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ON THE NORMAL NUMBER OF PRIME FACTORS OF $\varphi(n)$

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Dedicated to the memory of E. G. Straus and R. A. Smith

1. Introduction. Denote by $\Omega(n)$ the total number of prime factors of n, counting multiplicity, and by $\omega(n)$ the number of distinct prime factors of n. One of the first results of probabilistic number theory is the theorem of Hardy and Ramanujan that the normal value of $\omega(n)$ is loglog n. What this statement means is that for each $\varepsilon > 0$, the set of n for which

$$(1.1) \qquad |\omega(n) - \log\log n| < \varepsilon \log\log n$$

has asymptotic density 1. The normal value of $\Omega(n)$ is also loglog n.

A paricularly simple proof of these results was later given by Turán. He showed that

$$(1.2) \qquad \sum_{n \leqq x} (\omega(n) - \log\log x)^2 = x \log\log x + O(x)$$

from which (1.1) is an immediate corollary. The method of proof of the asymptotic formula (1.2) was later generalized independently by Turán and Kubilius to give an upper bound for the left hand side where $\omega(n)$ is replaced by an arbitrary additive function. The significance of the "loglog x" in (1.2) is that it is about $\sum_{p \leqq x} \omega(p)p^{-1}$, where p runs over primes. Similarly the expected value of an arbitrary additive function $g(n)$ should be about $\sum_{p \leqq x} g(p)p^{-1}$.

The finer distribution of $\Omega(n)$ and $\omega(n)$ was studied by many people, culminating in the celebrated Erdös-Kac theorem: for each $x \geqq 3$, u, let

$$G(x, u) = \frac{1}{x} \cdot \# \{n \leqq x: \Omega(n) \leqq \log\log x + u (\log\log x)^{1/2}\}.$$

Then

$$(1.3) \qquad \lim_{x \to \infty} G(x, u) = G(u) \stackrel{\text{def}}{=} (2\pi)^{-1/2} \int_{-\infty}^{u} e^{-t^2/2}\, dt,$$

the Gaussian normal distribution. The corresponding statement with

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$\omega(n)$ is also true. Later Kubilius and Shapiro (independently) generalized the Erdös-Kac theorem to more general additive functions. In particular, they gave a simple criterion for the Gaussian normal distribution to be achieved.

The problem that we consider here is the corresponding problem for the additive function $\Omega(\varphi(n))$, where φ is Euler's function. Using the machinery of the Kubilius-Shapiro work, the issue devolves upon the estimation of the sums

$$\sum_{p \leq x} \Omega(p-1) \text{ and } \sum_{p \leq x} \Omega(p-1)^2.$$

Sums of this type were estimated already in 1951 by Haselgrove [**4**], but the proofs were complicated and not given. Our proofs are a simple application of the Bombieri-Vinogradov theorem and Brun's method. Of course, in 1951, the Bombieri-Vinogradov theorem did not yet exist.

What we prove is that

$$\lim_{x \to \infty} \frac{1}{x} \cdot \#\left\{n \leq x\colon \Omega(\varphi(n)) \leq \frac{1}{2}(\log\log x)^2 + \frac{u}{\sqrt{3}}(\log\log x)^{3/2}\right\} = G(u).$$

Thus the normal number of prime factors of $\varphi(n)$ is $1/2\,(\log\log n)^2$ and the "standard deviation" is $3^{-1/2}(\log\log n)^{3/2}$.

The situation with the function $\omega(\varphi(n))$ is the same, but the treatment is less routine, notably because $\omega(\varphi(n))$ is not additive. As one might expect, though, the difference $\Omega(\varphi(n)) - \omega(\varphi(n))$ is usually not large (compared with $\Omega(\varphi(n))$), so we can obtain the same result for $\omega(\varphi(n))$.

2. The number of prime factors of a shifted prime. For any y we define the completely additive function $\Omega_y(n)$, the total number of prime factors $p \leq y$ of n, counting multiplicity. Thus, for example, $\Omega_3(100) = 2$. The letters p, q, r always denote primes. Let $P(n)$ denote the largest prime factor of n.

