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SOME QUESTIONS RELATED TO ALMOST 2-FULLY NORMAL SPACES

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1. Introduction and history. For notational convenience, we adopt the following conventions. All spaces considered are regular Hausdorff spaces. If $\mathscr{C}$ is a cover of a space X, $\mathscr{C}^*$ denotes the cover $\{\mathrm{st}(x, \mathscr{C}): x \in X\}$. A cover of the form $\mathscr{C}^*$ is called a point-star cover, and a subset A of X is called a refiner of $\mathscr{C}$ provided that A is a subset of some member of $\mathscr{C}$. A cover of a space is directed (ω-directed) provided that the union of finitely (countably) many members is a refiner of the cover. Let m be a cardinal number greater than 1. A space X is almost m-fully normal (almost finitely fully normal) provided that if $\mathscr{C}$ is an open cover of X there is an open refinement $\mathscr{R}$ of $\mathscr{C}$ so that each refiner of $\mathscr{R}^*$ that has m or fewer elements (that is finite) is a refiner of $\mathscr{C}$. A space X is m-fully normal provided that if $\mathscr{C}$ is an open cover of X there is an open refinement $\mathscr{R}$ of $\mathscr{C}$ so that if $\mathscr{R}'$ is a subcollection of $\mathscr{R}$ for which $\bigcap\mathscr{R}' \neq \emptyset$ and $\mathrm{card}(\mathscr{R}') \leqq m$, then $\bigcup\mathscr{R}'$ is a refiner of $\mathscr{C}$.

Almost 2-fully normal spaces were first considered by H.J. Cohen **[4]**, who showed that every almost 2-fully normal space is collectionwise normal and characterized the almost 2-fully normal spaces as those spaces for which the filter of all neighborhoods of the diagonal is the fine uniformity. In **[19]**, M.J. Mansfield coined the terminology "(almost) m-fully normal" and began the systematic generalization of full normality in which the study of almost 2-fully normal spaces is now embedded. (In the literature almost 2-fully normal spaces are also called divisible **[7]**, strongly collectionwise normal **[10]**, doubly covered **[14]**, entirely normal **[13]**, and in the Russian literature almost pseudoparacompact **[25]**.) In particular, Mansfield raised the following questions.

Question 1. Is every almost 2-fully normal space 2-fully normal?

Question 2. Is there a finitely fully normal space that is not $\aleph_0$-fully normal?

Question 3. Is every finitely fully normal space countably paracompact?

Question 4. Is every almost 2-fully normal space almost n-fully normal for every integer n greater than 2?

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The first of these questions was answered negatively by H.H. Corson who showed that A.H. Stone's space F_0 is an almost $\aleph_0$-fully normal space that is not 2-fully normal [6]. Using Mansfield's result that every almost $\aleph_0$-fully normal space is countably paracompact, K.P. Hart recently answered Questions 2 and 3 in one fell swoop by showing that M.E. Rudin's Dowker space is finitely fully normal [11]. In [18], the authors showed that an example due to D.K. Burke and E. van Douwen provides an almost finitely fully normal countably paracompact space that is not almost $\aleph_0$-fully normal. Mansfield's last question remains unanswered, although a partial result was obtained by Corson [6, Lemma 3].

2. Cofinal Completeness. The earliest and in many ways the most important problem concerning almost 2-fully normal spaces was raised by J. Kelley in a conjecture, which may be rephrased as follows. A Hausdorff space is paracompact if and only if it is almost 2-fully normal and Dieudonné complete [17, 208]. As Corson pointed out in [5], the space F_0 provides a counterexample to Kelley's conjecture, and so we are left with the underlying problem considered in this paper, i.e., how can Kelley's conjecture be emended?

