\mathrm{R}^{d_k}$ with ${}^1\mathrm{R}^\infty \times 0 \subset \mathrm{R}^\infty$. Then, we have the commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, hook] \arrow[d, "\phi|"] & X_2 \arrow[r, hook] \arrow[d, "\phi|"] & X_3 \arrow[r, hook] \arrow[d, "\phi|"] & \cdots \\ {}^1\mathrm{R}^{d_0} \times {}^2\mathrm{R}^{d_1-d_0} \arrow[d, equal] & {}^1\mathrm{R}^{d_1} \times {}^2\mathrm{R}^{d_2-d_1} \arrow[d, equal] & {}^1\mathrm{R}^{d_2} \times {}^2\mathrm{R}^{d_3-d_2} \arrow[d, equal] & \\ \mathrm{R}^{d_1} \arrow[r, "i^1_0 \times i^2_0"] & \mathrm{R}^{d_2} \arrow[r, "i^1_1 \times i^2_1"] & \mathrm{R}^{d_3} \arrow[r, "i^1_2 \times i^2_2"] & \cdots . \end{tikzcd}$$

Let us consider the composition $p\phi : X \to {}^1\mathrm{R}^\infty$, where $p : \mathrm{R}^\infty \to {}^1\mathrm{R}^\infty \subset \mathrm{R}^\infty$ is the projection. From the proof of Lemma 2.1 and the condition (3), there is an embedding $\psi : X \to {}^1\mathrm{R}^\infty$ approximating $p\phi$ such that

(i) $\psi|X_k : X_k \to^1 \mathrm{R}^{d_{k-1}}$ is a 1-LCC embedding;

(ii) $\psi(X_{k+1} - X_k) \cap^1 \mathrm{R}^{d_{k-1}} = \emptyset$; and

(iii) there is a homotopy $H : \phi \simeq \psi$ in R^∞ such that $H(X_k \times I) \subset \mathrm{R}^{d_k}$ and $H((X_{k+1} - X_k) \times I) \cap \mathrm{R}^{d_k} = \emptyset$.

To complete the proof, we consider two cases.

Case 1. $\phi(X) \cap \phi(X) = \phi$. Again, by Lemma 2.1, we can assume additionally that $H|X_k \times I$ is a 1-LCC embedding into R^{d_k}. We will construct inductively a sequence of homeomorphisms $h_n : \mathrm{R}^{d_n} \to \mathrm{R}^{d_n} (n = 1, 2, 3, \cdots)$ such that $h_n\phi|X_n = \psi|X_n$ and $h_{n+1}|\mathrm{R}^{d_n} = h_n$. Then, the homeomorphism h of ${}^1\mathrm{R}^\infty \times^2 \mathrm{R}^\infty$ defined by $h(x) = h_n(x)$ if $x \in \mathrm{R}^{d_n}$ will satisfy the relation $h\phi = \psi$, and the proof of Case 1 will be complete.

First, from Addendum (2) to Theorem 1.1, it follows that there is an isotopy h_t^1 of R^{d_1} such that h_t^1 is an extension of $H_t^1 \equiv H_t\phi^{-1}|\phi(X_1)$ and $h_t^1 = \mathrm{id}$ on $\mathrm{R}^{d_1} - U$, where U is a relatively compact neighborhood of $H^1(\phi(X_1) \times I) = H(X_1 \times I)$ in R^{d_1}. Let us define $h_1 = H_1^1$. Then $h_1\phi(x) = H_1^1\phi(x) = H(\phi^{-1}\phi(x), 1) = H(x, 1) = \psi(x)$ if $x \in X_1$.

Second, to construct h_2, we observe that, since $H_t\phi^{-1}$ and h_t^1 agree on $\phi(X_1) \subset \mathrm{R}^{d_1}$, they define a homotopy $H_t^2 : \phi(X) \cup \mathrm{R}^{d_1} \to \mathrm{R}^\infty (t \in I)$ such that $H_t^2| : (\phi(X_2) \cup \mathrm{R}^{d_1}) \to \mathrm{R}^{d_2}$ fixes the complement of a compact set. So, $H^2(\phi(X_2) \times I) = H(X_2 \times I)$ is 1-LCC in R^{d_2}. Again, from Addendum (2) to Theorem 1.1, there is an isotopy h^2 of R^{d_2} such that (a) h_t^2 is an extension of $H_t^2|\phi(X_2)$; (b) consequently, $h_1^2\phi(x) = \psi(x)$ if $x \in X_2$; (c) $h_t^2 = \mathrm{id}$ on the complement of a compact neighborhood of $H(X_2 \times I)$ in R^{d_2}; and (d) $h_t^2|\mathrm{R}^{d_1} = h_1^1$. Define $h_2 = h_1^2$.