LEMMA 2.1. *If* $3 \leq y \leq x$, *then*

$$\sum_{p \leq x} \Omega_y(p-1) = \frac{x \log\log y}{\log x} + O\left(\frac{x}{\log x}\right),$$

where the implied constant is uniform.

PROOF. We have (where $\pi(x, k, l) = \sum_{\substack{p \leq x \\ p \equiv l\,(k)}} 1$)

$$\sum_{p \leq x} \Omega_y(p-1) = \sum_{p \leq x} \sum_{\substack{q^a \mid p-1 \\ q \leq y}} 1 = \sum_{\substack{q^a \\ q \leq y}} \pi(x, q^a, 1)$$

$$= \sum_{q \leq y} \pi(x, q, 1) + \sum_{\substack{q^a,\, a \geq 2 \\ q \leq y}} \pi(x, q^a, 1) = S_1 + S_2, \text{ say}.$$

For S_1 we consider two ranges for the prime q: $q \leqq \min\{y, x^{1/3}\}$ and $\min\{y, x^{1/3}\} < q \leqq y$. Of course, depending on the size of y, the latter range may be vacuous. We estimate the sum in the first range by the Bombieri-Vinogradov theorem by

$$\sum_{q \leq \min\{y, x^{1/3}\}} \pi(x, q, 1) = \sum_{q \leq \min\{y, x^{1/3}\}} li(x)/\varphi(q) + O\left(\frac{x}{\log^2 x}\right)$$
$$= \frac{x \operatorname{loglog} y}{\log x} + O\left(\frac{x}{\log x}\right).$$

It thus remains to show that the second range for q in S_1 and all of S_2 contribute only $O(x/\log x)$ to the sum.

The second range in S_1 can be estimated very simply (thanks are due to K. Murty for the suggestion)

$$\sum_{\min\{y, x^{1/3}\} < q \leq y} \pi(x, q, 1) \leqq \sum_{q > x^{1/3}} \pi(x, q, 1)$$
$$= \sum_{p \leq x} \sum_{\substack{q \mid p-1 \\ q > x^{1/3}}} 1 \leqq 2\pi(x) = O\left(\frac{x}{\log x}\right).$$

We also break S_2 into two ranges: $q^a \leqq x^{1/3}$ and $x^{1/3} < q^a \leqq x$. The first range is estimated by the Brun-Titchmarsh theorem to be

$$\sum_{\substack{q^a \leq x^{1/3}, a \geq 2 \\ q \leq y}} \pi(x, q^a, 1) \ll \frac{x}{\log x} \sum \frac{1}{\varphi(q^a)} \ll \frac{x}{\log x}.$$

The second part of S_2 is bounded using the trivial estimate

$$\sum_{\substack{q^a > x^{1/3}, a \geq 2 \\ q \leq y}} \pi(x, q^a, 1) \leqq \sum \frac{x}{q^a} \ll x^{5/6}.$$

We thus have proved the lemma.

LEMMA 2.2. *If* $3 \leqq y \leqq x$, *then*

$$\sum_{p \leq x} \Omega_y(p-1)^2 = \frac{x (\operatorname{loglog} y)^2}{\log x} + O\left(\frac{x \operatorname{loglog} y}{\log x}\right),$$

where the implied constant is uniform.