To begin with, let us consider properties that imply paracompactness in an almost 2-fully normal space. Since every almost 2-fully normal space is collectionwise normal, θ-refinability is one such property [24, Theorem (iii)]. In [13], N. Howes introduced the following property. A space X is cofinally Δ-complete provided that every cofinal Δ-Cauchy net has a cluster point, where a net $\phi: D \to X$ is a cofinal Δ-Cauchy net if for each open cover $\mathscr{U}$ of X there is a $p \in X$ and a cofinal subset C of D such that $\phi(C) \subset \mathrm{st}(p, \mathscr{U})$. A straightforward argument establishes a characterization of cofinal Δ-completeness in terms of open covers. A space X is cofinally Δ-complete provided that every directed open cover of X has an open point-star refinement. Howes proved that every metacompact space is cofinally Δ-complete and that every cofinally Δ-complete almost 2-fully normal space is paracompact. It is natural to modify Howes's definition by saying that a space X is cofinally θ-complete provided that, corresponding to each directed open cover $\mathscr{C}$ of X, there is a sequence $\langle \mathscr{G}_n \rangle$ of open covers of X so that for each $x \in X$ there exists a natural number $n(x)$ for which $\mathrm{st}(x, \mathscr{G}_{n(x)})$ is a refiner of $\mathscr{C}$. We also say that X is cofinally $\delta\theta$-complete provided that the conditions given above hold for ω-directed open covers. Arguments analogous to those given by Howes establish that every $(\delta)\theta$-refinable space is cofinally $(\delta)\theta$-complete and that every cofinally $\delta\theta$-complete almost 2-fully normal space is paracompact. The latter result is established in Proposition 3.7. Notice that the omission of the world "directed" in the covering characterization of cofinal Δ-

completeness (cofinal θ-completeness) yields full normality (subparacompactness). Seemingly, in light of Proposition 2.6, cofinal Δ-completeness is a suitable substitute for Dieudonné completeness in Kelley's conjecture However, we have the following questions posed by Y. Katuta, which remain unanswered.

Question 5 [**16**, Question 2.6(b)]. Is every cofinally Δ-complete space metacompact?

Question 6 [**16**, Question 2.7(b)]. Is every cofinally θ-complete space θ-refinable?

Since every cofinally Δ-complete space is almost expandable [**16**, Theorem 2.2], every θ-refinable cofinally Δ-complete space is metacompact. Thus an affirmative answer to Question 6 would yield an affirmative answer to Question 5; and, as H. Junnila observed in [**15**], an affirmative answer to Question 6 would also yield an affirmative answer to the following well-known outstanding question. Note that Junnila's observation follows from the fact that every strict p-space is cofinally θ-complete.

Question 7. Is every strict p-space θ-refinable?

The following two propositions support the point of view that Questions 5, 6 and 7 have affirmative answers. The first of these propositions is due to Junnila [**15**, Theorem 2.3]; the second is an immediate consequence of [**16**, Theorems 1.2 and 2.3].

PROPOSITION 2.1. *A space X is θ-refinable if and only if, corresponding to each open cover $\mathscr{C}$ of X, there is a sequence $\langle \mathscr{G}_n \rangle$ of open covers of X so that for each $x \in X$ there is a natural number $n(x)$ and a finite subcollection $\mathscr{C}(x)$ of $\mathscr{C}$ so that* $\operatorname{st}(x, \mathscr{G}_{n(x)}) \subset \bigcup \mathscr{C}(x)$ *and* $x \in \bigcap \mathscr{C}(x)$.

PROPOSITION 2.2. *Every cofinally θ-complete space is countably metacompact.*

There is some further evidence in support of an affirmative answer to Question 6. It is known, for example, that every locally compact cofinally θ-complete space is θ-refinable [**3**, Proposition 2]. In order to admit one further piece of supporting evidence, which is a straightforward adaptation of a result of Junnila [**8**, Lemma 5.39], we need the following definition.

A space X is preorthocompact provided that for each open cover $\mathscr{C}$ of X there is a reflexive relation V on X so that, for each $x \in X$, $V(x)$ is open and $V \circ V(x)$ is a refiner of $\mathscr{C}$.

The reader may verify the following proposition, using Proposition 2.1 and the method of proof of the result of Junnila mentioned above. It is useful, for the proof of Proposition 3.2, to note also that it follows from this method of proof that, in a cofinally θ-complete space, every interior-preserving open cover $\mathscr{C}$ has a sequence $\langle \mathscr{G}_n \rangle$ of open covers satisfying the conditions of Proposition 2.1.

