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METRIC TRANSFORMATIONS OF THE REAL LINE

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1. A metric transformation between two metric (or semi-metric) spaces M_1 and M_2 is defined to be a function f such that for some function $\rho: \mathbf{R}^+ \to \mathbf{R}^+$, called the scale function associated with f, $\rho(d_1(x, y)) = d_2(f(x), f(y))$, where $x, y \in M$. The set $f(M_1)$ is said to be a metric transform of M_1. In this paper all metric transforms from the real line in Euclidean n-space are characterized.

The notion of a metric transformation was introduced by Wilson **[10]** in 1935. In 1938 von-Neumann and Schoenberg **[8]** characterized all continuous metric transformations of the real line, $\mathbf{R}$, into Hilbert space. This powerful result shows that the scale functions ρ corresponding to such transformations are those, and only those, functions which satisfy the condition

$$\rho^2(t) = \left(\int_0^\infty \frac{\sin^2 tu}{u^2}\, d\alpha(u)\right),$$

where α is non-decreasing and $\int_1^\infty u^{-2} d\alpha(u) < \infty$. They also showed that, in order that $f(\mathbf{R})$ be embeddable in $\mathbf{E}^n$ (finite dimensional Hilbert space), it is necessary and sufficient that α increase at only a finite number of points. In this case

$$\rho^2(t) = \sum_1^m A_i^2 \sin^2 k_i t + c^2 t^2,$$

and in a suitable coordinate system,

(1) $\quad f(t) = (A_1 \cos k_1 t, A_1 \sin k_1 t, \ldots, A_m \cos k_m t, A_m \sin k_m t, \mathrm{ct})$

If $f(\mathbf{R})$ is embeddable in E^n, but not in E^{n-1}, then, for n odd, $2m = n - 1$ and $c \neq 0$, while $2m = n$ and $c = 0$ for n even. As a helix is typical, von-Neumann and Schoenberg refer to continuous metric transforms of $\mathbf{R}$ as screw curves.

Metric transformations, including the von-Neumann and Schoenberg result, have appeared in the literature of late in connection with a method of data analysis known as Multidimensional Scaling. (See **[1]**, **[3]**, **[6]** and **[7]**). Here one takes a semi-metric space M_1 and some other metric

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or semi-metric space M_2, such as E^n, and attempts to construct a metric transformation of M_1 into M_2 with the scale function ρ being strictly monotone. In this case M_1 is said to be order embeddable into M_2 and f is called an order transformation. Once this has been accomplished one might ask if the result, in some sense, is unique.

Beals, Krantz and Tversky [1] have given necessary and sufficient conditions for a semimetric space M_1 to be order embeddable into a convex metric space. They show the embedding is unique up to a similarity. Kelly is credited in [6] with the classification of all those semimetric spaces of $n + 2$ points which are order embeddable into E^n. Erdös and Kelly [3] have shown that, for m sufficiently large, there are semimetric spaces of m points not order embeddable into E^n. Lew [7] uses (1) to show that ℓ_m^∞ and ℓ_n^1 are not order embeddable in E^n, for any n, and von-Neuman and Schoenberg [8] use (1) to show that any continuous metric transformation of E^n into E^m is either a similarity or maps E^m to a single point of E^n. The characterization (1) would seem to be fundamental in the study of metric and order transformations, particularly for uniqueness properties.

In this paper we consider all metric transforms of **R** into E^n, including those which are discontinuous. To illustrate our result we present the following three examples, the first of which is due to Vogt [9].

EXAMPLE 1. Let f: $\mathbf{R} \to \mathbf{R}$ be a group homomorphism. Then $d(f(x), f(y)) = |f(x) - f(y)| = |f(|x-y|)| = \rho(d(x, y))$, showing that f is a metric transformation.

More generally, let M be any normed linear space, and let f: $\mathbf{R} \to M$ be a group homomorphism. Then $\|f(x) - f(y)\| = \|f(|x - y|)\|$, again showing that f is a metric transformation.

