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WITT RINGS AND K-THEORY

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Dedicated to the memory of Gus Efroymson

ADDED IN PROOF. A version of this article was distributed by the author at the AMS-NATO Summer conference on Ordered Fields and Real Algebraic Geometry, held at Boulder, Colorado, July 3–9, 1983. A few months later, N. Schwartz discovered counterexamples to some of the Propositions. It turned out that what was needed was a better definition of the ring $C(X)$ of §3. N. Schwartz provided such a definition. Also, H. Delfs and M. Coste helped clarify the matter.

Here is the correction needed. In the notation of §3, an element of $C(X)$ should be a constructible continuous section $s: X \to X_{A[T]}$ such that the image $s(X)$ is relatively closed in $\pi^{-1}(X) \subset X_{A[T]}$. In fact, without this closedness condition, Proposition 3.3 is false in general, since clearly 3.3 implies $s(X)$ is closed. With the better definition, Proposition 3.7 and 3.9, as stated below, can also be improved. Specifically, Proposition 3.7 is true for any ring homomorphism $\gamma: A \to B$, as hoped for in Remark 3.8. Also, Proposition 3.9 is true for any integral extension $\gamma: A \to B$, so $C(X_A) \to C(X_{A/I})$ is surjective for any ideal $I \subset A$.

1. Introduction. The primary purpose of these notes is to outline a proof of the following result.

THEOREM. *Let A be any commutative ring, X_A its real spectrum. Then there is a natural ring homomorphism $W(A) \to KO(X_A)$ with both kernel and cokernel 2-torsion groups.*

Here, $W(A)$ is the Witt ring of A, and $KO(X_A)$ is something like the real K-theory of a topological space. Much of the paper is devoted to the definition and properties of $KO(X)$ for any constructible subset $X \subseteq X_A$. In fact, four "definitions" are given, all isomorphic, and each a direct analogue of a construction of the classical real K-theory of a compact (= quasi-compact and Hausdorff topological) space X. More about these four constructions below.

Here are some special cases and applications of the theorem. First, if $A = k$ is a field, then it turns out that $KO(X_k) = \mathrm{Cont}(X_k, \mathbf{Z})$, the ring of

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continuous maps from the space X_k of orders on k to the integers. More generally, for any A, there is a dimension homomorphism $KO(X_A) \to \mathrm{Cont}(X_A, \mathbf{Z})$, which is trivially surjective. The composition $W(A) \to KO(X_A) \to \mathrm{Cont}(X_A, \mathbf{Z})$ is just the global signature. Thus the theorem includes the result of Pfister that the global signature $W(k) \to \mathrm{Cont}(X_k, \mathbf{Z})$ is an isomorphism modulo 2-torsion for a field k, and includes the result of Mahé that the global signature $W(A) \to \mathrm{Cont}(X_A, \mathbf{Z})$ is surjective modulo 2-torsion for any A.

At this point, I want to pause and record my primary debt in this whole project. Namely, the results and methods of Mahé really come very close to establishing the theorem above. Not only that, but when I was in Rennes in 1981, Mahé already mentioned to me the likelihood of such a relationship between Witt rings and K-theory for real varieties. Mahé's result, of course, established the Knebusch conjecture that the components of a real affine variety could be separated by signatures of quadratic forms. If $V \subset \mathbf{R}^n$ is any real affine variety, and $A = A(V) = \mathbf{R}[x_1 \cdots x_n]/I(V)$ is its affine coordinate ring, then it turns out that $KO(X_A) = KO(V)$, the ordinary topological real K-theory of V. Thus the theorem "computes" $W(A(V))$, up to 2-torsion, namely, $W(A(V)) \otimes \mathbf{Z}[1/2] \cong KO(V) \otimes \mathbf{Z}[1/2]$.

In fact, algebraic K-theorists define higher Witt groups, $W_n(A)$, $n \geqq 0$, with $W_0(A) = W(A)$. Karoubi has axiomatically characterized the functors which assign to A the localized groups $W_n(A) \otimes \mathbf{Z}[1/2]$, $n \geqq 0$. Once $KO(X_A)$ is defined, it is not difficult to continue imitating topological constructions and define higher KO-groups, $KO^{-n}(X_A)$, $n \geqq 0$. Then Karoubi's axioms can be verified for the functors $KO^{-n}(X_A) \otimes \mathbf{Z}[1/2]$, hence $W_n(A) \otimes \mathbf{Z}[1/2] \cong KO^{-n}(X_A) \otimes \mathbf{Z}[1/2]$ for all A and all $n \geqq 0$. Again, if $A = A(V)$ is the affine coordinate ring of a real variety, this gives the computation $W_n(A(V)) \otimes \mathbf{Z}[1/2] \cong KO^{-n}(V) \otimes \mathbf{Z}[1/2]$ of the higher Witt rings of $A(V)$, up to 2-torsion, in terms of the classical real K-theory of (suspensions of) V. Since the theory applies to all commutative rings, for example fields, this extension of the theorem also implies the known computations of, say $W_n(k) \otimes \mathbf{Q}$, k a field. (The formula is $W_n(k) \otimes \mathbf{Q} = 0$ if $n \not\equiv 0 \pmod 4$, and $W_n(k) \otimes \mathbf{Q} = W(k) \otimes \mathbf{Q}$ if $n \equiv 0 \pmod 4$.) If k is a number field, then the rank of $W(k)$ is equal to the number of real places of k. In the context of the theorem, these results correspond to the fact that for a number field k, X_k is the finite, discrete space of real embeddings of k.

Here are some comments about the four constructions of $KO(X)$, $X \subseteq X_A$ constructible, and a rough outline of part of the paper. The first construction is based on copying the definition of real vector bundle, as a locally trivial family of real affine spaces $p^{-1}(\alpha)$, $p\colon E \to X$, $\alpha \in X$. The trivial bundle over X_A is, by definition, $\pi\colon X_{A[T_1 \cdots T_n]} \to X_A$, induced by the inclusion $A \to A[T_1 \cdots T_n]$. The trivial bundle over a subset $X \subseteq X_A$

is π: $\pi^{-1}(X) \to X$. The fibre over $\alpha \in X_A$ is, thus, the real spectrum $X_{k(\alpha)[T_1 \cdots T_n]}$, where the point $\alpha \in X_A$ corresponds to a homomorphism $A \to k(\alpha)$, $k(\alpha)$ a real closed field, algebraic over image (A). Of course, there is a standard inclusion of the usual affine space $k(\alpha)^n \subset X_{k(\alpha)[T_1 \cdots T_n]}$. For a locally trivial bundle over X, we need a suitable cover $X = \bigcup_i U_i$ and trivial bundles over the U_i, glued together over $U_i \cap U_j$. The gluing data amounts to a suitable choice of matrices $\varphi_{ij}(\alpha) \in GL_n(k(\alpha))$, $\alpha \in U_i \cap U_j$, varying continuously in α. (Actually, the fibre dimension n depends on α also, but is at least locally constant.) Note any matrix $\varphi \in GL_n(k(\alpha))$ induces a ring automorphism of $k(\alpha)[T_1 \cdots T_n]$, hence a space automorphism of $X_{k(\alpha)[T_1 \cdots T_n]}$which extends the standard linear automorphism of $k(\alpha)^n$ defined by the matrix φ.

Since a matrix over $k(\alpha)$ is just an array of elements of $k(\alpha)$, the definition of vector bundle sketched above pretty much takes care of itself, once the proper analogue of continuous real valued function is isolated. In our abstract context the proper notion is provided by the continuous, constructible sections of the projection $X_{A[T]} \to X_A$ over various constructible subsets $X \subseteq X_A$. Constructibility implies such sections take values in the fields $k(\alpha) \subset X_{k(\alpha)[T]}$. Thus, such sections form a ring $C(X)$, although seemingly simple results such as continuity of a sum or product are quite a bit more subtle than one might first imagine. The necessary properties of $C(X)$ are listed in §3 and the definition of vector bundles and $KO(X)$ is given in §4.

Strictly speaking, the notation, especially for the rings $C(X)$, should probably include the ring A. As indicated above, we use the sheaf-like nature of the collection of rings $C(U)$, $U \subseteq X$ open constructible, to define $KO(X)$. But it is plausible, and I think even rather likely, that $KO(X)$ depends only on the lattice of open sets in X. Specifically, a cover $\mathscr{U}$ of X has a nerve $N(\mathscr{U})$, a simplicial complex, and a refinement of covers $\mathscr{V} < \mathscr{U}$ induces a simplicial map $N(\mathscr{V}) \to N(\mathscr{U})$. I conjecture $KO(X) \cong \lim_{\mathscr{U}} KO(N(\mathscr{U}))$, the direct limit of the ordinary real K-theory of nerves of covers of X.