PROPOSITION 2.3. *Every preorthocompact cofinally θ-complete space is* θ-refinable.

It is known that the space F_0 is not preorthocompact; there are two ways to obtain this result. First, we may use the result of Corson, that F_0 is almost 2-fully normal and not 2-fully normal, and the result of Junnila [**14**, Theorem 2.2.10] that every almost 2-fully normal preorthocompact space is 2-fully normal. A second approach is given in [**8**, Example 5.22]. Following S.A. Peregudov [**20**], but departing from his terminology, we say that a space X is semi-metacompact provided that every open cover $\mathscr{C}$ of X has an open refinement $\mathscr{R}$ so that no non-empty open subset of X is a subset of infinitely many members of $\mathscr{R}$. It is known that F_0 is semi-metacompact, and Junnila showed that every semi-metacompact preorthocompact space is metacompact [**14**]. Thus F_0 is a semi-metacompact space that is neither cofinally θ-complete nor preorthocompact. Curiously, even the following special part of Question 5 appears to be unanswered.

Question 8. Is every cofinally Δ-complete space semi-metacompact?

A space X is nearly metacompact [**12**] provided that for each open cover $\mathscr{C}$ of X there is a dense set D and an open refinement $\mathscr{R}$ of $\mathscr{C}$ so that $\mathscr{R}$ is point finite on D. Evidently, every nearly metacompact space is semi-metacompact; but, as the space F_0 indicates, the converse fails (see Proposition 2.4). A well-known construction of B. Scott ([**8**, §5.14] and [**12**, 234–235]) associates with any infinite space X a nearly metacompact space $\mathscr{G}(X)$ that contains X as a closed subspace and is metacompact whenever X is metacompact.

The following proposition makes use of a characterization of almost 2-full normality due to G. Aquaro [**1**]. A normal space X is almost 2-fully normal if and only if for each open cover $\mathscr{C}$ of X there exists a locally finite open cover $\mathscr{V}$ of X so that every two-element refiner of $\mathscr{V}$ is a refiner of $\mathscr{C}$.

PROPOSITION 2.4. *Every almost* 2-*fully normal nearly metacompact space is paracompact.*

PROOF. Let X be an almost 2-fully normal nearly metacompact space and let $\mathscr{C}$ be an open cover of X. There is a dense set D and an open refinement $\mathscr{R}$ of $\mathscr{C}$ so that $\mathscr{R}$ is point finite on D. There is a locally finite open cover $\mathscr{G}$ of X such that every two-element refiner of $\mathscr{G}$ is a refiner of $\mathscr{R}$. Let $G \in \mathscr{G}$. There exists $d \in G \cap D$. Let $x \in G$. Then there exists $R \in \mathscr{R}$ such that $\{x, d\} \subset R$. It follows that, for each $G \in \mathscr{G}$, there is a finite subcollection $\mathscr{C}(G)$ of $\mathscr{C}$ so that $G \subset \bigcup \mathscr{C}(G)$. Thus $\{G \cap C: C \in \mathscr{C}(G)$ and $G \in \mathscr{G}\}$ is a locally finite open refinement of $\mathscr{C}$.

The method of proof of Proposition 2.4 establishes the following proposition.

PROPOSITION 2.5. *Every separable almost 2-fully normal space is countably paracompact.*

PROOF. Let X be a separable almost 2-fully normal space, let $\mathscr{C} = \{C_n: n \in \omega\}$ be a countable open cover of X, and let $D = \{d_n: n \in \omega\}$ be a countable dense subset of X. For each $j \in \omega$, set $R_j = C_j - \{d_n: n \leqq j$ and $d_n \in \bigcup_{i<j} C_i\}$. Then $\mathscr{R} = \{R_n: n \in \omega\}$ is an open refinement of $\mathscr{C}$ that is point finite on D; and, as in the preceding proof, $\mathscr{C}$ has a locally finite open refinement.

With regard to the previous proposition, it is interesting to note that there is a separable almost 2-fully normal space that is neither weakly orthocompact nor almost realcompact **[18]**.