PROOF. Let u range over the integers with $\omega(u) = 2$ and $P(u) \leqq y$. Then

$$\sum_{p \leq x} \Omega_y(p-1)^2 = \sum_{p \leq x} \sum_{\substack{q^a \| p-1 \\ q \leq y}} a^2 + 2 \sum_{p \leq x} \sum_{\substack{u \mid p-1 \\ (u, (p-1)/u) = 1}} 1$$
$$= S_3 + S_4, \text{ say.}$$

We have

$$\begin{aligned} S_3 &= \sum_{p\leqq x} \Omega_y(p-1) + \sum_{p\leqq x} \sum_{\substack{q^a \| p-1 \\ q\leqq y,\, a\geqq 2}} (a^2-a) \\ &\leqq \sum_{p\leqq x} \Omega_y(p-1) + \sum_{\substack{q^a\leqq x^{1/3} \\ q\leqq y,\, a\geqq 2}} (a^2-a)\pi(x,q^a,1) + \sum_{\substack{q^a> x^{1/3} \\ q\leqq y,\, a\geqq 2}} (a^2-a)\pi(x,q^a,1) \\ &= O\left(\frac{x \operatorname{loglog} y}{\log x}\right), \end{aligned}$$

where we used Lemma 2.1 for the first sum, the Brun-Titchmarsh theorem for the middle sum, and a trival estimate for the last sum.

For S_4, we reverse the order of summation obtaining

$$\begin{aligned} S_4 &= 2\sum_{u\leqq x^{1/6}} \sum_{d\mid u} \mu(d)\,\pi(x,du,1) + 2\sum_{u> x^{1/6}} \sum_{d\mid u} \mu(d)\pi(x,du,1) \\ &= S_{4,1} + S_{,2}, \text{ say.} \end{aligned}$$

For $S_{4,1}$, the main term, we use the Bombieri-Vinogradov theorem to estimate

$$\begin{aligned} S_{4,1} &= 2\, l\, i(x) \sum_{u\leqq x^{1/6}} \sum_{d\mid u} \frac{\mu(d)}{\varphi(du)} + O\left(\frac{x}{\log^2 x}\right) \\ &= 2\, l\, i(x) \sum_{u\leqq x^{1/6}} \frac{1}{u} + O\left(\frac{x}{\log^2 x}\right) \\ &= \frac{x\,(\operatorname{loglog} y)^2}{\log x} + O\left(\frac{x \operatorname{loglog} y}{\log x}\right). \end{aligned}$$

Finally, for $S_{4,2}$, we have the larger prime power factor of du exceeding $x^{1/12}$, so that

$$\begin{aligned} S_{4,2} &\leqq 2 \sum_{\substack{q^a<r^b,\, q^a r^b > x^{1/6} \\ q,\, r\leqq y}} \pi(x, q^a r^b, 1) \\ &\leqq 2\sum_{\substack{q^a \\ q\leqq y}} \sum_{\substack{r^b > x^{1/12} \\ b\geqq 2}} \pi(x,q^a r^b,1) + 2\sum_{\substack{q^a \\ q\leqq y}} \sum_{r> x^{1/12}} \pi(x,q^a r,1) \\ &\leqq 2x \sum_{\substack{q^a \\ q\leqq y}} \sum_{\substack{r^b > x^{1/12} \\ b\geqq 2}} \frac{1}{q^a r^b} + 2\sum_{p\leqq x} \sum_{\substack{q^a \mid p-1 \\ q\leqq y}} \sum_{\substack{r\mid p-1 \\ r> x^{1/12}}} 1 \\ &\ll x^{23/24} \operatorname{loglog} y + \sum_{p\leqq x} \sum_{\substack{q^a\mid p-1 \\ q\leqq y}} 1 \\ &= x^{23/24} \operatorname{loglog} y + \sum_{p\leqq x} \Omega_y(p-1) \\ &\ll \frac{x \operatorname{loglog} y}{\log x}, \end{aligned}$$

by Lemma 2.1.

LEMMA 2.3. *If* $3 \leqq y \leqq x$, *then*

$$\sum_{p\leqq x} \Omega_y(p-1)/p = \operatorname{loglog} x \operatorname{loglog} y - \frac{1}{2}(\operatorname{loglog} y)^2 + O(\operatorname{loglog} x),$$

where the implied constant is uniform.