Inductively, we can similarly define h_n after h_{n-1} has been constructed as the last stage h_1^{n-1} of an isotopy h_t^{n-1} of $\mathrm{R}^{d_{n-1}}$ which satisfies the similar properties (a)-(d) above. Then the proof of Case 1 is complete.

Case 2. $\phi(X) \cap \psi(X) \neq \emptyset$. From the dimension condition (3) and from the 1-LCC properties (2) of ϕ and (i) of ψ, it follows that $\phi(X_k) \cup \psi(X_k)$ is 1-LCC in R^{d_k} for each k by Lemma 1.2. Hence, if we follow the proof of Lemma 2.1, we can define inductively $\theta_k : X_k \to \mathrm{R}^{d_k}$ such that (α) if θ is defined by $\theta(x) = \theta_k(x)$ for $x \in X_k$, then θ satisfies properties similar to (1) and (2) of ϕ; (B) θ is α-close to ϕ, and (γ) $\theta(X) \cap [\phi(X) \cup \psi(X)] = \emptyset$. By use of the convex structure of R^∞, we can assume that θ is α-homotopic to ϕ, this general case will follow from using Case 1 twice.

ADDENDUM TO LEMMA 2.2. *By use of the relative approximation Lemma* 1.3 *and the relative version of Theorem* 1.1, *we can choose the above homeomorphism* $h : {}^1\mathrm{R}^\infty \times {}^2\mathrm{R}^\infty \to^2 R^\infty$ such that

(1) $h(0) = 0$, *and*

(2) $h\phi(X) \subset^1 \mathrm{R}^\infty$.

THEOREM 2.3. *Let* (X, X_0) *be a pair of closed subsets of* R^∞ *and let* M *be an* R^∞*-manifold. Given a map* $f : X \to M$ *such that* $f|X_0$ *is an* R^∞*-deficient embedding and given an open cover* α *of* M, *there is an* R^∞*-deficient embedding* $g : X \to M$ *such that*

(1) $g|X_0 = f|X_0$,

(2) g *is* α*-close to* f.

PROOF. We will prove that the closed embedding g constructed in Lemma 2.1 is R^∞-deficient. We identify M with $M \times \mathrm{R}^\infty$ and assume $g(X_0) \subset M \times 0$. Observe that the map $p_{\mathrm{R}^\infty} g$ induces an embedding $\phi = \bar{g} : X/X_0 \to \mathrm{R}^\infty$ that satisfies Lemma 2.2 and $\phi(X_0) = \{0\}$. Let h be a homeomorphism of R^∞ from Addendum to Lemma 2.2 for ϕ, and define $\tilde{h} = \mathrm{id}_M \times h$. Then, $\tilde{h}$ is a homeomorphism of $M \times \mathrm{R}^\infty = M \times^1 \mathrm{R}^\infty \times^2 \mathrm{R}^\infty$ such that $\tilde{h}g(X) \subset M \times^1 \mathrm{R}^\infty \times \{0\}$; hence, g is an R^∞-deficient embedding and the proof is complete.

LEMMA 2.4. *Let (X, X_0) be a pair of closed subsets of I^∞, $F : X \to N \equiv M \times I^\infty$ such that $f|X_0$ is an I^∞-deficient embedding with $f(X_0) \subset M \times 0$. Given an open cover α of N and W an open neighborhood of $f(X)$ in N, there is a closed map g such that*

(1) *$g|X_0 = f|X_0$,*

(2) *g is α-close to f, and*

(3) *$g(X) \subset W$.*

PROOF. The proof is the same as that of Lemma 3.1 in [**16**] with the following modification. Recall that $(I^\infty, 0) \approx (\mathrm{R}^\infty, 0)$.

(1). Being an open subset of the I^∞-manifold $M \times 0, W \cap (M \times 0)$ is paracompact. Therefore, the proof of Lemma 2.7 in [**16**] shows that there is a nbd-finite [**7**] open cover $\{V_a | a \in A\}$ of $W \cap (M \times 0)$ and a sequence of maps $\varepsilon^n : W \cap (M \times 0) \to (0,1), n = 1, 2, \cdots$, such that $\{(V_a; \varepsilon^1(x), \varepsilon^2(x), \cdots) | a \in A, x \in V_a\}$ is a refinement of $\{\omega \cap W | \omega \in \alpha\}$, where $(V_a; \varepsilon^1(x), \varepsilon^2(x), \cdots)$ denotes $(V_a \times [0, \varepsilon^1(x)) \times [0, \varepsilon^2(x)) \times \cdots) \cap (V_a \times I^\infty)$.