The second construction of $KO(X)$ is analogous to the homotopy classification of vector bundles in topology, specifically, every bundle over X is a subbundle of a trivial bundle. This provides maps from X to a certain space, roughly, the space of affine subspaces of the fibres $X_{k(\alpha)[T_1 \cdots T_N]}$ of $X_{A[T_1 \cdots T_N]} \to X_A$. Now, the Grassmann manifold $G_{n,N}(k(\alpha))$ of n-planes in N-space over $k(\alpha)$ can be identified with rank N-n symmetric idempotent matrices, or orthogonal projection operators P on N-space, by assigning to P the n-plane ker(p). Thus $G_{n,N}$ is defined in any affine N^2-space by equations over $\mathbf{Z}$. The topological notion of homotopy classes of maps to $G_{*,N}(\mathbf{R}) = \bigcup_{0 \leq n \leq N} G_{n,N}(\mathbf{R})$ is replaced by homotopy classes of continuous, constructible $G_{*,N}$-valued sections of $X_{A[T_1 \cdots T_{N^2}]} \to X_A$.

Homotopy makes perfect sense, because if $X \subseteq X_A$ is any constructible set, then there is an obvious constructible set in $X_{A[T]}$ which goes by the name $X \times I$. With these interpretations, we show isomorphism classes of vector bundles Vect(X) coincides with $\lim_{N\to\infty}[X, G_{*,N}]$. This result is covered in §5.

Proofs, as well as definitions, in many situations follow the lines of known arguments in topology. For the result just mentioned, what is needed is bundle covering homotopy theorem, which implies bundles over the constructible called $X \times I$ are induced from bundles over X, via the projection $X \times I \to X$. This result applies not only to vector bundles, but to all constructible fibre bundles whose fibres are constructible subsets of affine space defined universally, that is, over **Z**. The point is, the bundle covering homotopy theorem really follows from properties of covers of $X \times I$. In topology the step of refining a cover of $X \times I$ to a cover of the form $\bigcup_{i,j} U_i \times [t_j, t_{j+1}]$, where $X = \bigcup_i U_i$, is trivial. In our context, the constructible set called $X \times I$ is not so simple. Nonetheless, we can formulate a suitable result about covers of $X \times I$, which becomes the most subtle part of the proof of the bundle covering homotopy theorem.

The third construction of $KO(X)$ is provided by the Serre-Swan theorem which says $KO(X) \cong K_0(C(X))$, where $K_0(C(X))$ is the Grothendieck group of the monoid of finitely generated, projective, $C(X)$-modules. The arguments of Swan apply rather routinely in our context, once the basic properties of our $C(X)$ and our vector bundles are established.

In §6, we bring in Witt rings. In topology, there is an isomorphism $KO(X) \cong W(C(X))$ for compact spaces, due to Lusztig and Gelfand and Mischenko. Just like the Serre-Swan theorem, the topological proof can be carried over to our context and provides the fourth construction of $KO(X)$. The details here require development in our context of more of the general theory of fibre bundles mentioned above.

The main theorem now becomes a comparison of two Witt rings, namely, that $W(A) \to W(C(X_A))$ is an isomorphism modulo 2-torsion. This is accomplished roughly as follows. First, Witt rings may be replaced by free Witt rings W', defined in terms of non-degenerate forms on free modules only, since $W' \to W$ is always an isomorphism modulo 2-torsion. Then $W'(C(X_A))$ has a homotopy theoretic interpretation, namely, as homotopy classes of sections with values symmetric, non-singular matrices, a universal open set in affine space. We enlarge A to a ring Γ^∞, by adjoining inverses and square roots of functions everywhere positive on X_A. Then $W'(A) \to W'(\Gamma^\infty)$ is an isomorphism modulo 2-torsion, by the method of Mahé. Now, to close the gap between $W'(\Gamma^\infty)$ and $W'(C(X_A))$ we do two things. On the one hand, a careful look at the geometry of the space of non-singular, symmetric matrices allows us up to homotopy to replace matrices with entries in $C(X_A)$ by matrices with entries which are piecewise-Γ^∞ functions. On the other hand, we prove in §7 a kind of Stone-Weierstrass theorem, which allows arbitrarily close approximations to piecewise-Γ^∞ functions by Γ^∞ functions. This gives surjectivity of $W'(\Gamma^\infty) \to W'(C(X_A))$. Injectivity is proved by applying the homotopy approximation and Stone-Weierstrass results to an element of $W'(C(X_A \times I))$, then proving that symmetric matrices over Γ^∞, which are homotopic through matrices in Γ^∞, are stably Γ^∞-conjugate, hence represent the same Witt ring elements. These final steps are given in §8.

Although all results are formulated for constructible sets over any commutative ring, there is a key special case which roughly bridges the gap between this generality and topology. Namely, if $A = k[x_1 \cdots x_n]$ is the polynomial ring over a real closed field, then the constructibles $X \subseteq X_A$ correspond to semi-algebraic sets in affine space k^n, and $C(X)$ is exactly the ring of continuous, k-valued, semi-algebraic functions over these affine semi-algebraic sets. For example, our Stone-Weierstrass theorem in this case is essentially already contained in work of Efroymson. A general procedure is to try to formulate and understand results in the semi-algebraic case, then see if one cannot really prove something "more general".

The reason this works is pretty clear. When dealing with constructible sets, one is dealing with the algebra of finite boolean combinations of formulas $f_i > 0$, $f_i \in A$, and their consequences. These consequences, by the Prestel-Stengle Positivstellensatz, are expressed by formulas in A involving the f_i and sums of squares. Elementary statements about the real spectrum X_A are thus statements about finitely generated algebras over $\mathbf{Z}$ which map to A. Nonetheless, it would be a mistake to deal only with the classical semi-algebraic case. For example, many results about general fields would be overlooked, but are treated simultaneously by the arbitrary ring approach.

Another special case which is included in our arbitrary ring approach is the classical topological theory of vector bundles over compact spaces X. If A is the ring of continuous real valued functions on X, then $A \cong C(X_A)$ (although X_A is pretty complicated). Arbitrary real vector bundles over X are equivalent to constructible vector bundles in our sense over X_A. Again, if one studies carefully the role of rings of continuous, real valued functions in the classical theory, one finds the whole ring plays no essential role. Instead, one works with finitely many functions at a time (for example, a matrix P with $P^2 = P$), and one exploits the facts that inverses of nowhere zero functions and square roots of nowhere negative functions exist in the ring. In our abstract setting, we can carry out these came arguments in rings constructed from any A, such as $C(X_A)$ and in some cases Γ^∞. Theorems in classical K-theory, including the best theorem of all, Bott

Periodicity, are thus given more algebraic treatments. (This is really an observation about known proofs of Bott Periodicity, not an essentially new proof.)

Here is another point. Although real spectra X_A for bizarre A might be pretty messy, we are dealing primarily with a relative situation, $X_{A[T_1 \cdots T_n]} \to X_A$, with affine space fibres. Topologists and algebraic geometers find the same thing, namely that many theorems are results about morphisms $X \to B$ with control on the fibres, but with B pretty arbitrary, rather than merely results about nice objects X.

This paper is based entirely on real spectrum methods. Nonetheless, and I think this is an important point, I refer only to constructible subsets, never the topology generated by the constructible open sets. Notice for instance that quasi-compactness, which is crucial, is really a property of any basis, even subbasis, of a topology. Thus the Tychonoff theorem, on which quasi-compactness of real spectra depends, never really requires 'all'' the open sets of any topology to formulate and prove. Another example of avoiding dependence on topology is provided by needed facts about real spectra of fields. Such X_k are quasi-compact, Hausdorff, totally disconnected spaces. All spaces with these properties have covering dimension 0, that is, any open cover has a refinement consisting of disjoint open sets. This explains why $KO(X_k) \cong \mathrm{Cont}(X_k, \mathbf{Z})$. Despite the words, if one checks behind the scenes, no "topology" is needed. In the case of the X_k, the point is simply that closed formulas $f \geqq 0, f \in k^*$, are obviously equivalent to open formulas $f > 0$. Finally, although we talk about continuous sections of $X_{A[T]} \to X_A$, we only mean constructible sections which happen to be continuous. This is an elementary notion, and images and inverse images of constructible sets remain constructible.

I now wish to record some debts of gratitude and list some references. I have already mentioned the crucial role played by Mahé's work **[15]** in this project. I studied Mahé's paper carefully in a seminar while visiting the University of Hawaii in Autumn 1982. I am grateful to that institution for inviting me and to Ron Brown for several very helpful conversations. Before that, I benefitted from a lengthy visit to Stanford by Gilbert Stengle, during which I finally understood the importance of the Positivstellensatz for general real spectra, not just semi-algebraic sets. Obviously, I am greatly indebted to Michel Coste and Marie-Francoise Roy for their foundational work on the real spectrum, **[4]**. In addition to their writings, I found the survey article of Lam **[13]** very valuable. I will refer to that article for background and other references concerning the real spectrum. On the topological side, I studied K-theory and general bundle theory long ago. The classical books of Atiyah **[1]** and Steenrod **[17]** contain the basic topological theory I need to imitate. Of great importance for this project was the work of Karoubi. His book **[9]**, the paper **[10]**,

and other writings are filled with details and careful treatment of the similarities and differences between algebraic K-theory and topological K-theory. Not only that, but his work on Witt rings and localization **[12]** provided one of the most important steps in Mahé's work and therefore also in mine. I also benefitted from some conversations with Jack Wagoner about higher algebraic K-groups and Witt groups. In a related direction, the two papers of Swan **[18]**, **[19]** clearly are right on the line between algebra and topology. The second paper especially, which was motivated by the work of Lønstad **[14]**, Evans **[6]**, and Fossum **[7]**, even brings in modern real algebra, namely the Nullstellensatz and Positivstellensatz. Finally, I am greatly indebted to Gus Efroymson. His ideas on approximation of semi-algebraic functions by Nash functions and the construction of Γ^∞ near partitions of unity with desirable properties in **[5]**, as well as ealier work with Bochnak on separation of sets by Γ^∞ functions in **[3]** were very useful to me.