PROOF. This result follows immediately from Lemma 2.1 and partial summation. We have

$$\sum_{p\le x}\frac{\Omega_y(p-1)}{p}=\frac{1}{x}\sum_{p\le x}\Omega_y(p-1)+\int_2^x\frac{1}{t^2}\sum_{p\le t}\Omega_y(p-1)\,dt$$
$$=O\left(\frac{\operatorname{loglog} y}{\log x}\right)+\int_2^y\frac{\operatorname{loglog} t}{t\log t}dt+\int_y^x\frac{\operatorname{loglog} y}{t\log t}dt+O\left(\int_2^x\frac{dt}{t\log t}\right)$$
$$=\operatorname{loglog} x\operatorname{loglog} y-\frac{1}{2}(\operatorname{loglog} y)^2+O(\operatorname{loglog} x).$$

LEMMA 2.4. *If* $3\leqq y\leqq x$, *then*

$$\sum_{p\le x}\Omega_y(p-1)^2/p=\operatorname{loglog} x(\operatorname{loglog} y)^2-\frac{2}{3}(\operatorname{loglog} y)^3$$
$$+O(\operatorname{loglog} x\operatorname{loglog} y),$$

where the implied constant is uniform.

PROOF. This result is derived from Lemma 2.2 and partial summation.

LEMMA 2.5. *If* $2\leqq k\leqq x$, *then*

$$\sum_{\substack{p\le x\\ p\equiv 1\,(k)}}\frac{1}{p}=\frac{\operatorname{loglog} x}{\varphi(k)}+O\left(\frac{\log k}{\varphi(k)}\right),$$

where the implied constant is uniform.

This result can be found in Norton [**5**] and Pomerance [**6**].

3. The normal number of prime factors of $\varphi(n)$. An additive function $f(n)$ is called strongly additive if $f(p^a)=f(p)$ for all $a\geqq 1$. If $f(n)$ is real-valued and strongly additive, let

$$A(x)=\sum_{p\le x}f(p)/p,\qquad B(x)=\Big(\sum_{p\le x}f(p)^2/p\Big)^{1/2}.$$

Suppose for each $\varepsilon>0$, we have

$$\lim_{x\to\infty}\frac{1}{B(x)^2}\sum_{\substack{p\le x\\ |f(p)|>\varepsilon B(x)}}\frac{f(p)^2}{p}=0. \tag{3.1}$$

The theorem of Kubilius-Shapiro (see Elliott [1], Theorem 12.2) states that if (3.1) holds, then for each real number u,

$$\lim_{x\to\infty}\frac{1}{x}\cdot\#\{n\leqq x: f(n)-A(x)\leqq uB(x)\}=G(u), \tag{3.2}$$

where $G(u)$ is defined in (1.3). That is, if (3.1) holds, then the normal value for $n\leqq x$ of $f(n)$ is $A(x)$ and the standard deviation is $B(x)$.

We would like to apply the Kubilius-Shapiro theorem to the additive function $\Omega(\varphi(n))$, but it is not strongly additive. Instead, we define

$$f(n) = \sum_{p\mid n} \Omega(p-1). \tag{3.3}$$

Then $f(n)$ is strongly additive and does not differ very much from $\Omega(\varphi(n))$.

THEOREM 3.1. *For every real number u we have*

$$\lim_{x\to\infty} \frac{1}{x} \cdot \#\left\{n \leqq x: \Omega(\varphi(n)) - \frac{1}{2}(\operatorname{loglog} x)^2 \leqq \frac{u}{\sqrt{3}}(\operatorname{loglog} x)^{3/2}\right\} = G(u), \tag{3.4}$$

where $G(u) = (2\pi)^{-1/2} \int_{-\infty}^{u} e^{-t^2/2}\, dt$.