(2). We only need the property (ii) of μ_m on p. 297 in [**16**] to show that $g^{(m)}(X_m - X_{m-1}) \cap (M \times I^{m-1} \times 0) = \emptyset$ since μ_m is positive on $W \cap (M \times I^{m-1} \times 0)$ (refer to the proof of Case 2 of (i) on p. 295 in [**16**].)

3. Unknotting theorem in R^∞-manifolds.

LEMMA 3.1. *Let X be a closed subset of an R^∞-manifold M and let $f : X \to M$ be a closed embedding. If f is homotopic to the inclusion $X \subset M$, say by $\tilde{H}$, then, given an open neighborhood W of $H(X \times I)$ in $M \times R^\infty (H = j\tilde{H}$ where $j : M \to M \times R^\infty$ is defined by $j(x) = (x, 0))$, there is a homeomorphism F of $M \times R^\infty$ such that*

(i) $F = \mathrm{id}$ *on the complement of W:*

(ii) *$F(x, 0) = (f(x), 0)$ for all $x \in X$; and*

(iii) *if H is limited by an open cover α of $M \times R^\infty$, then F can be chosen to be $St^4(\alpha)$-close to id.*

PROOF. Case 1. $X \cap f(X) = \emptyset$. By Lemma 2.4 and Theorem 2.3, we can assume that H is an R^∞-deficient embedding into $M \times \mathrm{R}^\infty$ and it is $\mathrm{St}(\alpha)$-limited since $H|X \times \{0,1\}$ is an R^∞-deficient embedding. Thus, without loss of generality, we can assume $H(X \times I) \subset M \times 0$. Let $M = \lim_\rightarrow M_k^{m_k}$ as in Lemma A. Set $Z_n = H^{-1}(M_n \times 0)$. Then Z_n is a compact subset of $X \times I$. Hence, $X_n = p_X(Z_n)$ is compact and we can assume by Lemma B that $H(X_n \times I) \subset M_{n+1} \times 0$. Let $\{d_n\}$ be an increasing sequence of positive integers such that
(*) $m_{n+1} \leq m_n + d_n$, and
(**) $4(m_n + d_n) + 5 \leq m_{n+1} + d_{n+1}$.
We will construct a tower of compact-manifold subsets $\{N_n : n = 1, 2, \cdots\}$ of $M \times \mathrm{R}^\infty$ such that
(1) $\mathrm{Int} N_n$ is homeomorphic to an open subset of $\mathrm{R}^{2(m_n+d_n)+2}$;
(2) $H(X_n \times I)$ is a 1-LCC subset of $\mathrm{Int}\, N_n$ and $H(X \times I) \cap N_n = H(X_n \times I)$;
(3) N_{n-1} is a 1-LCC subset of $\mathrm{Int} N_n$; and
(4) $\cup\{N_n : n = 1, 2, \cdots\} = M \times \mathrm{R}^\infty$.
Let B_k^d denote $\prod_{i=1}^d [-k, k]_i$. To fix ideas let $n = 1$ and let $Y = H(X_1 \times I) \cup (M_1 \times B_1^{d_1})$. Then Y is a 1-LCC subset of $\mathrm{Int}(M_2 \times B_2^{d_2})$ by Lemma 1.2. Since $\mathrm{Int}(M_2 \times B_2^{d_2})$ is an absolute neighborhood retract, there is a compact PL-manifold neighborhood U of a 1-LCC copy of Y in $\mathrm{R}^{2(m_1+d_1)+2}$[**3**, p.49] and a map $g : U \to \mathrm{Int}(M_2 \times B_2^{d_2})$ such that $g(x) = x$ for each $x \in Y$. Moreover, since $H(X \times I) \cap \mathrm{Int}(M_2 \times B_2^{d_2}) \subset M_2 \times 0$ is 1-LCC in $\mathrm{Int}(M_2 \times B_2^{d_2})$ and since

$$\begin{aligned} m_2 + 2(m_1 + d_1) + 3 &\leq 3(m_1 + d_1) + 3 \quad \text{by } (*) \\ &< 2(2(m_1 + d_1) + 2) + 1 \\ &\leq m_2 + d_2 \quad \text{by } (**), \end{aligned}$$

We can assume by Lemma 1.3 that g is a 1-LCC embedding such that $g(U) \cap H((X - X_1) \times I) = \emptyset$. Define $N_1 = g(U)$. Then, N_1 satisfies the properties (1)-(4).

Similarly, to define N_n, we use $Y = H(X_n \times I) \cup (M_n \times B_n^{d_n})$ which contains N_{n-1}. Then, we can verify that the sequence $\{N_n : n = 1, 2, \cdots\}$ enjoys the properties (1)-(4). Infact, N_{n-1} is a 1-LCC subset of $\mathrm{Int} N_n$ since $N_{n-1} \subset Y$ and $\dim Y \leq \dim N_n - 3$.