REMARK. A group homomorphism f: $\mathbf{R} \to M$ (M a vector space) is simply a function satisfying $f(a + b) = f(a) + f(b)$. G. Hamel [4] showed that one method of constructing such functions is to consider **R** and M as vector spaces over the rationals. If $A \subseteq \mathbf{R}$ is a basis for **R**, as a vector space over the rationals, and f: $A \to M$ is arbitrarily defined, then f can be extended by linearity to **R**. The resulting function is clearly a group homomorphism, and hence a metric transformation. Of interest is that if $B \subseteq M$ is a basis for M, as a vector space over the rationals, and if the cardinality of A and B are the same (the cardinality of the continuum) then there are functions from A onto B and hence metric transformations from **R** onto M. In particular there are metric transformations of **R** onto any separable normed linear space.

Halperin [4] used the above type of construction to show that there are discontinuous functions f: $\mathbf{R} \to \mathbf{R}$ which satisfy the intermediate value theorem. In fact he produced a group homomorphism f: $\mathbf{R} \to \mathbf{R}$ such that $f((a, b)) = \mathbf{R}$ for $a < b$. It follows that there are metric transformations of $\mathbf{R}$ onto any separable normed linear space M such that $f((a, b)) = M$ for $a < b$.

EXAMPLE 2. Let C be the unit circle in E^2, and let θ: $\mathbf{R} \to \mathbf{R}/2\pi$ be any group homomorphism. Then the function $t \to (\cos \theta(t), \sin \theta(t))$ is a metric transformation of $\mathbf{R}$ to C. The scale function ρ is given by $\rho(d) = 2 \sin(\theta(d)/2)$.

EXAMPLE 3. Let f_i: $\mathbf{R} \to V_i$, $i = 1, \ldots, n$ be a metric transformation from $\mathbf{R}$ to a normed linear space V_i, let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$, with norm given by $\|(v_1, v_2, \ldots, v_n)\|^2 = \|v_1\|^2 + \|v_2\|^2 + \cdots \|v_n\|^2$, and let $f = (f_1, f_2, \ldots, f_n)$. Then f: $\mathbf{R} \to V$ is a metric transformation with scale function ρ, where $\rho^2 = \sum \rho_i^2$. Note that Example's 1 and 2 may be combined in this way.

In Theorem 1 we classify all metric transformations from $\mathbf{R}$ into E^n, whether continuous or not. In this respect, then, the result is stronger than the corresponding result of von-Neumann and Schoenberg [**8**] which assumes continuity; however, (general) Hilbert space has been replaced by E^n.

2. Definitions. A set H is said to be an m-flat of E^n if it is a translate of an m-dimensional subspace of E^n. A set S is said to span E^n it if lies in no $(n - 1)$-flat.

A (rigid) motion T of a metric space M is defined to be an isometry from M onto M. For $M = E^n$, it is a standard theorem of linear algebra that such a function can be written as $T(x) = U(x) + T(0)$ where U is an orthogonal transformation (that is, a linear norm preserving transformation of E^n). A set of motions $\{T_s | s \in \mathbf{R}\}$ of a metric space M satisfying $T_s \circ T_r = T_{s+r}$ is called a one-parameter subgroup of motions of M.

It follows immediately that for any one-parameter subgroup of motions $\{T_s | s \in \mathbf{R}\}$ and any s, $r \in \mathbf{R}$ we have $T_s \circ T_r \equiv T_r \circ T_s$, $(T_s)^n = T_{n \cdot s}$ and $(T_{s+r})^n = T_{n \cdot s} \circ T_{n \cdot r}$.

LEMMA 1. *If $B \subseteq E^n$ spans E^n, and $\tilde{T}$: $B \to E^n$ is an isometry, then there is a unique motion T: $E^n \to E^n$ such that $T|B = \tilde{T}$.*

PROOF. See [**2**, §38].