PROOF. We apply the Kubilius-Shapiro theorem to the strongly additive function $f(n)$ defined in (3.3). We have

$$A(x) = \sum_{p\leqq x} \Omega(p-1)/p = \frac{1}{2}(\operatorname{loglog} x)^2 + O(\operatorname{loglog} x)$$

by Lemma 2.3 (with $y = x$). Also

$$B(x)^2 = \sum_{p\leqq x} \Omega(p-1)^2/p = \frac{1}{3}(\operatorname{loglog} x)^3 + O((\operatorname{loglog} x)^2),$$

by Lemma 2.4 (with $y = x$). Thus to apply the Kubilius-Shapiro theorem to $f(n)$ it remains to verify (3.1). Let $\varepsilon > 0$ be fixed and let $T = \varepsilon/\sqrt{3}\,\cdot$ $(\operatorname{loglog} x)^{3/2}$. From Erdös and Sárközy [3], it follows that for any $y \geqq 2$,

$$\sum_{\substack{n\leqq y\\ \Omega(n)\geqq T}} 1 \ll 2^{-T}T^4 y \log y,$$

so that

$$\begin{aligned}
\sum_{\substack{p\leqq x\\ \Omega(p-1)\geqq T}} \Omega(p-1)^2/p &\leqq \sum_{\substack{n\leqq x\\ \Omega(n)\geqq T}} \Omega(n)^2/n = x^{-1} \sum_{\substack{n\leqq x\\ \Omega(n)\geqq T}} \Omega(n)^2 + \int_2^x t^{-2} \sum_{\substack{n\leqq t\\ \Omega(n)\geqq T}} \Omega(n)^2\, dt \\
&\ll x^{-1}(\log x)^2 \sum_{\substack{n\leqq x\\ \Omega(n)\geqq T}} 1 + \int_2^x t^{-2}(\log t)^2 \sum_{\substack{n\leqq t\\ \Omega(n)\geqq T}} 1\, dt \\
&\ll 2^{-T}\, T^4(\log x)^3 + 2^{-T}\, T^4 \int_2^x t^{-1}(\log t)^3\, dt \\
&\ll 2^{-T}\, T^4(\log x)^4 = o(1).
\end{aligned}$$

Thus (3.1) is verified and, by the Kubilius-Shapiro theorem, we have (3.4) with $f(n)$ in place of $\Omega(\varphi(n))$. But $\Omega(\varphi(n)) = f(n) + \Omega(n) - \omega(n)$ and $\Omega(n) - \omega(n)$ is normally $o(\operatorname{loglog} n)$ by the Hardy-Ramanujan theorem. (In fact, for each $\varepsilon > 0$ there is a k_ε such that the asymptotic density of the n with $\Omega(n) - \omega(n) \geqq k_\varepsilon$ is at most ε. Thus, if $h(n)$ tends to infinity arbitrarily slowly, then the set of n with $\Omega(n) - \omega(n) \leqq h(n)$ has asymptotic density 1.) We therefore may replace $f(n)$ with $\Omega(\varphi(n))$, obtaining (3.4).

THEOREM 3.2. *For every real number u we have*

$$\lim_{x\to\infty} \frac{1}{x} \cdot \# \{x \leqq x : \omega(\varphi(n)) - \frac{1}{2}(\text{loglog } x)^2 \leqq \frac{u}{\sqrt{3}}(\text{loglog } x)^{3/2}\} = G(u).$$

PROOF. This result will follow immediately from Theorem 3.1 if we can show that, but for $o(x)$ choices of $n \leqq x$,

$$\Omega(\varphi(n)) - \omega(\varphi(n)) = o((\text{loglog } x)^{3/2}).$$

In fact, we shall show the stronger result, that but for $o(x)$ choices of $n \leqq x$

$$\Omega(\varphi(n)) - \omega(\varphi(n)) = O(\text{loglog } x \text{ logloglogloglog } x). \tag{3.5}$$

Let $\omega_y(n)$ denote the number of distinct prime factors of n which do not exceed y. From now on we always take

$$y = (\text{loglog } x)^2. \tag{3.6}$$

Our strategy is to show that but for $o(x)$ choices of $n \leqq x$

$$\Omega(\varphi(n)) - \Omega_y(\varphi(n)) = \omega(\varphi(n)) - \omega_y(\varphi(n)). \tag{3.7}$$

We then will be able to restrict ourselves to bounding $\Omega_y(\varphi(n)) - \omega_y(\varphi(n))$.