For each integer $n \geq 1$, let α_n denote the open cover $\{V \cap N_n | V \in \alpha\}$ of N_n. Now, since $\mathrm{Int}N_1$ is homeomorphic to an open subset of $\mathrm{R}^{2(m_1+d_1)+2}$, it follows from Addendum (3) to Theorem 1.1 that there is a $\mathrm{St}(\alpha_1)$-isotopy $h_t^1 : N_1 \to N_1$ which is the identity on $N_1 - W$ and extends $H_t|X_1$ for each $t \in I$. Then, by (2), (3) and Lemma 1.2, it follows that $N_1 \cup H(X_2 \times I)$ is a 1-LCC subset of $\mathrm{Int}N_2$. On the other hand, the map

$$h^1 \cup (H|X_2 \times I) : (N_1 \cup X_2) \times I \to N_2$$

is a well defined $\mathrm{St}(\alpha_2)$-homotopy, and

$$\begin{aligned} 2\dim(N_1 \cup X_2) + 4 &= 2\max\{\dim N_1, \dim X_2\} + 4 \\ &\leq 2\max\{2(m_1 + d_1) + 2, m_2\} + 4 \\ &\leq 4(m_1 + d_1) + 8 \quad \text{by } (*) \\ &\leq m_2 + d_2 + 3 \quad \text{by } (**) \\ &< 2(m_2 + d_2) + 2 \quad \text{by } (1) \\ &= \dim N_2. \end{aligned}$$

Hence, from Addendum (3) to Theorem 1.1, there is a $\mathrm{St}(\alpha_2)$-isotopy $h_t^2 : N_2 \to N_2$ which is the identity on $N_2 - W$ and which extends $H|(X_2 \times I)$ such that $h_1^2|N_1 = h_1^1$. And so, we can obtain inductively a sequence of $\mathrm{St}(\alpha_n)$-isotopies $h_t^n : N_n \to N_n$ such that

(1) $h_t^n|(N_n - W) = \mathrm{id}$;
(2) $h^n|(X_n \times I) = H|(X_n \times I)$; and
(3) $h_1^n|N_{n-1} = h_1^{n-1}$.

Finally, we define $F(x) = h_1^n(x)$ if $x \in N_n$. Then, F is a well-defined homeomorphism and it has all three desired properties. (F is $\mathrm{St}(\alpha)$-close to the identity of $M \times \mathrm{R}^\infty$.)

Case 2. (General case). We think of R^∞ as $I^\infty \times J^\infty$. It follows from Theorem 2.3 and local convexity of $M \times I^\infty$ as an open subset of R^∞ [**12**] that there is a closed embedding $g : X \to M \times I^\infty$ such that (i) $g(X) \cap (M \times 0) = \emptyset$ and (ii) g is α-homotopic to the inclusion; hence, (iii) g is $\mathrm{St}(\alpha)$-homotopic to f. Then, observing the definition of F in Case 1, we can construct homeomorphisms F_1 and F_2 of $M \times \mathrm{R}^\infty$ fixing $(M \times \mathrm{R}^\infty) - W$ such that

(*) F_1 is $\mathrm{St}(\alpha)$-close to the identity and $F_1(x, 0) = (g(x), 0)$ for all $x \in X$,

(**) F_2 is $\mathrm{St}^2(\alpha)$-close to the identity and $F_2(g(x), 0) = (f(x), 0)$ for all $x \in X$.

Define $F = F_2 F_1$. T then F is a homeomorphism of $M \times \mathrm{R}^\infty$ satisfying (i), (ii) and (iii) as we desired; therefore, the lemma follows

Let X_0 be a closed subset of X and assume that H is stationary on X_0. By Lemma 2.1, we can assume that $H(X - X_0) \times I) \cap (X_0 \times 0) = \emptyset$. If we choose the isotopies $h^1, h^2, \cdots$ in the proof of Case 1 with further restriction by use of Addendum (3) to Theorem 1.1 corresponding to open sets $W_0 \cap N_1, W_0 \cap N_2, \cdots$, where W_0 is a prechosen open neighborhood of $H((X - X_0) \times I)$ in $(M \times \mathrm{R}^\infty) - (X_0 \times 0)$, then we obtain the following addendum.