Since F_0 is not preorthocompact, it appears that a reasonable start toward righting Kelley's conjecture is to add (pre)orthocompactness as a hypothesis. Indeed, while F_0 is a Dieudonné complete almost 2-fully normal space that is not preorthocompact and ω_1 is an orthocompact almost 2-fully normal space that is not Dieudonné complete, the authors have not found a preorthocompact space in which Kelley's conjecture fails. The following proposition indicates a surprising parallel between cofinal Δ-completeness and Dieudonné completeness in the class of almost 2-fully normal spaces.

PROPOSITION 2.6. *Let X be an almost 2-fully normal space. Then X is Dieudonné complete if and only if whenever $\mathscr{U}$ is an ultrafilter without a cluster point, the directed open cover $\{X - \bar{U}: U \in \mathscr{U}\}$ has an open point-star refinement.*

3. Point-star Covering Properties. A topological space X is point-star (compact, paracompact, metacompact, orthocompact) provided that, if $\mathscr{C}$ is an open cover of X, there is a finite (locally finite, point-finite, interior-preserving) open refinement $\mathscr{R}$ of $\mathscr{C}^*$ so that, for each $x \in X$, there exists $R(x) \in \mathscr{R}$ so that $x \in R(x) \subset \operatorname{st}(x, \mathscr{C})$.

There is an interesting characterization, in terms of quasi-uniformities, of the point-star covering properties we are considering. We let $\mathscr{S}$ denote the collection of all neighborhoods of the diagonal of a space X, and as usual we let $\mathscr{P}$, $\mathscr{PF}$, $\mathscr{LF}$, and $\mathscr{FT}$ denote the Pervin, point-finite, locally finite, and fine transitive quasi-uniformities of X.

PROPOSITION 3.1. *A space X is pointwise star-compact (-paracompact,*

-metacompact, -orthocompact) if and only if $\mathscr{S} \subset \mathscr{P}$ ($\mathscr{S} \subset \mathscr{L}\mathscr{F}$, $\mathscr{S} \subset \mathscr{P}\mathscr{F}$, $\mathscr{S} \subset \mathscr{F}\mathscr{T}$).

In **[14]**, Junnila defined a space X to be discretely orthocompact provided that, whenever $\mathscr{F}$ is a discrete family of closed subsets of X and for each $F \in \mathscr{F}$, U_F is an open set containing F, there exists an interior-preserving open family $\{V_F: F \in \mathscr{F}\}$ such that $F \subset V_F \subset U_F$ for each $F \in \mathscr{F}$. By **[14**, Proposition 2.3.8], a θ-refinable space is orthocompact if and only if it is discretely orthocompact. The following lemma, which is a strengthening of **[2**, Theorem 3], makes use of **[14** Proposition 2.3.8] and is comparable to it.

LEMMA. *A θ-refinable space is orthocompact if and only if it is point-star orthocompact.*

PROOF. Only one implication requires proof. Let X be a θ-refinable point-star orthocompact space. It suffices to show that X is discretely orthocompact. Let $\mathscr{F}$ be a discrete family of closed subsets of X and, for each $F \in \mathscr{F}$, let U_F be an open set containing F. We assume, without loss of generality, that for each $F \in \mathscr{F}$, U_F meets only one member of $\mathscr{F}$. Set $\mathscr{C} = \{U_F: F \in \mathscr{F}\} \cup \{X - \bigcup \mathscr{F}\}$. There is an interior-preserving open refinement $\mathscr{R}$ of $\mathscr{C}^*$ so that, for each $x \in X$, there exists $R(x) \in \mathscr{R}$ such that $x \in R(x) \subset \operatorname{st}(x, \mathscr{C})$. For each $F \in \mathscr{F}$ set $V(F) = \bigcup\{R(x): x \in F\}$. The collection $\{V(F): F \in \mathscr{F}\}$ is the required interior-preserving family.