We apply the Turán-Kubilius inequality (Elliott [1], Lemma 4.1) to the additive function $\Omega_y(\varphi(n))$. We have

$$\begin{aligned} E_y(x) &\overset{\text{def}}{=} \sum_{p^k \leq x} \frac{\Omega_y(\varphi(p^k))}{p^k}\left(1 - \frac{1}{p}\right) \\ &= \sum_{p \leq x} \frac{\Omega_y(p-1)}{p} + O\Bigg(\sum_{\substack{p^k \leq x \\ k>1}} \frac{\Omega((p-1)p^{k-1})}{p^k}\Bigg) \\ &= \text{loglog } x \text{ loglog } y - \frac{1}{2}(\text{loglog } y)^2 + O(\text{loglog } x), \end{aligned}$$

by Lemma 2.3 and

$$\begin{aligned} D_y(x)^2 &\overset{\text{def}}{=} \sum_{p^k \leq x} \frac{\Omega_y(\varphi(p^k))^2}{p^k} = \sum_{p \leq x} \frac{\Omega_y(p-1)^2}{p} + O\Bigg(\sum_{\substack{p^k \leq x \\ k>1}} \frac{\Omega(\varphi(p^k))^2}{p^k}\Bigg) \\ &= \text{loglog } x\,(\text{loglog } y)^2 - \frac{2}{3}(\text{loglog } y)^3 + O(\text{loglog } x \text{ loglog } y) \end{aligned}$$

by Lemma 2.4. Therefore, by the Turán-Kubilius inequality,

$$\sum_{n \leq x} (\Omega_y(\varphi(n)) - E_y(x))^2 \leqq 32x\, D_y(x)^2. \tag{3.8}$$

By (3.6), we have

$$E_y(x) = \text{loglog } x \text{ logloglogloglog } x + O(\text{loglog } x),$$

$$D_y(x)^2 = \text{loglog } x(\text{logloglog} \log x)^2 + O(\text{loglog } x \text{ loglogloglog } x).$$

Therefore, by (3.8), the number of $n \leqq x$ with $\Omega_y(\varphi(n)) > 2 \text{ loglog } x \cdot \text{loglogloglog } x$ is $O(x/\text{loglog } x) = o(x)$. We thus have but for $o(x)$ choices of $n \leqq x$

$$(3.9) \qquad 0 \leqq \Omega_y(\varphi(n)) - \omega_y(\varphi(n)) \leqq 2 \text{ loglog } x \text{ loglogloglog } x.$$

We now show that but for $o(x)$ choices of $n \leqq x$ we have (3.7). Suppose $p^2 | \varphi(n)$ where $p > y$ and $n \leqq x$. There are three possibilities:

(i) $p^3 | n$,

(ii) there is some $q | n$ with $q \equiv 1 \bmod p^2$,

(iii) there are distinct q_1, q_2 with $q_1 q_2 | n$ and $q_1 \equiv q_2 \equiv 1 \bmod p$.

The number of $n \leqq x$ in the first case is at most

$$\sum_{p>y} x/p^3 = o(x/y^2) = o(x).$$

The number of $n \leqq x$ in the second case is, by Lemma 2.5, at most

$$\sum_{p>y} \sum_{\substack{q \equiv 1 (p^2) \\ q \leqq x}} \frac{x}{q} = \sum_{p>y} \frac{x \text{ loglog } x}{\varphi(p^2)} + O\left(\sum_{p>y} \frac{x \log p}{p^2}\right)$$
$$= O\left(\frac{x \text{ loglog } x}{y \log y}\right) + O\left(\frac{x}{y}\right) = o(x).$$