ADDENDUM TO LEMMA 3.1. *If X_0 is a closed subset of X and if the homotopy $\tilde{H}$ is stationary on X_0, then the homeomorphism F can be additionally chosen to be the identity on $(M \times R^\infty) - W_0$, especially on X_0.*

LEMMA 3.2. *If X is an R^∞-deficient subset of an R^∞-manifold M, then there is a homeomorphism $\phi : M \to M \times R^\infty$ such that $\phi(x) = (x, 0)$ if $x \in X$.*

PROOF. Let $h : M \to M \times \mathrm{R}^\infty = M \times I^\infty \times \mathrm{R}^\infty$ be a homeomorphism such that $h(x) \subset M \times 0$. Let $\lambda : I^\infty \to I\infty \times \mathrm{R}^\infty$ be a homeomorphism with $\lambda(0) = (0, 0)$. Now, define ϕ to be $(h^{-1} \times \mathrm{id}_{\mathrm{R}^\infty}) \circ (\mathrm{id}_M \times \lambda \times \mathrm{id}_{\mathrm{R}^\infty}) \circ h$. Then, $\phi(x) = (x, 0)$ if $x \in X$.

THEOREM 3.3. (UNKNOTTING THEOREM). *Let X be an R^∞-deficient subset of an R^∞-manifold M, and let $f : X \to M$ be an R^∞-deficient embedding. If f is homotopic to the inclusion $X \subset M$, say by H, then there is a homeomorphism F of M such that $F(x) = f(x)$ for all $x \in X$.*

PROOF. Let $\phi, \psi : M \to M \times \mathrm{R}^\infty$ be homeomorphisms from Lemma 3.2 such that

(i) $\phi(x) = (x, 0)$ if $x \in X$

(ii) $\psi(y) = (y, 0)$ if $y \in f(X)$.

Let $j : M \to M \times \mathrm{R}^\infty$ be a map defined by $j(x) = (x, 0)$. Then $\overline{H} : X \times I \to M \times \mathrm{R}^\infty$ defined by $\overline{H}_t(x) = jH(x, t) = (H(x, t), 0)$ is a homotopy from ji to jf. Moreover, if $x \in X_0$, then $\overline{H}(x, t) = jH(x, t) = (x, o)$. Therefore, from Lemma 3.1, there is a homeomorphism $\tilde{F}$ of $M \times \mathrm{R}^\infty$ such that (1) $\tilde{F}(x, 0) = (x, 0)$ if $x \in X_0$ and (2)

$\tilde{F}(x,0) = (\tilde{F}ji)(x) = jf(x) = (f(x),0)$ for each $x \in X$. Now, define $F = \psi^{-1}\tilde{F}\phi$. Then, for each $x \in X, F(x) = \psi^{-1}\tilde{F}\phi(x) = \psi^{-1}\tilde{F}(x,0) = \psi^{-1}(f(x),0) = f(x)$. So, F is a homeomorphism that we desired and the proof is complete.

REMARK. The unknotting theorem for Z-embeddings in R^∞-manifolds does not hold true (see [**16**]).

4. Collar theorem. We begin the section with a lemma that is essential in the proof of the collar theorem for R^∞-deficient embeddings. Recall that $J = I = [0,1]$.

LEMMA 4.1. *There is a homeomorphism* $h : (I^\infty \times [0,1], I^\infty \times 0) \to (I^\infty \times J^\infty, I^\infty \times 0)$ *such that* $h(x,0) = (x,0)$ *for all* $x \in I^\infty$.

PROOF. Let us denote $I^n = I_1 \times \cdots \times I_n \times \{0\}$ and $J^m = J_1 \times \cdots \times J_m \times \{0\}$. We will construct a sequence of embeddings $\{h_k | k = 0,1,2,\cdots\}$ with the following properties:

(1) $h_k : I^{n_k} \times [0,1] \to I^{n_{k+1}} \times J^{m_{k+1}}$,
(2) $h_{k+1}|I^{n_k} \times [0,1] = h_k$,
(3) $h_{k+1}(I^{n_{k+1}} \times [0,1]) \supset I^{n_k} \times J^{m_k}$ for $k \geq 0$, and
(4) $h_k(x,0) = (x,0)$ for all $x \in I^{n_k}$.

Define h by $h(x) = h_k(x)$ if $x \in I^{n_k} \times [0,1]$. Then it will be a well-defined homeomorphism that we desired.

1. *Construction of* $h_0(n_0 = 1, m_0 = 0, n_1 = 6, m_1 = 9)$. Let $h_0 : I^1 \times [0,1] \xrightarrow{\cong} > I^1 \times J^1 \hookrightarrow I^6 \times J^9$ be the composition of the trivial PL-homeomorphism and the inclusion.

2. *Construction of* $h_1(n_1 = 6, m_1 = 9, n_2 = 32, m_2 = 35)$. Let $A_1 = (I^6 \times \{0\}) \cup (I^1 \times J^1)$ be a subpolyhedron of $I^6 \times J^9$, and let $g_1 : A_1 \to \partial(I^6 \times [0,1])$ be the extension of $h_0^{-1}|I^1 \times J^1$ defined by $g_1(x,0) = (x,0)$ for each $x \in I^6$. It is clear that g_1 is a PL-embedding.