PROPOSITION 3.2. *Let X be a cofinally θ-complete point-star orthocompact space. Then X is θ-refinable and hence orthocompact.*

PROOF. Let $\mathscr{U}$ be a well-monotone open cover of X. The proof proceeds as in the proof of Theorem 3.2 of **[15]** except that the definition of the open covers $\mathscr{V}_s$ is modified in the following way. Since X is pointwise orthocompact for each open cover $\mathscr{L}_s$, there is a transitive relation T_s so that, for each $x \in X$, $T_s(x)$ is an open set about x contained in $\operatorname{st}(x, \mathscr{L}_s)$. In the definition of $V_\alpha(s \oplus \kappa)$ and $V_{\gamma+\alpha}(s \oplus \kappa)$ replace

$$\text{“}\operatorname{St}(X \sim \bigcup_{\beta \neq \gamma+\alpha} V_\beta(s), \mathscr{L}_{s\oplus k})^{\circ}\text{” by “}T_{s\oplus k}(X \sim \bigcup_{\beta \neq \gamma+\alpha} V_\beta(s))\text{”}$$

and

$$\text{“}\operatorname{St}(X \sim \bigcup_{\beta < \gamma+\alpha} V_\beta(s), \mathscr{L}_{s\oplus k})^{\circ}\text{” by “}T_{s\oplus k}(X \sim \bigcup_{\beta < \gamma+\alpha} V_\beta(s))\text{”}.$$

The proof given by Junnila also establishes that the modified collection $\mathscr{V}'_{s\oplus k}$ is an open cover, which is obviously interior preserving. It follows from the remark preceding Proposition 2.3 that the covers $\mathscr{L}_s$ may be chosen open and that the induction argument remains valid. Just as in **[15]**, $\bigcup\{\mathscr{V}'_s$: s is a finite sequence of natural numbers$\}$ is a θ-refinement of $\mathscr{U}$.

The authors believe that, of the point-star covering properties we have been considering, only point-star orthocompactness is well known ([**2**] and [**9**]); but the concept of point-star covering properties originates with Junnila [**14**]. Indeed, Junnila defines the following property, which we call point-star preorthocompactness. A space X is point-star preorthocompact provided that, if $\mathscr{C}$ is an open cover of X, there is a reflexive relation V on X so that, for each $x \in X$, $V(x)$ is open and $V^2(x) \subset \mathrm{st}(x, \mathscr{C})$. Two important results of [**14**] relating point-star preorthocompactness and metacompactness are that a completely regular space X is metacompact if and only if $X \times \beta X$ is point-star preorthocompact and that every cofinally Δ-complete point-star preorthocompact space is metacompact [**14**, Corollary 1.2.9 and Theorem 1.2.6]. Neither preorthocompactness nor point-star preorthocompactness has a natural characterization in terms of quasi-uniformities. It is known that each point-star covering property under discussion is strictly weaker than its corresponding covering property and that no two of these properties coincide. Using Aquaro's characterization of almost 2-full normality, we see that every almost 2-fully normal space is point-star paracompact; our interest in the point-star covering properties is primarily motivated by this result.

In the theory of almost 2-fully normal spaces, the following generalization of preorthocompactness is important because, unlike preorthocompactness, it is implied by almost 2-full normality. We say that a space X is (countably) almost preorthocompact provided that, if $\mathscr{C}$ is a (countable) open cover of X, there is a reflexive relation V on X so that, for each $z \in X$, $V(z)$ is open and whenever $y \in V \circ V(z)$ and $x \in V(z)$, $\{x, y\}$ is a refiner of $\mathscr{C}$. It is easy to verify that every almost preorthocompact space is point-star preorthocompact.

Propositions 2.4 and 2.5 follow readily from Proposition 3.3 and the corollary to Proposition 3.4. Whereas the proofs of Propositions 2.4 and 2.5 are relatively straightforward, our proof of Proposition 3.3, which generalizes Lemma 1 of [**12**], makes use of one of the deep results of Junnila mentioned above.