2. Preliminaries on Real Spectra. A is a commutative ring, X_A its real spectrum. The points of X_A are homomorphisms $\alpha\colon A \to k(\alpha)$, up to isomorphism over A, where $k(\alpha)$ is a real closed field, algebraic over the subring $\alpha(A)$. Thus, $k(\alpha)$ admits a unique ordering and no non-trivial automorphisms over A. We can define subsets of X_A by inequalities such as $U(f) = \{\alpha \in X_A | f(\alpha) > 0 \text{ in } k(\alpha)\}$, $f \in A$. Here $f(\alpha)$ means $\alpha(f)$, but we write $f(\alpha)$ because we want to think of elements of A as "functions" on X_A. This representation is faithful if A is a real ring, that is, if $\sum f_i^2 = 0$ in A implies each $f_i = 0$.

The constructible sets in X_A are the members of the smallest Boolean algebra of subsets which contains the sets $U(f)$. We write $W(f) = \{\alpha \in X_A | f(\alpha) \geqq 0\}$, and if $\{f_i\}$ is a finite subset of A, we write $U\{f_i\} = \bigcap_i U(f_i)$, and $W\{f_i\} = \bigcap_i W(f_i)$. The open constructible sets are the finite unions of $U\{f_i\}$ and the closed constructible sets are their complements, which are thus finite unions of $W\{g_j\}$, since $X_A - U(f) = W(-f)$.

There are other useful interpretations of the points of X_A. For example, the point α can be identified with the subset of A given by $T_\alpha = \{f \in A \mid f(\alpha) \geqq 0\} = \alpha^{-1}(k(\alpha)^2)$. These subsets $T_\alpha \subset A$ are characterized by the properties $T + T \subset T$, $T \cdot T \subset T$, $A^2 \subset T$, $-1 \notin T$, and $fg \in -T$ implies $f \in T$ or $g \in T$. Note $\alpha \in W(f)$ is the same as $f \in T_\alpha$ and $\alpha \in U(f)$ is $-f \notin T_\alpha$. In this way, X_A is identified with a closed subset of 2^A, the power set of A with the Tychonoff topology, and all constructible subsets of X_A are clopen in this Tychonoff topology on X_A. Thus,

Proposition 2.1. *An arbitrary cover of a constructible subset of X_A by constructible sets admits a finite subcover.*

If α, $\beta \in X_A$, then we say β is a specialization of α if $T_\alpha \subseteqq T_\beta$, that is, if $g(\alpha) \geqq 0$ always implies $g(\beta) \geqq 0$, $g \in A$. This is, of course, equivalent to $f(\beta) > 0$ implying $f(\alpha) > 0$, so any $U\{f_i\}$ containing β also contains α. Thus, whenever specializations occur, X_A is not Hausdorff. On the other hand, distinct maximal points, in fact, any two points which are not comparable, can be separated by an element of A. Namely, if $g_\alpha \in T_\alpha - T_\beta$ and $g_\beta \in T_\beta - T_\alpha$, let $g = g_\alpha - g_\beta$. Then $g(\alpha) > 0$ and $g(\beta) < 0$. In particular, α and β cannot both be specializations of some γ. The following result makes use of this. Let $\overline{\{\alpha\}}$ denote the set of specializations of α.

PROPOSITION 2.2. *If $X \subseteq X_A$ is any constructible and $\alpha \in X$, then $\overline{\{\alpha\}} \cap X$ is totally ordered by inclusion and contains a unique maximal point in X (not necessarily maximal in X_A). In particular, any open cover of the maximal points of X necessarily covers all of X.*

REMARK 2.3. The maximal points $\alpha \in X_A$ are the points $\alpha\colon A \to k(\alpha)$ such that every element of $k(\alpha)$ is bounded in absolute value by some element of the subring $\alpha(A)$.

As another application of the separability of non-comparable points, together with quasi-compactness, one proves that constructible subsets of real spectra are normal. (Perhaps quasi-normal would be a better term.) That is, disjoint closed sets can be surrounded by disjoint open sets. This is a special case of the following shrinking lemma, which is useful just as it is in topology.

PROPOSITION 2.4. *If $X \subseteq X_A$ is any constructible and $X \subseteq \bigcup U_i$ is a finite, constructible open cover of X, then there are constructible open sets V_i and constructible closed sets K_i with $V_i \cap X \subset K_i \cap X \subset U_i$ and $X \subseteq \bigcup V_i$.*

REMARK. In the semi-algebraic case, one can take $K_i = \bar{V}_i$, the closure of V_i in X_A, which will be constructible because of the Finiteness theorem. In general, I don't know which constructible sets have constructible closures.

3. Rings of Constructible, Continuous Sections. Let $\pi\colon X_{A[T]} \to X_A$ be the projection induced by the inclusion $A \to A[T]$ of A into the ring of polynomials in one indeterminate. If $\alpha \in X_A$, then the fibre $\pi^{-1}(\alpha)$ is exactly the real spectrum $X_{k(\alpha)[T]}$, [4]. There is a natural inclusion $k(\alpha) \subset \pi^{-1}(\alpha)$. Given $t \in k(\alpha)$, the map $\alpha\colon A \to k(\alpha)$ extends to $\alpha_t\colon A[T] \to k(\alpha)$ by $\alpha_t(T) = t$.)

If $X \subseteq X_A$ is constructible, we define $C(X)$ to be the set of constructible continuous sections $s\colon X \to X_{A[T]}$. Constructible section just means a constructible subset of $X_{A[T]}$ which projects by π bijectively onto X. All the maps π, π^{-1}, s, s^{-1} take constructible sets to constructible sets. Continuity means of course that $s^{-1}(U(g(T)))$ is relatively open in X, for all $g(T) \in A[T]$. Because of the following proposition we have $s^{-1}(U(g(T))) = \{\alpha \in X \mid g(s(\alpha)) > 0 \text{ in } k(\alpha)\}$.

PROPOSITION 3.1. *If $s \in C(X)$, then $s(\alpha) \in k(\alpha) \subset \pi^{-1}(\alpha)$, all $\alpha \in X$.*

This holds because a constructible subset of $X_{A[T]}$ on which no $f(T) \in A[T]$ vanished would contain intervals in some $k(\alpha)$.

It is easy to see that elements of A define elements of $C(X_A)$. Namely, given $a \in A$, take the "graph" of a, $a(\alpha) = \alpha(a) \in k(\alpha)$. The image $a(X_A)$ is the constructible set $Z(T - a)$ of zeros of $T - a$ in $X_{A[T]}$. If $Y \subseteq X$ are constructible sets in X_A, there is the restriction map $C(X) \to C(Y)$.

It is surprisingly difficult to see that sums and products of elements of $C(X)$ belong to $C(X)$. Constructibility is easy enough, but continuity is tricky. The ring homomorphisms s, m: $A[T] \to A[T_1, T_2]$ with $s(T) = T_1 + T_2$, $m(T) = T_1T_2$, induce continuous maps σ, μ: $X_{A[T_1,T_2]} \to X_{A[T]}$. Thus, the fact that $C(X)$ is a ring is a consequence of the following result, which is also the key to extending our notion of continuous sections to vector valued sections.

PROPOSITION 3.2. *Suppose s: $X \to X_{A[T_1 \cdots T_n]}$ is a constructible section over $X \subseteq X_A$. Then $s(\alpha) = (s_1(\alpha), \ldots, s_n(\alpha)) \in k(\alpha)^n \subset X_{k(\alpha)[T_1 \cdots T_n]}$ for all $\alpha \in X$ and each s_i is constructible. Moreover, s is continuous if and only if each s_i is continuous.*

The hard part is the if part. I advise you, dear reader to try to prove this now. It will give you a new perspective on what you thought you understood about continuous real functions.

Here is another such result. If $s \in C(X)$, $s(X) \subset \pi^{-1}(X) \subset X_{A[T]}$ is like the "graph of a function". Let $s^+(X)$ and $s^-(X)$ denote the parts of $\pi^{-1}(X)$ above and below $s(X)$, respectively. (This makes sense, because each fiber $X_{k(\alpha)[T]}$ is a totally ordered set.)

PROPOSITION 3.3. *If $s \in C(X)$, then $s^+(X)$ and $s^-(X)$ are constructible (relatively) open subsets of $\pi^{-1}(X) \subset X_{A[T]}$.*

This result is rather useful. For example, it implies that if $s_1, s_2 \in C(X)$, then $U(s_1 - s_2) = \{\alpha \in X \mid s_1(\alpha) > s_2(\alpha)\}$ is an open constructible in X.