PROPOSITION 1. *Let f: $\mathbf{R} \to E^n$ be a metric transformation with scale function ρ, and assume that $f(\mathbf{R})$ spans E^n. Then there is a unique one-parameter subgroup of motions $\{T_s | s \in \mathbf{R}\}$ such that $f(s) = T_s(f(0))$.*

PROOF. For each $s \in \mathbf{R}$ define $\tilde{T}_s$: $f(\mathbf{R}) \to E^n$ by $\tilde{T}_s(f(t)) = f(t + s)$. As f is a metric transformation it follows that $\|\tilde{T}_s(f(t_1)) - \tilde{T}_s(f(t_2))\| = \|f(t_1 + s) - f(t_2 + s)\| = \rho(|t_1 - t_2|)$. By hypothesis, $f(\mathbf{R})$ spans E^n, hence

Lemma 1 shows $\tilde{T}_s$ can be uniquely extended to a motion T_s of E^n. Because of this uniquencess, and because $T_s \circ T_r(f(t)) = f(t + s + r) = T_{s+r}(f(t))$ it follows that $T_s \circ T_r = T_{s+r}$. Thus $\{T_s | s \in \mathbf{R}\}$ forms a one-parameter subgroup of motions of E^n, such that $f(s) = T_s(f(0))$.

If $\{R_s | s \in \mathbf{R}\}$ is any other one-parameter subgroup of motions such that $f(s) = R_s(f(0))$, then

$R_s(f(t)) = R_s R_t(f(0)) = R_{s+t} f(0)) = f(s + t) = T_{s+t}(f(0)) = T_s(f(t))$. As $f(\mathbf{R})$ spans E^n, Lemma 1 shows $R_s = T_s$.

The proof of the following result is straightforward. However since it is crucial to our argument, we include the details.

PROPOSITION 2. *If $\{T_s | s \in \mathbf{R}\}$ is a one-parameter subgroup of motions of E^n, then $T_s = U_s + T_s(0)$ where U_s is an orthogonal transformation of E^n, $\{U_s | s \in \mathbf{R}\}$ form a one parameter subgroup of motions of E^n, and for any $s, r \in \mathbf{R}$,*

$$(I - U_r)T_s(0) = (I - U_s)T_r(0). \tag{2}$$

PROOF. As mentioned earlier, T_s can be written as $T_s(x) = U_s(x) + T_s(0)$, where U_s is an orthogonal transformation.

As $T_r \circ T_s \equiv T_{s+r}$, it follows that $(U_{s+r} - U_s U_r)x = U_s(T_r(0)) + T_s(0) - T_{s+r}(0) =$ constant. As this is true for all x, the constant is 0, and hence $U_s U_r = U_{s+r}$ and $T_{s+r}(0) = U_s(T_r(0)) + T_s(0)$.

Similarly $U_r U_s = U_{s+r}$ and $T_{s+r}(0) = U_r(T_s(0)) + T_r(0)$. Thus $U_s(T_r(0)) + T_s(0) = U_r(T_s(0)) + T_r(0)$ or $(I - U_r)T_s(0) = (I - U_s)T_r(0)$, where I is the identity transformation.

PROPOSITION 3. *If $\{U_s | s \in \mathbf{R}\}$ is a one parameter subgroup of orthogonal transformations of E^n, then E^n can be written as $E^n = V_1 \oplus V_2 \oplus \cdots \oplus V_m \oplus W$, where V_j are two dimensional subspaces of E^n, V_j and W are invariant under U_s, for all s and j, $U_s | W = I_W$ for all s and $U_s | V_j$ has in any positively oriented orthonomal basis the matrix form*

$$M_{sj} = \begin{pmatrix} \cos \theta_j(s) & -\sin \theta_j(s) \\ \sin \theta_j(s) & \cos \theta_j(s) \end{pmatrix}.$$

For each j, there is an s, call it s_j, such that $U_{s_j} | V_j \neq I$, and the functions θ_j: $\mathbf{R} \to \mathbf{R}/2\pi$ are group homomorphisms.

PROOF. The bulk of the proof consists of applying standard techniques of linear algebra to the transformations $\{U_s | s \in \mathbf{R}\}$, so we shall omit it. That $\theta_j(s + r) = \theta_j(s) + \theta_j(r)$ (modulo 2π) follows from the fact that $M_{sj} M_{rj} = M_{(s+r)j}$.