PROPOSITION 3.3. *A nearly metacompact space is metacompact if and only if it is almost preorthocompact.*

PROOF. Let X be an almost preorthocompact nearly metacompact space. Since X is point-star preorthocompact, in order to show that X is metacompact it suffices to show that X is cofinally Δ-complete. To this end, let $\mathscr{C}$ be a directed open cover of X and let $\mathscr{R}$ be an open refinement of $\mathscr{C}$ that is point finite on a dense set D. There is a reflexive relation V on X such that, for each $x \in X$, $V(x)$ is open and such that $\{y, z\}$ is a refiner of $\mathscr{R}$ whenever there is an $x \in X$ so that $y \in V(x)$ and $z \in V^2(x)$. Set $\mathscr{V} = \{V(x): x \in X\}$ and let $x \in X$. Let $d \in V(x) \cap D$ and let $z \in \mathrm{st}(x, \mathscr{V})$. Since

$\{x, z\} \subset V(a)$ and $d \in V^2(a)$ for some $a \in X$, $\{d, z\}$ is a refiner of $\mathscr{R}$. It follows that $\mathrm{st}(x, \mathscr{V}) \subset \mathrm{st}(d, \mathscr{R})$ and so $\mathrm{st}(x, \mathscr{V})$ is a refiner of $\mathscr{C}$.

The following proposition is comparable with Proposition 5.13 of **[8]** and Remark 1 of **[12]**.

PROPOSITION 3.4. *Every countably nearly metacompact countably almost preorthocompact space is countably metacompact.*

PROOF. Let $G_0 = \varnothing$ and let $\mathscr{G} = \{G_0, G_1, G_2, \ldots\}$ be a countable increasing open cover of a countably nearly metacompact countably almost preorthocompact space X. In order to show that X is countably metacompact, it suffices to see, as in the proof of Proposition 3.3, that $\mathscr{G}$ has an open point-star refinement. For if $\mathscr{V}$ is an open point-star refinement of $\mathscr{G}$, then $\{G_n \cap \mathrm{st}(X - G_{n-1}, \mathscr{V})\colon n \in \omega\}$ is a point-finite open refinement of $\mathscr{G}$.

COROLLARY. *Every separable countably almost preorthocompact space is countably metacompact.*

PROOF. Every separable space is countably nearly metacompact.

PROPOSITION 3.5. *Let X be a normal space. The following statements are equaivalent.*

(a) *For each open cover $\mathscr{C}$ of X there is a closed cover $\{F_\alpha\colon \alpha \in A\}$ and an open locally finite cover $\{G_\alpha\colon \alpha \in A\}$ so that, for each $x \in X$, there exists $\alpha(x) \in A$ such that $x \in F_{\alpha(x)} \subset G_{\alpha(x)} \subset \mathrm{st}(x, \mathscr{C})$.*

(b) *X is almost preorthocompact and point-star paracompact.*

(c) *X is almost 2-fully normal.*

PROOF. (a) $\Rightarrow$ (b). It is evident that condition (a) implies that X is point-star paracompact. We show that X is almost preorthocompact. Let $\mathscr{C}$ be an open cover of X. By hypothesis, there is a closed cover $\{F_\alpha\colon \alpha \in A\}$ and an open locally finite cover $\{G_\alpha\colon \alpha \in A\}$ so that for each $x \in X$ there exists $\alpha(x) \in A$ so that $x \in F_{\alpha(x)} \subset G_{\alpha(x)} \subset \mathrm{st}(x, \mathscr{C})$. For each $\alpha \in A$ set $B_\alpha = \{x \in F_\alpha\colon G_\alpha \subset \mathrm{st}(x, \mathscr{C})\}$ and set $V = \bigcup_{x \in X}\{x\} \times [\bigcap\{G_\alpha\colon x \in \bar{B}_\alpha\} - \bigcup\{\bar{B}_\alpha\colon x \notin \bar{B}_\alpha\}]$. Let $z \in X$, let $x \in V(z)$, and let $y \in V \circ V(z)$. There exists $b \in V(z)$ so that $y \in V(b)$. Let $\alpha \in A$. Since $y \in V(b)$, if $y \in \bar{B}_\alpha$ so is $b \in \bar{B}_\alpha$; and, since $b \in V(z)$, if $b \in \bar{B}_\alpha$ so is $z \in \bar{B}_\alpha$. Thus $x \in V(z) \subset \bigcap\{G_\alpha\colon z \in \bar{B}_\alpha\} \subset \bigcap\{G_\alpha\colon y \in \bar{B}_\alpha\}$. There is an $\alpha \in A$ so that $y \in B_\alpha \subset \bar{B}_\alpha \subset F_\alpha \subset G_\alpha \subset \mathrm{st}(y, \mathscr{C})$. Thus $x \in \mathrm{st}(y, \mathscr{C})$ and so $\{x, y\}$ is a refiner of $\mathscr{C}$.