The following proposition constructs some other elements of $C(X)$. For the inverse, exploit the obvious automorphism of $X_{A[T,T^{-1}]} \subset X_{A[T]}$ defined by mapping T to T^{-1}, $A[T, T^{-1}] \to A[T, T^{-1}]$. For the square root, exploit the closed mapping $X_{A[\sqrt{T}]} \to X_{A[T]}$ induced by the extension $A[T] \to A[\sqrt{T}]$, which identifies $W(\sqrt{T}) \subset X_{A[\sqrt{T}]}$ with $W(T) \subset X_{A[T]}$.

PROPOSITION 3.4. *Suppose $s \in C(X)$, $s(\alpha) \neq 0$ for all $\alpha \in X$. Then $s^{-1} \in C(X)$. If $s(\alpha) \geqq 0$, for all $\alpha \in X$, then $\sqrt{s} \in C(X)$.*

Next, we have a sheaf property.

PROPOSITION 3.5. *If $X = \bigcup X_i$ is either a finite relatively open cover or a finite relatively closed cover of X by constructible subsets, and if $s_i \in C(X_i)$ are such that $s_i = s_j$ in $C(X_i \cap X_j)$, then there exists a unique $s \in C(X)$ with $s = s_i$ in $C(X_i)$.*

Using these two results and Proposition 2.3, we can construct partitions of unity subordinate to any open cover $X \subset \bigcup_i U_i$.

PROPOSITION 3.6. *If $X \subset \bigcup_j U_i$, choose V_i and K_i as in 2.3. Then there exists. $\varphi_i \in C(X)$ with $0 \leqq \varphi_i \leqq 1$, $\sum \varphi_i = 1$ and $U(\varphi_i) = V_i$.*

The point is, if $V_i = \bigcup_j U\{f_{ijk}\}$ let $f_{ij} = \prod_k f_{ijk}^2$. Then $p_{ij} \in C(X)$ can be defined by $p_{ij} = f_{ij}$ on $W\{f_{ijk}\}$, $p_{ij} = 0$ on $X - U\{f_{ijk}\}$. Now take $p_i = \sum_j p_{ij}$ and $\varphi_i = p_i / \sum p_i$.

There is a more useful functorial behavior of the rings $C(X)$ than that provided by simple restriction. If $\gamma: A \to B$ is a ring homomorphism, then there is the induced continuous map $\gamma^*: X_B \to X_A$. If $X \subseteq X_A$ is constructible, so is $(\gamma^*)^{-1}X \subseteq X_B$. If $\gamma^*(\beta) = \alpha$, then we have $\alpha = \beta\gamma: A \to B \to k(\beta)$, hence $k(\alpha) \subset k(\beta)$. We find in this situation the next proposition.

PROPOSITION 3.7. *If $\gamma: A \to B$ is an integral extension or if A is finitely generated over $\mathbf{Z}$, and if $s \in C(X)$, $Y = (\gamma^*)^{-1}X$, then $\gamma_* s \in C(Y)$, where $\gamma_* s(\beta) = s(\gamma^*(\beta)) = s(\alpha) \in k(\alpha) \subset k(\beta)$. The image $\gamma_* s(Y)$ is the constructible $(\tilde{\gamma}^*)^{-1} s(X) \subset X_{B[T]}$, where $\tilde{\gamma}: A[T] \to B[T]$ is the obvious extension of $\gamma: A \to B$.*

REMARK 3.8. Clearly, when defined, $\gamma_*: C(X) \to C(Y)$ is a ring homomorphism. Niels Schwartz has an example of $\gamma: A \to B$ and $s \in C(X_A)$ with $\gamma_*(s) \notin C(X_B)$, that is, $\gamma_*(s)$ fails to be continuous without some hypotheses. This strongly suggests that our definition of $C(X_A)$ might not be the best, since one would like the above functoriality for any $\gamma: A \to B$.

In the development of algebraic topology in our context, the following Tietze extension theorem is useful. In a special case, it says that if $W \subset X \subseteq X_A$ are constructibles with W relatively closed in X, then $C(X) \to C(W)$ is surjective.

PROPOSITION 3.9. *Suppose $\gamma: A \to B$ is a finitely presented integral extension $X \subseteq X_A$ $Y = (\gamma^*)^{-1}X \subset X_B$ constructible. Let $C_X(Y) \subset C(Y)$ be the subring of sections over Y which are constant over each $\alpha \in X$. (If $\gamma^*(\beta) = \alpha$, then $k(\beta) = k(\alpha)$, since B is integral over A.) Then $\gamma_*: C(X) \to C_X(Y)$ is surjective.*

If $W = X \cap \bigcup_j W\{g_{ij}\}$, take $B = \prod_j A[\{\sqrt{g_{ij}}\}]$. Then $C_X(Y) \cong C(W)$. As another example, let $B = A/I$, I a finitely generated ideal, $X = X_A$.

Then $\gamma^*: X_B \to X_A$ is a $1-1$ mapping onto $Z(I)$, so $C_X(Y) = C(X_{A/I})$ and the proposition says $\gamma_*: C(X_A) \to C(X_{A/I})$ is surjective. Note $Z(I) \subset X_A$ may not be constructible if I is not finitely generated. Niels Schwartz has an example which shows $C(X_A) \to C(X_{A/I})$ is not always surjective. Again, an alternate definition of $C(X_A)$ might have better properties.

We will make essential use of certain subrings of $C(X)$. Denote by $A^\infty(X)$ the smallest subring of $C(X)$ which contains the image of A and which contains $\sqrt{s}$ whenever it contains s with $s > 0$ on X. Denote by $\Gamma^\infty(X)$ the smallest such subring which in addition contains s^{-1} whenever it contains such s.

Both $A^\infty(X)$ and $\Gamma^\infty(X)$ are easily described as direct limits of a countable sequence of extensions of A. In fact, at each step, the Positivstellensatz describes algebraically the set of elements whose square roots or inverses are to be adjoined. For example, if $f \in A$, then $f > 0$ on X_A precisely when there is an equation $(1+p)f = 1+q$, with p, q sums of squares in A. In any ring, let $S(1)$ denote the multiplicative set of elements of the form $1 + \sum x_i^2$.

PROPOSITION 3.10. *If $X \subseteq X_A$ is any constructible, then $\Gamma^\infty(X) = A^\infty(X)_{S^\infty(X)}$, where $S^\infty(X) = \{f \in A^\infty(X) \mid f > 0$ on $X\}$. If X is closed, then $S^\infty(X) = S(1) \subset A^\infty(X)$, so $\Gamma^\infty(X) = A^\infty(X)_{S(1)}$.*

The first statement is easy. The point of the second is that for closed X, the real spectrum of $A^\infty(X)$ (and the real spectrum of $\Gamma^\infty(X)$ for any X) can be identified with a sort of halo around X, specifically, the intersection of all open neighborhoods of X in X_A which is the same as the set of points with a specialization in X. We will discuss this further in §7.

A more complicated subring of $C(X)$ is the subring $N(X)$ of Nash functions on X. Roughly, $N(X)$ consists of sections which are locally elements of rings obtained by adjoining simple roots of polynomials to A. Clearly, $A \subset A^\infty(X) \subset \Gamma^\infty(X) \subset N(X) \subset C(X)$ (even though it is not clear what $N(X)$ is precisely).

We conclude this section with some examples of the rings $C(X)$.

EXAMPLE 1. (The semi-algebraic case) If $A = k[x_1 \cdots x_n]$ is a polynomial ring over a real closed field k, then a constructible subset $X \subseteq X_A$ is determined by the semi-algebraic subset $X_0 = X \cap k^n$ of affine space over k. If $C_0(X_0)$ is the ring of continuous, k-valued, semi-algebraic functions on X_0, then $C(X) \simeq C_0(X_0)$ is an isomorphism. It is easy enough to define a constructible section $s: X \to X_{A[T]}$ in terms of the graph of s_0: $X \to k$, $s_0 \in C_0(X_0)$, but continuity of s is a little tricky and seems to me to require the Finiteness theorem.

EXAMPLE 2. (Fields) As mentioned previously, if k is any field, all constructibles in X_k are clopen. Elements of $C(X)$, $X \subseteq X_k$, are in a certain strong sense locally constant. Namely, for each $s \in C(X)$, one can find a finite cover $X = \bigcup U_i$ by disjoint open sets, finite algebraic extensions $\gamma_i\colon k \to k_i$ with $\gamma_i^*(X_{k_i}) = U_i \subset X_k$, and elements $\sigma_i \in k_i$ with $\gamma_*(s|_{U_i}) = \sigma_i \in k_i \subset C(X_{k_i})$. In particular, $N(X) = C(X)$ in this case.

EXAMPLE 3. (Topology) Suppose X is a completely regular topological space, A the ring of continuous, real valued functions on X. Then $A \simeq C(X_A)$ is an isomorphism.