PROPOSITION 4. *Let $\{T_s | s \in \mathbf{R}\}$ and $\{U_s | s \in \mathbf{R}\}$ be as in Proposition 2, and $V_1, \ldots, V_m$ and W as in Proposition 3. For each s, let $T_s(0) = T_{s1}(0) +$*

$\cdots + T_{sm}(0) + T_{sw}(0)$, *where* $T_{sj}(0) \in V_j$, $T_{sw}(0) \in W$. *Then there is a* $v \in V_1 \oplus \cdots \oplus V_m$ *such that for all* s, $T_s(x) = U_s(x - v) + v + T_{sw}(0)$.

PROOF. Let $U_{sj} = U_s | V_j$. It follows from Proposition 3 that $U_s | W = I$. Define $T_{sj}(x)$ and $T_{sw}(x)$ by

$$T_{sj}(x) = U_{sj}(x) + T_{sj}(0) \text{ and } T_{sw}(x) = x + T_{sw}(0).$$

If $x = x_1 + \cdots + x_m + x_w$, $x_j \in V_j$ and $x_w \in W$ it follows that

$$\begin{aligned} T_s(x) &= \sum T_{sj}(x_j) + T_{sw}(x_w). \\ &= \sum T_{sj}(x_j) + x_w + T_{sw}(0). \end{aligned}$$

By Proposition 3, for each j, there is an $s = s_j$ with $U_{s_j j} \neq I$. Thus $(I - U_{s_j j})^{-1}$ exists and we define $v_j = (I - U_{s_j j})^{-1} T_{s_j j}(0)$. It follows that $T_{s_j j}(x_j) = U_{s_j j}(x_j - v_j) + v_j$.

Using equation (2) and the fact that U_{s_j} and U_s commute for all s, it can now be shown that $T_{sj}(x_j) = U_{sj}(x_j - v_j) + v_j$ for all s, j. Letting $v = v_1 + \cdots + v_m$, it follows that $T_s(x) = U_s(x - v) + v + T_{sw}(0)$.

We are now prepared for the main Theorem of this paper.

THEOREM 1. *Let* $f\colon \mathbf{R} \to E^n$ *be a metric transformation such that* $\{f(t)\colon t \in \mathbf{R}\}$ *span* E^n. *Then there are complementary subspaces* V *and* W, *with orthogonal projections* $P_v\colon E^n \to V$ *and* $P_w\colon E^n \to W$ *respectively, and a vector* $u \in E^n$ *such that, if* $\tilde f(t) = f(t) - u$, $\tilde f_v = P_v \circ \tilde f$ *and* $\tilde f_w = P_w \circ \tilde f$, *then*

$$\tilde f_v(t) = (A_1 \cos \theta_1(t), A_1 \sin \theta_1(t), \ldots, A_m \cos \theta_m(t), A_m \sin \theta_m(t)),$$

where $A_j \geqq 0$ *are constants,* $\theta_j\colon \mathbf{R} \to \mathbf{R}/2\pi$ *are group homomorphisms, and* $\tilde f_w(t)$ *is a group homomorphism from* $\mathbf{R}$ *into* W.

Conversely, if $f\colon \mathbf{R} \to E^n$ *and there are complemetary subspaces* V *and* W *of* E^n *such that*

$$P_v \circ f(t) = (A_1 \cos \theta_1(t), A_1 \sin \theta_1(t), \ldots, A_m \cos \theta_m(t), A_m \sin \theta_m(t)),$$

where $\theta_j\colon \mathbf{R} \to \mathbf{R}/2\pi$ *are group homomorphisms and* $f_w = P_w \circ f$ *is a group homomorphism form* $\mathbf{R}$ *into* W, *then* f *is a metric transformation.*

PROOF OF CONVERSE. Let t_1 and t_2 be in $\mathbf{R}$. Then

$$\begin{aligned} \|f(t_1) - f(t_2)\|^2 &= \sum_{j=1}^m 4A_j^2 \sin^2\left(\frac{\theta_j(t_1) - \theta_j(t_2)}{2}\right) + \|f_w(t_1) - f_w(t_2)\|^2 \\ &= \sum_{j=1}^m 4A_j^2 \sin^2\left(\frac{\theta_j(t_1 - t_2)}{2}\right) + \|f_w(t_1 - t_2)\|^2 \\ &= \sum_{j=1}^m 4A_j^2 \sin^2\left(\pm\frac{\theta_j(|t_1 - t_2|)}{2}\right) + \| \pm f_w(|t_1 - t_2|)\|^2 \\ &= \sum_{j=1}^m 4A_j^2 \sin^2\left(\frac{\theta_j(|t_1 - t_2|)}{2}\right) + \|f_w(|t_1 - t_2|)\|^2. \end{aligned}$$