(b) $\Rightarrow$ (c). Let $\mathscr{C}$ be an open cover of X. There is a reflexive relation V on X so that, for each $x \in X$, $V(x)$ is an open set and such that, whenever $y \in V \circ V(z)$ and $x \in V(z)$, the set $\{x, y\}$ is a refiner of $\mathscr{C}$. Set $\mathscr{V} = \{\mathscr{V}(x)\colon x \in X\}$. There is a locally finite open refinement $\mathscr{R} = \{R_\alpha\colon \alpha \in A\}$ of $\mathscr{V}^*$ such that, for each $x \in X$, there is an $\alpha(x) \in A$ so that $x \in R_{\alpha(x)} \subset \mathrm{st}(x, \mathscr{V})$. For each $\alpha \in A$, set $B_\alpha = \{y \in R_\alpha\colon R_\alpha \subset \mathrm{st}(y, \mathscr{C})\}$ and set $\mathscr{B} = \{\mathrm{int}(B_\alpha)\colon$

$\alpha \in A\}$. We first show that $\mathscr{B}$ is a (locally finite open) cover of X. Let $x \in X$. There is an $\alpha \in A$ so that $x \in R_\alpha \subset \mathrm{st}(x, \mathscr{V})$. We show that $V(x) \cap R_\alpha \subset B_\alpha$, whence $x \in \mathrm{int}(B_\alpha)$. Let $y \in V(x) \cap R_\alpha$ and let $a \in \bigcup\{V(z): x \in V(z)\} = \mathrm{st}(x, \mathscr{V})$. There exists $z \in X$ so that $\{a, x\} \subset V(z)$ and $y \in V(x)$. Thus $\{a, y\}$ is a refiner of $\mathscr{C}$ and so $\mathrm{st}(x, \mathscr{V}) \subset \mathrm{st}(y, \mathscr{C})$. Thus $y \in B_\alpha$.

Evidently every two-element refiner of the cover $\mathscr{B}$ is a refiner of $\mathscr{C}$ and so, by the characterization of Aquaro, X is almost 2-fully normal.

(c) $\Rightarrow$ (a). Let $\mathscr{C}$ be an open cover of X and let $\mathscr{R} = \{R_\alpha: \alpha \in A\}$ be a locally finite open cover such that every two-element refiner of $\mathscr{R}$ is a refiner of $\mathscr{C}$. Since X is normal, there is an open cover $\mathscr{H} = \{H_\alpha: \alpha \in A\}$ so that, for each $\alpha \in A$, $\bar{H}_\alpha \subset R_\alpha$. Thus, for each $x \in X$ there exists $\alpha \in A$ so that $x \in \bar{H}_\alpha \subset R_\alpha \subset \mathrm{st}(x, \mathscr{C})$.

The collectionwise normal space attributed to R.H. Bing by H.J. Cohen **[4]** is not point-star paracompact. Indeed, this space is not point-star preorthocompact. Nonetheless, point-star paracompactness and collectionwise normality are related concepts.