EXAMPLE 4. (Idempotence) Presumably, for any A, the map $C(X_A) \simeq C(X_{C(X_A)})$ is an isomorphism, since, presumably, $X_{C(X_A)} \cong X_A$ (this requires a variant of the Substitution Theorem), $X_{C(X_A)[T]} \cong X_{A[T]}$, and constructibility over A and $C(X_A)$ coincide. For any constructible $X \subseteq X_A$, I believe $C(X_{C(X)}) \cong C(X)$. (For locally closed X, M. Coste informs me that already $X_{C(X)} \cong X$, but this does not hold for all X.)

4. Constructible Vector Bundles and KO(X). As usual, let X be a constructible subset of some real spectrum X_A. We write $GL_n(C(X))$ for the group of invertible $n \times n$ matrices over $C(X)$. Note that there is an action of $GL_n(C(X))$ on affine n-space over X, that is, on $\pi^{-1}(X)$ where $\pi\colon X_{A[T_1 \cdots T_n]} \to X_A$, which is contained in the map $X_{A[S_{ij}, T_k]} \to X_{A[T_k]}$, induced by $A[T_k] \to A[S_{ij}, T_k]$ which sends T_k to $\sum_{j=1}^{n} S_{kj}T_j$. More generally, $m \times n$ matrices $M_{m,n}(C(X))$ map affine n-space to affine m-space over X.

By a constructible, real, n-dimensional vector bundle (n-bundle) over X, we mean a map $p\colon E \to X$, together with a maximal family of pairwise compatible charts $p^{-1}(U_i) \simeq \pi^{-1}(U_i)$, where $U_i \subseteq X$ are relatively open constructibles which cover X and $\pi\colon X_{A[T_1 \cdots T_n]} \to X_A$. Charts $p^{-1}(U_i) \simeq \pi^{-1}(U_i)$ and $p^{-1}(U_j) \simeq \pi^{-1}(U_j)$ are compatible if the induced map $\pi^{-1}(U_i \cap U_j) \simeq \pi^{-1}(U_i \cap U_j)$ is defined by an element $\varphi_{ij} \in GL_n(C(U_i \cap U_j))$. It is clear that a vector bundle is determined by any pairwise compatible family of charts over U_i, where $X = \bigcup U_i$ is a finite, constructible, open cover of X. (We will sometimes write "open" instead of "relatively open.")

The above definition requires the fibre dimension n of an n-bundle to be constant over X. By a vector bundle over X, we mean a collection of n_i-bundles over U_i, where $X = \bigcup U_i$ is a finite constructible cover by disjoint open sets. Thus, the fibre dimension of a vector bundle is always locally constant.

A fibre preserving map $\varphi\colon E \to F$ between vector bundles over X is said to be a vector bundle morphism if in suitable local coordinate charts covering X, φ is induced by elements of $M_{m,n}(C(U))$. A vector bundle morphism $\varphi\colon E \to F$ is a strict bundle morphism if rank (φ) is locally constant. A morphism $\varphi\colon E \to F$ is an isomorphism if φ is bijective and $\varphi^{-1}\colon F \to E$ is also a morphism. By Cramer's formula for the inverse of a matrix, any morphism which is bijective on each fibre is an isomorphism.

Let Vect(X) denote the set of isomorphism classes of vector bundles over X. Whitney sums Vect(X) $\times$ Vect(X) $\to$ Vect(X) are defined using coordinate charts as in topology, and Vect(X) becomes an abelian monoid. Let $KO(X)$ denote the associated Grothendieck group. Tensor products of vector bundles are also defined just as in topology, and $KO(X)$ becomes a ring. The locally constant dimension of a vector bundle extends to a ring homomorphism dim: $KO(X) \to \text{Cont}(X, Z)$. We denote by $\mathscr{E}^n$ the trivial n-bundle π: $X_{A[T_1 \cdots T_n]} \to X_A$ (or its restriction to any constructible subset $X \subseteq X_A$). A simple construction in $KO(X)$ (subtracting various trivial bundles over disjoint open subsets) shows dim: $KO(X) \to \text{Cont}(X, \mathbf{Z})$ is surjective.

The following result is proved just as in topology, exploiting the partitions of unity constructed in §3.

PROPOSITION 4.1. *For any vector bundle E over X, there exist vector bundle epimorphisms $\mathscr{E}^N \to E$ for suitably large N.*

Next we want to deal with subbundles. The only way the following proposition differs from its topological counterpart is in some extra care needed to interpret the meaning of the terms. (Our vector bundle fibres are real spectra $X_{k(\alpha)[T_1 \cdots T_n]}$, not simple affine spaces $k(\alpha)^n$.)

PROPOSITION 4.3. *If φ: $E \to F$ is a strict vector bundle morphism over X, then* Ker(φ) *and* Im(φ) *are vector bundles over X.*

It is relatively routine to construct quotient bundles with respect to subbundles. We thus also have a bundle Coker(φ), with the usual isomorphisms $\text{Im}(\varphi) \simeq E/\text{Ker}(\varphi)$ and $\text{Coker}(\varphi) \simeq F/\text{Im}(\varphi)$.

The notion of a symmetric bilinear form on a vector bundle E over X is not hard to define. Basically, this means a morphism $E \otimes E \to \mathscr{E}^1$, invariant under the switching automorphism of $E \otimes E$. So, locally a symmetric bilinear form on an n-bundle is given by a symmetric $n \times n$ matrix over some $C(U)$. We say a form is non-singular (respectively, positive definite) if the defining $n \times n$ matrix over $C(U)$ is non-singular (respectively, positive definite) at each point $\alpha \in U$. Partitions of unity again easily gives us the following.

PROPOSITION 4.3. *Every vector bundle over X admits a positive definite form.*

Given a bundle E over X, the dual bundle E^* is defined in the usual way. More generally, given E, F, there is a bundle Hom(E, F), with $F^* =$ Hom(E, $\mathscr{E}^1$). A symmetric bilinear form $E \otimes E \to \mathscr{E}^1$ is the same as a self-adjoint bundle morphism $E \to E^*$. Given a subbundle $F \subset E$, we define the orthogonal subbundle (relative to the form) $F^{\perp} = \mathrm{Ker}(E \to E^* \to F^*)$.

PROPOSITION 4.4. If $b\colon E \otimes E \to \mathscr{E}^1$ is a symmetric non-singular form so that $b|_F\colon F \otimes F \to \mathscr{E}^1$ is also non-singular (for example, if b is positive definite), then $F \cap F^{\perp} = (0)$ and $F \oplus F^{\perp} \simeq E$ is an isomorphism.

In particular, if $\varphi\colon \mathscr{E}^N \to E$ is a bundle epimorphism, then we get direct sum decompositions $K \oplus E \simeq \mathscr{E}^N$, where $K = \mathrm{Ker}(\varphi)$. Specifically, given a positive definite form on $\mathscr{E}^N$, φ induces an isomorphism $K^{\perp} \simeq E$. More generally, short exact sequences of bundle morphisms always split.

From our basic definition of a vector bundle $p\colon E \to X$, constructible open subsets of E, and general constructible subsets, are defined by exploiting a finite trivializing cover. In particular, constructible, continuous sections over any constructible in X are defined. From Proposition 3.2, the set $\Gamma(E, X)$ of such global sections forms a $C(X)$-module. If $E \cong \mathscr{E}^n$ is trivial, then $\Gamma(E, X) \cong C(X)^n$ is free. Also $\Gamma(E \oplus F, X) \cong \Gamma(E, X) \oplus \Gamma(F, X)$.

The propositions above prove that every bundle E over X is a direct summand of a trivial bundle. Thus $\Gamma(E, X)$ is always a finitely generated, projective $C(X)$-module. A bundle morphism $E \to F$ induces a $C(X)$-module homomorphism $\Gamma(E, X) \to \Gamma(F, X)$. In fact, it is not hard to see that vector bundle morphisms $E \to F$ coincide with global sections $\Gamma(\mathrm{Hom}(E, F), X)$ and that the map $\Gamma(\mathrm{Hom}(E, F), X) \simeq \mathrm{Hom}_{C(X)}(\Gamma(E, X), \Gamma(F, X))$ is an isomorphism of $C(X)$-modules. (This is checked locally, then patching arguments are used.) This discussion proves the Serre-Swan theorem.

PROPOSITION 4.5. *The category of vector bundles over X is isomorphic to the category of finitely generated, projective $C(X)$-modules. In particular, $KO(X) \cong K_0(C(X))$.*

Obviously, $KO(X)$ has certain functorial properties. First, if $Y \subseteq X$, then we get restriction $\mathrm{Vect}(X) \to \mathrm{Vect}(Y)$. More generally, if $\gamma\colon A \to B$ is a ring homomorphism $X \subseteq X_A$, $Y = (\gamma^*)^{-1}X \subseteq X_B$, then there is a pull-back construction $\gamma_*\colon \mathrm{Vect}(X) \to \mathrm{Vect}(Y)$, inducing $\gamma_*\colon KO(X) \to KO(Y)$. This ring homomorphism coincides with the map $K_0(C(X)) \to K_0(C(Y))$ induced by $\gamma_*\colon C(X) \to C(Y)$ in the context of the Serre-Swan theorem. We also have the following bundle form of the Tietze extension theorem, Proposition 3.9.