The number of $n \leqq x$ in the third case is at most, by Lemma 2.5,

$$\sum_{p>y} \sum_{\substack{q_1 \equiv q_2 \equiv 1 (p) \\ q_1 < q_2 \leqq x}} \frac{x}{q_1 q_2} \leqq \frac{1}{2} x \sum_{p>y} \left(\sum_{\substack{q \equiv 1 (p) \\ q \leqq x}} \frac{1}{q}\right)^2$$
$$= \frac{1}{2} x \sum_{p>y} \left(\frac{\text{loglog } x}{\varphi(p)} + O\left(\frac{\log p}{p}\right)\right)^2$$
$$= O\left(\frac{x(\text{loglog} x)^2}{y \log y}\right) + O\left(\frac{x \text{ loglog } x}{y}\right)$$
$$+ O\left(\frac{x \text{ loglog } y}{y}\right) = o(x).$$

This estimate completes the proof that (3.7) holds for all but $o(x)$ choices of $n \leqq x$. Combined with (3.9), we have (3.5) and thus the theorem.

4. Further comments. Let $\lambda(n)$ denote the least common multiple of the $\lambda(p^a)$ for $p^a | n$ where $\lambda(p^a) = \varphi(p^a)$ for $p > 2$, $p^a = 2, 4$ and $\lambda(2^a) = 2^{a-2}$ for $a \geqq 3$. Then $\lambda(n)$, also called the Carmichael function, is the universal exponent for group of residues mod n coprime to n. That is, if gcd $(a, n) = 1$, then $a^{\lambda(n)} \equiv 1 \bmod n$, and no smaller positive exponent works for all such a. We evidently have

$$\prod_{p | \varphi(n)} p \mid \lambda(n), \qquad \lambda(n) \mid \varphi(n).$$

Therefore,

$$\omega(\varphi(n)) = \omega(\lambda(n)) \leqq \Omega(\lambda(n)) \leqq \Omega(\varphi(n))$$

for all n. Thus as a corollary to the theorems of §3, we have

THEOREM 4.1. *For each real number u,*

$$\lim_{x\to\infty} \frac{1}{x} \cdot \#\{n \leqq x: \Omega(\lambda(n)) - \frac{1}{2}(\operatorname{loglog} x)^2 \leqq \frac{u}{\sqrt{3}}(\operatorname{loglog} x)^{3/2}\} = G(u)$$

and the same holds for $\omega(\lambda(n))$ in place of $\Omega(\lambda(n))$.

In Erdös [**2**], the following two theorems were stated without proof:

THEOREM A. *For each $\varepsilon > 0$ and k we have for all $x \geqq x_0(\varepsilon, k)$,*

$$\frac{x}{\log x}(\operatorname{loglog} x)^k \leqq \frac{1}{x}\sum_{n\leqq x} \lambda(n) \leqq \frac{x}{(\log x)^{1-\varepsilon}}.$$

THEOREM B. *The normal value of* $\log(n/\lambda(n))$ *is* $\operatorname{loglog} n \operatorname{logloglog} n$.
Theorem B can be restated the following way. For each $\varepsilon > 0$, the n for which

$$\frac{n}{(\log n)^{(1+\varepsilon)\operatorname{logloglog} n}} < \lambda(n) < \frac{n}{(\log n)^{(1-\varepsilon)\operatorname{logloglog} n}}$$

have asymptotic density 1.

The proofs of these theorems are not easy. In a forthcoming paper we shall present the details.

If $\gcd(a, n) = 1$, let $l_a(n)$ denote the exponent to which a belongs mod n. Then $l_a(n) | \lambda(n)$. Almost certainly we have

$$\lim_{x\to\infty} \frac{a}{\varphi(a)x} \cdot \#\left\{n \leqq x: \gcd(a, n) = 1, \Omega(l_a(n)) - \frac{1}{2}(\operatorname{loglog} x)^2 \leqq \frac{u}{\sqrt{3}}(\operatorname{loglog} x)^{3/2}\right\} = G(u)$$

for any value of $a \neq 0, \pm 1$. The same should be true for $\omega(l_a(n))$, but we have been unable to prove either statement.

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