Let N_1 be a regular neighborhood $g_1(A_1)$ in $\partial(I^6 \times [0,1])$. Since A_1 is collapsible, N_1 is PL-homeomorphic to the PL 6-ball by Corollary 3.27 [**19**]. Hence, there is a PL-homeomorphism

$$\theta_1 : I^6 \times [0,1] \to N_1 \times [0,1]$$

that carries x to $(x,0)$ for $x \in N_1$. Similarly, if N_1' is a regular neighborhood of A_1 in $\partial(I^6 \times J^9)$, there is a PL-homeomorphism

$$\theta_2 : I^6 \times J^9 \to N_1' \times [0,1]$$

that carries x to $(x, 0)$, for $x \in N_1'$. Now, from the embedding theorem 5.4 in [**19**], we can extend g_1^{-1} to a PL-embedding $\tilde{h}_1 : N_1 \to N_1'$. Then, define h_1 to be the composition $i\theta_2^{-1}(\tilde{h}_1 \times \mathrm{id})\theta_1$

$$\begin{aligned} h_1 : I^6 \times [0,1] \to^{\theta_1} N_1 \times [0,1] \to^{\tilde{h}_1 \times \mathrm{id}} N_1' \times [0,1] \\ \to^{\theta_2^{-1}} I^6 \times J^9 \hookrightarrow^{i} I^{32} \times J^{35}. \end{aligned}$$

It is easy to show that h_1 satisfies the properties (1)-(4).

3. *Construction of* h_2. $(n_3 = 69, m_3 = 72)$. Let A_2 denote $(I^{32} \times 0) \cup (I^6 \times J^9) \subset \partial(I^{32} \times J^{35}) \subset I^\infty \times J^\infty$. Observe that A_2 is collapsible and that $h_1(x, 0) = (x, 0)$ for all $x \in I^6$ since $g_1(x, 0) = (x, 0)$ for all $x \in I^6$; hence, $(I^{32} \times 0) \cap (\mathrm{Im}(h_1)) = I^6 \times 0$ and we can define a PL-embedding

$$\tilde{g}_2 : B \equiv (I^{32} \times 0) \bigcup h_1(I^6 \times [0,1]) \to \partial(I^{32} \times [0,1])$$

by

$$\tilde{g}_2(z) = \begin{cases} z & \text{if } z \in I^{32} \times 0 \\ h_1^{-1}(z) & \text{if } z \in h_1(I^6 \times [0,1]). \end{cases}$$

Then, let $\overline{g}_2 : A_2 \to \partial(I^{32} \times [0,1])$ be an extension of $\tilde{g}_2$. Since $I^{32} \times 0$ is a codimension-zero PL-ball in $\partial(I^{32} \times [0,1])$, first we can assume that $\overline{g}_2(A_2 - (I^{32} \times 0))$ does not contain the center $(c, 0)$ of $I^{32} \times 0$. Then, by use of a radial structure of $I^{32} - \{c\} \approx \partial I^{32} \times [0,1)$, we can push $\overline{g}_2 A_2 - (I^{32} \times 0))$ off $\mathrm{Int} I^{32} \times 0$. Finally, by use of a collar of $\partial I^{32} \times 0$ in $\partial(I^{32} \times [0,1]) - (\mathrm{Int}\, I^{32} \times 0)$ we push $\overline{g}_2(A_2 - (I^{32} \times 0))$ off $\partial I^{32} \times 0$. Therefore, without loss of generality, we can assume that $\overline{g}_2^{-1}(I^{32} \times 0) = I^{32} \times 0$. Now, since $2 \dim(A_2 - (\mathrm{Int}\, I^{32} \times 0)) + 2 = 32 = \dim \partial(I^{32} \times [0,1])$, it follows from Theorem 5.4 [**19**] that there is a PL-embedding $g_2 : A_2 \to \partial(I^{32} \times [0,1])$ such that $g_2(x) = \overline{g}_2(x)$ for $x \in B$. Consequently, g_2 is an extension of $\tilde{g}_2$. Finally, similar to the construction of h_1, we can define a PL-embedding

$$h_2 : I^{32} \times [0,1] \to I^{32} \times J^{35} \hookrightarrow^{i} I^{69} \times J^{72}$$

which is an extension of g_2^{-1}. Observe that h_2 is an extension of $(h_1^{-1})^{-1} = h_1$, that $h_2(I^{m_2} \times [0,1]) \supset I^6 \times J^9 = I^{n_1} \times J^{m_1}$, and that $h_2(x, 0) = (x, 0)$ if $x \in I^{32}$. So, h_2 satisfies all inductive hypotheses.