Proposition 3.6. *Every normal point-star paracompact space is collectionwise normal.*

Proof. Let X be a normal point-star paracompact space, let $\{F_\alpha: \alpha \in A\}$ be a discrete collection of closed sets, and let $F = \bigcup\{F_\alpha: \alpha \in A\}$. For each $\alpha \in A$ set $G_\alpha = X - \bigcup\{F_\beta: \beta \neq \alpha\}$. Since X is normal, for each $\alpha \in A$ there is an open set H_α so that $F_\alpha \subset H_\alpha \subset \bar{H}_\alpha \subset G_\alpha$. Let $\mathscr{C} = \{X - F\} \cup \{H_\alpha: \alpha \in A\}$. There is a locally finite open cover $\mathscr{L}$ so that for each $x \in X$ there exists $L_x \in \mathscr{L}$ so that $x \in L_x \subset \mathrm{st}(x, \mathscr{C})$. Set $\mathscr{L}' = \{L_x: x \in F\}$, for each $\alpha \in A$ set $R_\alpha = X - \overline{\mathrm{st}(\bigcup_{\beta\neq\alpha} F_\beta, \mathscr{L}')}$ and set $B_\alpha = R_\alpha \cap \mathrm{st}(F_\alpha, \mathscr{L}')$. Then for each $\alpha \in A$, $F_\alpha \subset B_\alpha$ and $B_\alpha \cap B_\beta = \varnothing$ whenever $\alpha \neq \beta$.

The terminology of **[14]** suggests that a space is almost preorthocompact if it is point-star preorthocompact. A negative answer to our last question would provide a counterexample to such a conjecture as a by-product. A straightforward modification of the proofs of Lemma 2.3.7 and Proposition 2.3.8 of **[14]** establishes that preorthocompactness and point-star preorthocompactness coincide in θ-refinable spaces. Thus the conjecture holds in θ-refinable spaces. Note that Proposition 2.4 holds with "point-star paracompact" replacing "almost 2-fully normal".

Question 9. Is every collectionwise normal point-star paracompact space almost 2-fully normal?

Proposition 3.7. *Every cofinally $\delta\theta$-complete point-star paracompact space is paracompact.*

Proof. Let $\mathscr{C}$ be an open cover of a cofinally $\delta\theta$-complete point-star paracompact space X. There is a sequence $\langle \mathscr{G}_n \rangle$ of open covers of X so that for each $x \in X$ there is $n(x) \in \omega$ and a countable subset $\mathscr{C}(x)$ of $\mathscr{C}$ so that $\operatorname{st}(x, \mathscr{G}_{n(x)}) \subset \bigcup \mathscr{C}(x)$. Without loss of generality, we assume that each $\mathscr{C}(x)$ is countably infinite and index the members thereof as $\{C(i, x): i \in \omega\}$. For each $n \in \omega$, there is a locally finite open cover $\mathscr{H}_n$ so that, for each $x \in X$, there is $H(n, x) \in \mathscr{H}_n$ so that $x \in H(n, x) \subset \operatorname{st}(x, \mathscr{G}_n)$. For each $i \in \omega$, set $\mathscr{H}'_i = \{H(i, x): x \in X \text{ and } n(x) = i\}$. For each $H \in \mathscr{H}'_i$ choose one $x \in X$ so that $H \subset \bigcup\{C(j, x): j \in \omega\}$ and set $x = x(H)$. For each $(i, j) \in \omega \times \omega$ set $M(i, j) = \{H \cap C(j, x(H)): H \in \mathscr{H}'_i\}$. Then $\bigcup\{M(i, j): i, j \in \omega\}$ is a σ-locally finite open refinement of $\mathscr{C}$.

The previous proposition is a slight extension of J. Worrell's assertion that every 2-fully normal $\delta\theta$-refinable space is paracompact [**23**, 431] (see also [**22**]). K.P. Hart has shown that M.E. Rudin's Dowker space is orthocompact and almost 2-fully normal [**10**, **11**] and, using $\diamondsuit^{++}$, M.E. Rudin has recently constructed another Dowker space that is collectionwise normal and metaLindelöf [**21**]. It follows that a space may be orthocompact and 2-fully normal or cofinally $\delta\theta$-complete and collectionwise normal and yet fail to be countably paracompact.

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ADDED IN PROOF. It is not difficult to see that in Proposition 2.5 and Corollary 3.4 the condition, separability, can be replaced (in a regular Hausdorff space) by the condition of being weakly Lindelöf. The reader may also find the following two papers pertinent:

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