PROPOSITION 4.6. *Suppose $\gamma\colon A \to B$ is an integral extension $X \subseteq X_A$, $Y = (\gamma^*)^{-1}X \subset B_B$, E a vector bundle over X. Let $\Gamma_X(\gamma_*E, Y) \subset \Gamma(\gamma_*E, Y)$ be the $C_X(Y)$-submodule of sections of γ_*E which are constant over each $\alpha \in X$. Then the natural map $\Gamma(E, X) \to \Gamma_X(\gamma_*E, Y)$ is surjective.*

As special cases, we have surjections $\Gamma(E, X) \to \Gamma(E, Y)$, $Y \subseteq X$ a closed constructible, and $\Gamma(E, X_A) \to \Gamma(\gamma_* E, X_{A/I})$, $\gamma\colon A \to A/I$, E a bundle over X_A.

5. Homotopy Classification of Vector Bundles. Let $p\colon E \to X$ be a vector bundle over $X \subseteq X_A$. From §4, $E \cong \mathrm{Ker}(P)$, where $P\colon \mathscr{E}^N \to \mathscr{E}^N$ is an orthogonal projection, that is, $P^2 = P = P^*$, $\mathscr{E}^N = \pi^{-1}(X) \subset X_{A[T_1 \cdots T_N]}$, some N. The operator P is defined by a matrix in $M_{N,N}(C(X))$.

Consider $\rho\colon X_{A[S_{ij}, T_k]} \to X_{A[S_{ij}]}$, $1 \leqq i, j, k \leqq N$, as the trivial N-bundle over $X_{A[S_{ij}]}$, with the vector bundle endomorphism $\mathfrak{B}$ over $X_{A[S_{ij}]}$ defined by the action of the matrix (S_{ij}) on the vector (T_k). This self-map is a strict morphism (that is, locally constant rank) when restricted to $\rho^{-1}(G_{*,N})$, where $G_{*,N} \subset X_{A[S_{ij}]}$ is the constructible defined by $(S_{ij})^2 = (S_{ij})$, $S_{ij} = S_{ji}$. Thus $\rho\colon \mathrm{Ker}(\mathfrak{B}) = K \to G_{*,N}$ is a vector bundle, of fibre dimension n over the subset $G_{n,N}$ of $G_{*,N}$ defined by the additional conditions rank $(S_{ij}) = N - n$. Moreover, $\rho\colon K \to G_{*,N}$ is a universal bundle in the sense that given $X \subseteq X_A$ and symmetric $P \in M_{N,N}(C(X))$ with $P^2 = P$, then P defines a constructible, continuous $G_{*,N}$-valued section s of $X_{A[S_{ij}]} \to X_A$ over X, with $E \cong s^*(K)$, where $E \to X$ is the bundle $\mathrm{Ker}(P)$, $P\colon \mathscr{E}^N \to \mathscr{E}^N$ over X. (In general, a constructible, continuous section s of $X_{A[R_k]} \to X_A$ over $X \subseteq X_A$ identifies X and $s(X)$, and $C(X) \cong C(s(X))$, where $C(s(X))$ is defined in terms of constructibles in $X_{A[R_k, T]}$. Alternatively, the coordinates of s belong to $C(X)$, hence there is a ring homomorphism $\sigma\colon A[R_k] \to C(X)$, with $X = X_{C(X)} = (\sigma^*)^{-1}(s(X))$. Thus, pull-backs with respect to a section are special cases of pull-backs with respect to ring homomorphisms.)

If $X \subseteq X_A$, then there is an obvious constructible called $X \times I \subset X_{A[T]}$, defined as $\{\beta \in X_{A[T]} \mid \pi(\beta) \in X, 0 \leqq T(\beta) \leqq 1\}$. The fibres of $\pi\colon X \times I \to X$ are the constructibles in $X_{k(\alpha)[T]}$ defined by $0 \leqq T \leqq 1$, where $\alpha \in X$. But these "intervals" vary with α, and $X \times I$ is much more complicated than a simple product. Nonetheless, 0 and $1 \in A$ define sections $X \to X \times I$, with images which we denote $X \times \{0\}$ and $X \times \{1\}$.

Suppose given sections s_0, s_1 of some $X_{A[R_k]} \to X_A$ over X, with values in some $Y \subseteq X_{A[R_k]}$. We say s_0 and s_1 and homotopic if there exists a section S of $X_{A[R_k, T]} \to X_{A[T]}$ over $X \times I$ with values in $\pi^{-1}(Y)$ where $\pi\colon X_{A[R_k, T]} \to X_{A[R_k]}$, such that $S|_{X \times \{0\}} = s_0$ and $S|_{X \times \{1\}} = s_1$.

The main result of this section is that the correspondence defined above from $G_{*,N}$-valued sections of $X_{A[S_{ij}]} \to X_A$ over X to $\mathrm{Vect}(X)$ depends only on the homotopy class of the section. This amounts to proving that any bundle E over $X \times I$ is isomorphic to $\pi^* E_0$, where $\pi\colon X \times I \to X$ and $E_0 = E|_{X \times \{0\}}$, by an isomorphism which extends the identity isomorphism over $X \times \{0\}$. This in turn is a special case of a more general bundle covering homotopy theorem. Namely, a vector bundle is determined by a covering by open sets, say V_i, and gluing data $\varphi_{ij} \in GL_n(C(V_i \cap V_j))$.

Such data makes sense if GL_n is replaced by any universal affine group, say G. From such data, one constructs principal G-bundles, or, if G acts universally on some affine constructible F, one constructs fiber bundles with fibre F and group G.

PROPOSITION 5.1. *Any bundle $E \to X \times I$ with fibre F and group G is isomorphic, as F-bundle with group G, to $\pi^* E_0$, by an isomorphism extending the identity over $X \times \{0\}$.*

The point is that the fibre and group do not really matter in the proof. Instead one just needs to know how to refine an arbitrary open cover of $X \times I$.

PROPOSITION 5.2. *Suppose $X \times I = \bigcup V_i$ is an open cover. Then there is an open cover $X = \bigcup U_j$, and elements $a_{j,1}, \ldots, a_{j,n(j)} \in C(U_j)$, with $0 = a_{j,1} < a_{j,2} < \cdots < a_{j,n(j)} = 1$, such that each set $U_j \times [a_{j,k}, a_{j,k+1}] = \{\beta \in X_{A[T]} | \pi(\beta) \in U_j,\ a_{j,k}(\pi(\beta)) \leqq T(\beta) \leqq a_{j,k+1}(\pi(\beta))\}$ is contained in some V_i.*

Now one proves the bundle covering homotopy theorem 5.1 in the usual way, [**17**], by choosing a partition of unity $\sum p_j = 1$ subordinate to (a shrinking of) the cover $X = \bigcup U_j$, and constructing the bundle isomorphism $\pi^* E_0 \simeq E$ piece-by-piece, first on or below the graph of p_1 in $X \times I$, then on or below the graph of $p_1 + p_2$, etc. At each stage, one is working over one of the $U_j \subset X$ and one handles this stage also in steps, between the functions $a_{j,k}$ and $a_{j,k+1}$, $k = 1, 2, \ldots, n(j) - 1$, successively, where one is in a trivializing neighborhood for both E and $\pi^* E_0$.

Let $G_{*,N} \to G_{*,N+1}$ be the map corresponding to matrix stabilization $P \mapsto \left(\begin{smallmatrix} P & 0 \\ 0 & 1 \end{smallmatrix}\right)$. (On the ring level, this is induced by $A[S_{i,j}]_{1 \leqq i, j \leqq N+1} \to A[S_{i,j}]_{1 \leqq i, j \leqq N}$ sending $S_{N+1,N+1}$ to 1, $S_{N+1,j}$ and $S_{i,N+1}$ to 0 if $1 \leqq i, j \leqq N$, and $S_{i,j}$ to $S_{i,j}$ if $1 \leqq i, j \leqq N$.) The bundle covering homotopy theorem and basic constructions gives us the following.