Inductively, we can construct a desired sequence of embeddings $\{h_k\}$, with $n_k = 2(n_{k-1} + m_{k-1}) + 2$ and $m_k = n_k + 3$. Finally, let us define g by $g(y) = g_{k+1}(y)$; if $y \in I^{n_k} \times J^{m_k}$, then we can show that g is the inverse of h. Therefore, the proof is complete.

ADDENDUM TO LEMMA 4.1. *There is a homeomorphism $h : (I^\infty \times [0,1), I^\infty \times 0) \to (I^\infty \times J^\infty, I^\infty \times 0)$ such that $h(x,0) = (x,0)$ for all $x \in I^\infty$.*

PROOF. Along the line of the proof of Lemma 4.1, we first start with $h_0 : I^1 \times [0, \frac{1}{2}] \to I^6 \times J^9$, next $g_1 : A_1 \to \partial(I^6 \times [0, \frac{3}{4}])$ and $h_1 : I^6 \times [0, \frac{3}{4}] \to I^{32} \times J^{35}$, then $g_2 : A_2 \to \partial(I^{32} \times [0, \frac{7}{8}])$ and $h_2 : I^{32} \times [0, \frac{7}{8}] \to I^{69} \times J^{72}$, and so on. We will obtain a sequence of embeddings $\{h_k | k = 0, 1, 2, \cdots\}$ with similar properties (1)-(4): and the result follows.

THEOREM 4.2. (COLLAR THEOREM). *Let $M \subset N$ be a pair of R^∞-manifolds. If M is an R^∞-deficient subset of N, then M is collared in N.*

PROOF. Similar to the proof of Theorem 2.3 in [**16**], it suffices to show this for the case $N = \mathrm{R}^\infty = I^\infty \times J^\infty$. By the triangulation theorem [**12**], there is a locally finite simplicial complex K such that $M \approx K \times \mathrm{R}^\infty$.

We first prove that cone $(K) \times \mathrm{R}^\infty \approx \mathrm{R}^\infty$. Let $K = \bigcup_1^\infty K_n$, where K_n is a finite complex contained in the interior of K_{n+1}. Then, cone $(K) = \lim_\to \mathrm{cone}(K_n)$ endowed with the direct limit topology. Let us consider the direct sequence

$$(*) \qquad \text{cone } (K_1) \times \mathrm{R}^1 \to \text{cone } (K_2) \times \mathrm{R}^2 \to \cdots$$

From Theorem 1.9 [**7**, p. 425], the natural bijection

$$\varinjlim(\text{cone } (K_n) \times \mathrm{R}^n) \to \text{cone } (K) \times \mathrm{R}^\infty$$

is continuous; furthermore, by local compactness, it is straightforward to verify that it is an open map; hence a homeomorphism. Moreover, $\lim_\to(\text{cone } (K_n) \times \mathrm{R}^n)$ is a contractible R^∞-manifold since the sequence (*) satisfies the Proposition in [**14**]. Hence, cone$(K) \times \mathrm{R}^\infty$ is homeomorphic to R^∞.

Next, we assume that M is the subset $K\times 0\times I^\infty$ of cone$(K)\times J^\infty\times I^\infty\approx \mathrm{R}^\infty$. Then we observe that (i) M is R^∞-deficient in R^∞, and (ii) M is collared in R^∞. To see (ii), from Lemma 4.1, there is a homeomorphism $h:[0,\frac{1}{2}]\times I^\infty\to[0,\frac{1}{2}]\times J^\infty\times I^\infty$ such that $h(0,z)=(0,0,z)$ for each $z\in I^\infty$. Therefore, the composition

$$f:K\times[0,\frac{1}{2})\times I^\infty\to^{\mathrm{id}\times\lambda}K\times[0,\frac{1}{2}]\times I^\infty\to^{\mathrm{id}\times h}$$
$$K\times[0,\frac{1}{2}]\times J^\infty\times I^\infty\to^{\mathrm{id}\times\mu\times\mathrm{id}}K\times[0,\frac{1}{2})\times J^\infty\times I^\infty$$
$$\hookrightarrow\text{cone }(K)\times J^\infty\times I^\infty\approx\mathrm{R}^\infty,$$

where $\lambda:[0,\frac{1}{2})\times I^\infty\to[0,\frac{1}{2}\times I^\infty$ is a homeomorphism such that $\lambda(0,0)=(0,0)$ and $\mu=\lambda^{-1}:[0,\frac{1}{2}]\times J^\infty\to[0,\frac{1}{2})\times J^\infty$ where J^∞ is a copy of I^∞, defines a collar of $K\times 0\times 0\times I^\infty$ in R^∞.