This shows that f is a metric transformation with scale function $\rho(d)$ satisfying

$$\rho^2(d) = \sum_{j=1}^{m} 4A_j^2 \sin^2 \frac{\theta_j(d)}{2} + \|g(d)\|^2.$$

PROOF OF THEOREM 1. Construct a one-parameter subgroup of motions $\{T_s | s \in \mathbf{R}\}$ and $\{U_s | s \in \mathbf{R}\}$ as in Proposition 1, and let v, $V_1, \ldots, V_m$, W, T_{sj}, T_{sw}, and U_{sj}, be as in Proposition 4. Let $V = V_1 \oplus \cdots \oplus V_m$, $f_w(0) = P_w(f(0))$ and $u = v + f_w(0)$. Consider the translation $\tilde{f}(\mathbf{R})$ of $f(\mathbf{R})$ given by $\tilde{f}(s) = f(s) - u$ and let $g(s) = T_{sw}(0)$. Note that $\tilde{f}$ is a metric transformation, and $g(s) \in W$. Then

$$\begin{aligned}\tilde{f}(s) &= T_s(f(0)) - v - f_w(0) \\ &= U_s(f(0) - v) + v + T_{sw}(0) - v - f_w(0) \\ &= U_s(\tilde{f}(0)) + g(s).\end{aligned}$$

It now follows from Proposition 1 that if $\tilde{T}_s(x) = U_s(x) + g(s)$, then $\{\tilde{T}(s) | s \in \mathbf{R}\}$ is the unique one-parameter subgroup of motions such that $\tilde{f}(s) = \tilde{T}_s(\tilde{f}(0))$.

Choose a positively oriented orthonormal basis in J_j, $j = 1, \ldots, m$ such that the projection of $\tilde{f}(0)$ into V_j has co-ordinates $(A_j, 0)$. Proposition 2 now shows that the matrix of U_{sj} in this basis is

$$\begin{pmatrix} \cos \theta_j(s) & -\sin \theta_j(s) \\ \sin \theta_j(s) & \cos \theta_j(s) \end{pmatrix}$$

for some $\theta_j(s)$, such that $\theta_j\colon \mathbf{R} \to \mathbf{R}/2\pi$ is a group homomorphism. Thus, $\tilde{f}_v(t) = (A_1 \cos \theta_1(t),\ A_1 \sin \theta_1(t),\ \ldots,\ A_m \cos \theta(t),\ A_m \sin \theta_m(t))$ and $\tilde{f}_w(t) = g(t)$. Using the fact that $\{\tilde{T}_s | s \in \mathbf{R}\}$ form a one parameter subgroup of motions such that $\tilde{T}_s(x) = x + g(s)$, for $x \in W$, it follows immediately that $g(s) + g(r) = g(s + r)$, and hence that $g\colon \mathbf{R} \to W$ is a group homomorphism.

REMARKS. The assumption in Theorem 1 that $\{f(t) | t \in \mathbf{R}\}$ spans E^n can easily be eliminated. For, otherwise, we need only consider the smallest flat in E^n containing $\{f(t) | t \in \mathbf{R}\}$ and perform the above analysis in that flat.

The von-Neumann-Schoenberg result in E^n, where $f(t)$ is continuous, follows easily from this. For, if f is continuous, then θ_j, $j = 1, \ldots, m$ and g must be continuous, in which case it is not difficult to conclude that $\theta_j(s) = ks$ (modulo 2π) and $g(s) = su$, u a fixed vector in W. This then gives the characterization of a metric transformation of R into E^n given in the von-Neumann-Schoenberg paper.