PROPOSITION 5.3. *For any $X \subseteq X_A$, the map $\lim_{N \to \infty}[X, G_{*,N}] \simeq \mathrm{Vect}(X)$ is a bijection, where $[X, G_{*,N}]$ denotes homotopy classes of $G_{*,N}$-valued sections of $X_{A[S_{ij}]} \to X_A$ over X.*

REMARK. There is another stabilization $G_{*,N} \to G_{*+1,N+1}$ induced by $P \mapsto \left(\begin{smallmatrix} P & 0 \\ 0 & 0 \end{smallmatrix}\right)$. On the level of bundles $E = \mathrm{Ker}(P)$, this takes E to $E \oplus \mathscr{E}^1$. We say two vector bundles E, F are stably isomorphic if $E \oplus \mathscr{E}^m \cong F \oplus \mathscr{E}^n$ for some m, n. If $\widetilde{\mathrm{Vect}}(X)$ denotes stable isomorphism classes of bundles, then $\lim_{N \to \infty}[X, G_{*,N}] \simeq \widetilde{\mathrm{Vect}}(X)$ is bijective, where now the direct limit is taken with respect to both stabilizations $P \mapsto \left(\begin{smallmatrix} P & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $P \mapsto \left(\begin{smallmatrix} P & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

REMARK. If we choose a basepoint $\alpha \in X$, we have dimension at α, $KO(X) \to \mathbf{Z}$. Let $\widetilde{KO}(X) = \mathrm{Ker}(KO(X) \to \mathbf{Z})$. Then easily $KO(X) \cong$

$KO(X) \oplus \mathbf{Z}$, and there is a bijection $\widetilde{\text{Vect}}(X) \cong \widetilde{KO}(X)$. In a slightly different direction, if we fix j, there is a bijection $\lim_{N\to\infty}[X, G_{j,N}] \simeq \text{Vect}_j(X)$, where $\text{Vect}_j(X)$ denotes isomorphism classes of j-bundles. Then we have further bijections $\lim_{\substack{N\to\infty \\ j\to\infty}}[X, G_{j,N}] \simeq \lim_{j\to\infty}\text{Vect}_j(X) \simeq \text{Ker}$ $(KO(X) \to \text{Cont}(X, \mathbf{Z})) = IO(X)$, the elements of $KO(X)$ of global dimension 0. The space $\lim_{N\to\infty} G_{j,N}$ is called $BO(j)$ by topologists (it is really a functor), and $\lim BO_{j\to\infty}(j)$ is called BO. We have $KO(X) \cong [X, \mathbf{Z} \times BO]$, corresponding to the decomposition $KO(X) \cong \text{Cont}(X, \mathbf{Z}) \oplus IO(X)$.

We will record in this section two other applications of the bundle covering homotopy theorem. The first will be crucial in the following-sections when we bring in Witt rings.

PROPOSITION 5.4. *Any two positive definite forms on a vector bundle over X are isometric.*

The point is that two positive definite forms are homotopic, because of convexity of the set of positive definite matrices. Now, a simple Gram-Schmidt argument shows that a choice of a positive definite form gives a reduction of the structure group of a vector bundle from GL_n to the orthogonal group O_n. Applying the bundle covering homotopy theorem to the O_n-bundle over $X \times I$, given by the obvious convex linear combination of two forms on X, shows the two corresponding O_n-bundles are isomorphic as O_n-bundles. But this means isometric as forms.

The second application is to relative K-theory, which we do not really need, so won't develop extensively here. However, it is needed to justify our claims in the Introduction that for all $n \geqq 0$, $W_n(A) \otimes \mathbf{Z}[1/2]$ is isomorphic to $KO^{-n}(X_A) \otimes \mathbf{Z}[1/2]$.

The first goal of relative K-theory is to define groups $KO^{-n}(X, Y)$ for certain pairs and establish a long exact sequence. Slightly more generally, one wants to work with certain ring homomorphisms $\varphi: A \to B$. The case $A \to A/I$, $I \subset A$ an ideal, is already very general, since we can work with big rings like our $C(X)$ and ideals $I(Y) \subset C(X)$ of sections which vanish on closed $Y \subset X$.

If J is a ring, possibly without unit, let $J^+ = J \times \mathbf{Z}$ with multiplication $(x, m)(y, n) = (xy + my + nx, mn)$. Then J^+ is a ring with unit $(0, 1)$, and projection $*: J^+ \to \mathbf{Z}$ is a ring homomorphism. If J does have a unit, then the real spectrum X_{J^+} is simply the disjoint union of X_J and the point $*$. In general we set $KO^0(X_J) = \text{Ker}(KO(X_{J^+}) \to KO(X_{\mathbf{Z}}) = \mathbf{Z})$, so the definition extends our $KO(X_A)$ to rings without unit. In particular, if $I \subset A$ is an ideal, then $KO^0(X_I)$ is defined, and the ring homomorphism $I^+ \to A^+$ induces $KO^0(X_I) \to KO^0(X_A) = KO(X_A)$.

PROPOSITION 5.5. *If $I \subset A$ is an ideal, then $KO^0(X_I) \to KO(X_A) \to KO(X_{A/I})$ is exact.*

This result requires quite a bit. First, setwise one can see X_{I^+} is X_A with the subset $Z(I) = \text{image}(X_{A/I} \to X_A)$ collapsed to the point $*$. Moreover, the Positivstellensatz can be used to show constructible neighborhoods of $* \in X_{I^+}$ are cofinal in the constructible neighborhoods of $Z(I)$ in X_A. Finally, The Tietze extension theorem for bundle sections, Proposition 4.6, is necessary.

Next, we define $KO^{-n}(X_J)$ for all $n \geqq 0$ and all rings J, possibly without unit. Let $A(n) = A[T_0 \cdots T_n]/(\sum T_i^2 - 1)$ if A has a unit. (So $X_{A(n)}$ is the constructible called $X_A \times S^n$ in $X_{A[T_0 \cdots T_n]}$.) The basepoint $(T_0 \cdots T_n) = (1, \ldots, 0)$ defines $J^+(n) \to J^+$, and $*: J^+ \to \mathbf{Z}$ defines $J^+(n) \to \mathbf{Z}(n)$. Define $KO^{-n}(X_J) = \text{Ker}(KO(X_{J^+(n)}) \to KO(X_{J^+}) \times_{KO(X\mathbf{Z})} KO(X_{\mathbf{Z}(n)}))$. (If $J = A$ already has a unit, this is analogous to the definition in topology $KO^{-n}(X) = KO(S^n \wedge (X^+))$, with $X = X_A$. If $J = I \subset A$ is an ideal with zero set $Z \subset X = X_A$, this is analogous to the topological definition $KO^{-n}(X, Z) = KO(S^n \wedge (X/Z))$. Here, $\wedge$ denotes smash product of pointed spaces $B \times C/(b_0 \times C \cup B \times c_0)$.) The following long exact sequence combines the ideas of Proposition 5.5 with homotopy classification.

PROPOSITION 5.6. *If $I \subset A$ is an ideal, then there is a long exact sequence for $n \geqq 0$,* $\cdots \to KO^{-n-1}(X_A) \to KO^{-n-1}(X_{A/I}) \to KO^{-n}(X_I) \to KO^{-n}(X_A) \to KO^{-n}(X_{A/I})$.

REMARK 5.7. Proposition 5.6 implies for the functors $KO^{-n}(X_J)$ (hence also $KO^{-n}(X_J) \otimes \mathbf{Z}[1/2]$), one of Karoubi's axioms **[10]**, characterizing the functors $W_n(J) \otimes \mathbf{Z}[1/2]$, namely, a long exact sequence associated to a short exact sequence of rings $0 \to J' \to J \to J'' \to 0$, possibly without units. Another axiom is homotopy invariance $W_n(J) \otimes \mathbf{Z}[1/2] \simeq W_n(J[T]) \otimes \mathbf{Z}[1/2]$ which follows for our functors $KO^{-n}(X_J)$ (hence also $KO^{-n}(X_J) \otimes \mathbf{Z}[1/2]$), from homotopy classification, Proposition 5.3. The third and final axiom is agreement for $n = 0$ with $W(J) \otimes \mathbf{Z}[1/2]$, which for $KO^0(X_J) \otimes \mathbf{Z}[1/2]$ is exactly our main theorem. We also remark that if A is the ring of continuous real valued functions on a compact space X, then it is possible to prove $KO^{-n}(X_A) \cong KO^{-n}(X)$, the usual topological KO-group. Thus, the functors $S_n(J) = KO^{-n}(X_J)$ seem to solve Karoubi's Problem 7 in **[11]**.

REMARK 5.8. Suppose $V \subset R^n$ is a real algebraic set. There are various ways to see why $KO^{-n}(X_{A(V)}) \cong KO^{-n}(V)$, where $A(V)$ is the affine coordinate ring of V, and $KO^{-n}(V)$ is the ordinary real K-theory of V (or suspensions of V if $n > 0$). For example, one can use homotopy classification of both sides of the equation and the fact that semi-algebraic homotopy classes of semi-algebraic maps from V or suspensions of V to the Grassmann manifolds coincide with topological homotopy classes of maps. Or, one can use the Serre-Swan theorem on both sides and then prove our ring $C(V)$ of continuous semi-algebraic functions has the same K_0 as the ring of all continuous functions on V and similarly for suspensions of V. This can be done even if V is non-compact, although a little more care is needed in this case. Anyway, V always admits a semi-algebraically strong deformation retract to a compact subspace, so non-compactness can be avoided altogether.

6. Witt Rings. If A is a commutative ring with 1, by a bilinear space over A, we mean a finitely generated, projective A-module E, and a symmetric A-bilinear pairing $b: E \times E \to A$ such that the adjoint map is an isomorphism $\hat{b}: E \simeq E^* = \mathrm{Hom}_A(E, A)$. The definitions of isomorphism of two bilinear spaces, direct sums $(E_1 \oplus E_2, b_1 \oplus b_2)$, products $(E_1 \otimes E_2, b_1 \otimes b_2)$ and base extension, or functoriality, $\varphi^*(E, b) = (B \otimes_A E, \cdot \otimes_A b)$ with respect to a ring homomorphism $\varphi: A \to B$, are all routine.