Finally, the existence of a collar on an arbitrary R^∞-deficient subset M of R^∞ will follow from the above special case and the unknotting theorem 3.3 for R^∞-deficient embeddings in R^∞.

COROLLARY 4.3. *Let $f_i:M_{i-1}\to M_i$ be an R^∞-deficient embedding of R^∞-manifolds, for each $i=1,2,\cdots n$. Then the composition $c_n\circ\cdots\circ c_2\circ c_1:M(f_1,f_2,\cdots,f_n)\to M_n$ of collapsing maps $c_1,c_2,\cdots,c_n$ is a near homeomorphism.*

LEMMA 4.4. *Let (N,M) be a pair of R^∞-manifolds. If M is collared in N, then $M\times I$ is R^∞-deficient in $N\times I$.*

PROOF. Similar to the proof of Lemma 2.6 in [**16**], there is a homeomorphism $\psi:N\times I\to N\times I$ such that $\psi(M\times I)\subset N\times 0$. So,Lemma 4.4 follows because we can show that $N\times 0$ is R^∞-deficient in $N\times I$ by use of Lemma 4.1 and the stability theorem for R^∞-manifolds [**12**].

Since R^∞-manifolds are paracompact, the proofs of Lemma 3.4 and 3.5 in [**16**] are also applicable for R^∞-manifolds by use of the approximation Theorem 2.3 above. We can state similar lemmas as follows.

LEMMA 4.5. *Let X be an R^∞-deficient subset of an R^∞-manifold M, α and open cover of M. Then there is an open cover β of M such that if a closed map $f:X\to M$ is β-close to the inclusion, then there is an R^∞-deficient embedding $g:X\to M$ such that*

(1) $g(X) \cap (X \cup f(X)) = \phi$,
(2) *g is α-close to both f and the inclusion.*

LEMMA 4.6. *Let X be an R^∞-deficient subset of an R^∞-manifold $M, f : X \to M$ a map and an open cover α of M. Then there is an α-close to f and R^∞-deficient embedding $g : X \to M$ such that $g(X) \cap X = \phi$.*

Then, by use of Theorem 4.2, Lemmas 4.4, 4.5 and 4.6 and Theorem 2.3, we can prove similarly the following lemma, the R^∞-manifold version of Lemma 3.6 in [**16**].

LEMMA 4.7. *Suppose that M is an R^∞-deficient submanifold of an R^∞-manifold N, let α be an open cover N, and let $f : M \to N$ is an R^∞-deficient embedding. If f is α-homotopic to the inclusion, then there is an isotopy $H_t : N \times I \to N \times I$ such that*
(1) $H_0 = \mathrm{id}_{N\times I}$,
(2) $H_1|M \times I = f \times \mathrm{id}_I$, *and*
(3) *H is limited by $St^6(\alpha \times \gamma)$ for any prechosen open cover γ of I.*

THEOREM 4.8. *Given an open cover α of an R^∞-manifold N and $f, g : X \to N$ two α-homotopic, R^∞-deficient embeddings, then for any prechosen open cover γ of I there is a $St^8(\alpha \times \gamma)$-isotopy $H_t : N \times I \to N \times I$ such that $H_1 \circ (f \times id_I) = g \times id_I$.*

PROOF. Similar to the proof of Theorem 3.7 in [**16**], we may assume that X is an R^∞-manifold. Then the isotopy that we desired will follow from Lemma 4.7.

5. Main theorem and consequences. Now, we are ready to prove the α-approximation theorem.

THEOREM 5.1. *Let N be an R^∞-manifold and α an open cover of N. There is an open cover β of N such that if M is an R^∞-manifold and $f : M \to N$ is a β-equivalence, then f is α-close to a homeomorphism.*

PROOF. It is similar to the proof of Thm. 4.1 in [**16**]. The ingredients of the proof consist of
(1) the unknotting theorem for R^∞-deficient embeddings (weak version, Theorem 4.8),

(2) the relative R^∞-deficient embedding approximation theorem (Theorem 2.3),

(3) the collar theorem (Theorem 4.2), and

(4) the projection map $p_N : N\times I \to N$ being a near homeomorphism, which can be derived from the stability theorem for R^∞-manifolds [**12**]

COROLLARY 5.2. *Every fine homotopy equivalence between R^∞-manifolds is a near homeomorphism.*

THEOREM 5.3. *Every R^∞-deficient subset X of an R^∞-manifold M is strongly negligible; i.e., the inclusion map $M - X \subset M$ is a near homeomorphism.*

REMARK. Recently, Sakai has distributed much shorter proofs for Corollary 5.2 and 5.3, Theorem 5.3; e.g., Theorem 2.3 and 2.5 in [**21**].

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