3. As mentioned earlier, this problem has arisen in connection with

Multidimensional Scaling. Specifically, if $f\colon M_1 \to E^n$ is a metric transformation, is $f(M_1)$ unique in some sense? For this type of question, a copy of $\mathbf{R}$ may not be available in M_1, hence Theorem 1 is not applicable. However, often M_1 contains an interval, that is a set isometric to an interval of $\mathbf{R}$. Thus it is natural to ask if Theorem 1 characterizes all metric transforms of intervals. Theorem 2 shows that indeed it does.

LEMMA 2. *Let E^m be an m-flat of E^n. Let $T\colon E^n \to E^n$ be an isometry which maps a spanning set of E^m into E^m. Then $T(E^m) = E^m$.*

PROOF. See [**2**, §40].

PROPOSITION 5. *Let $\{T_s \mid |s| < \delta\}$ be a set of motions of E^n satisfying $T_s \circ T_r = T_{s+r}$ whenever s, r and s + r are in $(-\delta, \delta)$. Then there is a unique one-parameter subgroup of motions, called $\{\bar{T}_s \mid s \in \mathbf{R}\}$ such that $\bar{T}_s \equiv T_s$, $|s| < \delta$.*

PROOF. For $s \in \mathbf{R}$ pick an integer m such that $s/m \in (-\delta, \delta)$, and define $\bar{T}_s$ by $\bar{T}_s = (T_{s/m})^m$. It is not hard to show that $\bar{T}_s$ is independent of the choice of m, and then that $\{\bar{T}_s\}$ is the unique set of motions extending $\{T_s\}$ to a one parameter subgroup.

THEOREM 2. *Let f be a metric transformation of $(-a, a)$ into E^n. Then f can be uniquely extended to a metric transformation $\bar{f}$ of $\mathbf{R}$ into E^n. If $E^m \subseteq E^n$, and $f((-a, a)) \subseteq E^m$, then $\bar{f}(\mathbf{R}) \subseteq E^m$.*

PROOF. *Case* I. Assume $f((-a, a))$ spans E^n. The case that it does not will be covered in II. Let $-a < t_0 \leqq t_1 \leqq \cdots \leqq t_n < a$ be such that $\{f(t_i)\}$ spans E^n, and let $\delta = \min\{a - t_n, t_0 + a\}$.

For each s, $|s| < \delta$, the function given by $f(t) \to f(t + s)$ is an isometry of $f([-a + \delta, a - \delta])$ into E^n, hence can be uniquely extended to motion T_s of E^n (Lemma 1). For s and r such that s, r, and $s + r$ are in $(-\delta, \delta)$,

$$T_s \circ T_r(f(t)) = f(t + s + r) = T_{s+r}(f(t)),$$

and hence $T_s \circ T_r = T_{s+r}$. Thus $\{T_s \mid |s| < \delta\}$ satisfies the hypotheses of Proposition 5, so there is a unique one parameter subgroup of motions $\{\bar{T}_s | s \in \mathbf{R}\}$ which extends $\{T_s \mid |s| < \delta\}$.

Define $\bar{f}(s)$ by $\bar{f}(s) = \bar{T}_s(f(0))$. Then it is easy to show that $\bar{f}$ is the unique extension of f to a metric transformation of $\mathbf{R}$ to E^n.

Case II. Consider now the case $f((-a, a))$ does not span E^n. Let E^n be the m-flat of E^n which contains, and is spanned by $f((-a, a))$. Let $\bar{f}$ be any extension of f to a metric transformation of $\mathbf{R}$ and assume $\bar{f}(\mathbf{R})$ spans the flat E'. (Case I shows there is at least one such extension.) As above, let δ be such that $f([-a + \delta, a - \delta)]$ spans E^m. As in Proposition

1, let $\{T_s\}$ be a one-parameter subgroup of motions of E' such that $T_s(f(t)) = f(t + s)$. For $|s| < \delta$,

$$T_s(f([-a + \delta, a - \delta])) = f([-a + \delta + s, a - \delta + s)] \subseteq E^m.$$

Thus, by Lemma 2, $T_s(E^m) = E^m$.

Since $\bar{f}(s) = T_s(f(0))$ and $f(0) \in E^m$ it follows that $E' = E^m$, and the uniqueness of the extension follows from Case I.

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