If $b_0: U_0 \times U_0 \to A$ is any symmetric bilinear pairing on a projective module U_0, define the metabolic bilinear space $M(U_0, b_0) = (U_0 \oplus U_0^*, \tilde{b}_0)$, where $\tilde{b}_0((u, \varphi), (v, \psi)) = b_0(u, v) + \varphi(v) + \psi(u) \in A$. The Witt ring $W(A)$ is defined to be the ring generated by isomorphism classes of bilinear spaces (E, b) modulo the ideal generated by metabolic spaces. If (E, b) is nonsingular, then $M(E, b) \cong (E, b) \oplus (E, -b)$ and $M(U_0, b_0) \otimes (E, b) \cong M(U_0 \otimes E, b_0 \otimes b)$. Thus, each element of $W(A)$ is represented by a bilinear space (E, b) and $[(E, b)] = [(E', b')]$ is $W(A)$ if and only if there are metabolic spaces M, M' with $E \oplus M \cong E' \oplus M'$ as bilinear spaces. A ring homomorphism $\varphi: A \to B$ induces a homomorphism of Witt rings $W(A) \to W(B)$.

For foundational material on Witt rings, we refer to **[2]**. Actually, we will be mostly working modulo 2-torsion, so we can be pretty casual with foundations. For example, we might as well assume $1/2 \in A$. Then bilinear spaces and quadratic forms coincide and metabolic forms are all hyperbolic, which is the metabolic case with $b_0 = 0$, $H(U_0) = M(U_0, 0)$.

A bilinear form on the free module A^n is just a symmetric $n \times n$ matrix $b = (b_{ij})$ with $\det(b_{ij}) \in A^*$, the units in A. Matrices b and c represent isomorphic forms if they are conjugate, that is, $c = {}^t mbm$ for some matrix m, where ${}^t m$ is the transpose of m. Projective modules over local rings are free. If $1/2 \in A$ and A is local, then any bilinear space (E, b) is even diagonalizable, that is, $(E, b) \cong (A^n, \langle a_1 \cdots a_n \rangle)$, where $\langle a_1 \cdots a_n \rangle(\bar{x}, \bar{y}) = \sum a_i x_i y_i$, $a_i \in A^*$. Slightly more generally, we have

PROPOSITION 6.1. *If (E, b) is a bilinear space over any A and $P \subset A$ is a prime ideal with $2 \notin P$, then there exists $d \notin P$ such that (E_d, b_d) is diagonalizable, where $(E_d, b_d) = \varphi_*(E, b)$, $\varphi: A \to A_d = A[1/d]$.*

Now let $X \subseteq X_A$ be a constructible subset, We want to sketch a proof of the following result, which was proved in the compact, topological case by Lusztig and by Gelfand and Mischenko. A proof can be found in [**8**]. In fact, we will simply imitate that proof in our setting.

PROPOSITION 6.2. *There is a natural isomorphism of rings $W(C(X)) \simeq KO(X)$.*

Essentially by the Serre-Swan theorem (See Proposition 4.5) an element of $W(C(X))$ is represented by a vector bundle $E \to X$ together with a symmetric non-singular form $b\colon E \otimes E \to \mathscr{E}^1$. Proposition 6.2 follows from

PROPOSITION 6.3. *Given a non-singular form b on a vector bundle E over X, there exist bundle decompositions $E \cong E_+ \oplus E_-$, such that $b|_{E_+}$ is positive definite and $b|_{E_-}$ is negative definite.*

Note that the isomorphism class of E_+ depends only on (E, b), because, if we fix E_- and find a second E'_+, then $E'_+ \cap E_- = (0)$, by definiteness of b on these subbundles, hence $E'_+ \cong E/E_- \cong E_+$. The map $W(C(X)) \to KO(X)$ of Proposition 6.2 is then given by $(E, b) \mapsto [E_+] - [E_-]$. This map is well defined, because a hyperbolic form splits into isomorphic positive definite and negative definite summands. The map is surjective because of the existence of positive definite (or negative definite) forms on any bundle (Proposition 4.3.). The map is injective because of the uniqueness of positive definite forms, up to isometry (Proposition 5.4.).

Now back to Proposition 6.3. Note that such splittings certainly exist locally, for example, by Proposition 6.1 applied to $C(X)$. Also, we only need to globally define E_+, locally of maximum possible dimension, since then we can take $E_- = E_+^\perp \subset E$. Associated to the vector bundle $E \to X$, there is a bundle $G \to X$, with the same structure group and with fibres the total Grassmann manifolds $G_{*,n}$ of affine subspaces of fibres of E. (Strictly speaking, we should work separately over disjoint open subsets of X on which E has constant fibre dimension.) Sections of $G \to X$ correspond precisely to subbundles of E, everything constructible, of course. Now, the form b reduces the structure group of E to various orthogonal groups, preserving the standard forms $\begin{pmatrix} I^k & 0 \\ 0 & -I^{n-k} \end{pmatrix}$ of various signatures (again, different groups over disjoint open subsets of X). There is then a subbundle $\mathfrak{B} \subset G$ over X, with fibres the subsets $P \subset G_{*,n}$ of the Grassmann manifolds, consisting of subspaces of maximum dimension on which b is positive definite. But these sets P can be identified with convex open sets in a affine space, hence are contractible. Now, exploiting partitions of unity and the Tietze extension theorem, one can prove any such constructible bundle $\mathfrak{B} \to X$ with contractible fibres has a section.

Such a section defines the desired subbundle $E_+ \subset E$, proving Proposition 6.3.

Here are some other results we will use in the proof of the Main Theorem.

PROPOSITION 6.4. *For any $q \in W(A)$, $2q$ is represented by a form on a free A-module.*

PROPOSITION 6.5. *Let $s_n = 1 + x_1^2 + \cdots + x_n^2$, $x_i \in A$. Then in $W(A_{S(1)})$, $2^n\langle s_n\rangle = 2^n\langle 1\rangle$.*

PROPOSITION 6.6. *For any A, the map $W(A) \to W(A_{S(1)})$ has 2-torsion kernel and cokernel.*

The last two results are due to Karoubi. And the first is easy. Proofs can be found in Mahé's paper **[15]**, or **[12]**. We also want to look at $W(A) \to W(\Gamma^\infty)$, where $\Gamma^\infty = \Gamma^\infty(X_A) \subset C(X_A)$ was defined in §3, by adjoining inverses and square roots of functions strictly positive on X_A. In this direction, we need the following result, also from Mahé's paper.

PROPOSITION 6.7. *Suppose 2, $d \in A^*$, and let $A' = A[y]/(y^2 - d)$, i: $A \to A'$ the inclusion. Then the trace map $\text{trace}_{A'/A}$: $A' \to A$ induces a non-singular form $\text{trace}_{A'/A}(a'b')$: $A' \times A' \to A$, and a transfer tr_*: $W(A') \to W(A)$, $\text{tr}_*(E', b') = (E', \text{trace} \circ b')$, with $\text{tr}_*(i_*(q)) = \langle 1, d\rangle q$, $q \in W(A)$, and $i_*(\text{tr}_*(q')) = q' + \tau_* q'$, $q' \in W(A')$, τ: $A' \to A'$ the involution over A with $\tau(y) = -y$.*

Now suppose j: $A \to B$ is a ring homomorphism, with $\sqrt{d} \in B$. Extend j to j': $A' \to B$, $j'(y) = \sqrt{d}$. Then we have

PROPOSITION 6.8. *For any $q' \in W(A')$, $j_*(\text{tr}_*(\langle 1, y\rangle q')) = \langle 1, \sqrt{d}\rangle j'_*(q') + \langle 1, -\sqrt{d}\rangle j'_*(\tau_* q')$. In particular, if $\sqrt[4]{d} \in B$, with $\sqrt{d} = (\sqrt[4]{d})^2$, then $j_*(\text{tr}_*(\langle 1, y\rangle q')) = 2 j'_*(q')$.*

The whole point is $j_*(\text{tr}_*(\langle 1, y\rangle q')) = j'_* i_*(\text{tr}_*(\langle 1, y\rangle q')) = j'_*(\langle 1, y\rangle q' + \tau_*(\langle 1, y\rangle q')) = j'_*(\langle 1, y\rangle q' + \langle 1, -y\rangle \tau_* q') = \langle 1, \sqrt{d}\rangle j'_*(q') + \langle 1, -\sqrt{d}\rangle j'_*(\tau^* q')$. Of course, we have in mind $d > 0$ on X_A, $B = \Gamma^\infty(X_A)$. Such d are units in $A_{S(1)}$, by the Positivstellensatz. We iterate the results 6.5, 6.6, 6.7, 6.8, that is construct a sequence of formal localizations and formal square root adjunctions which map to $\Gamma^\infty(X_A)$, and conclude

PROPOSITION 6.9. *$W(A) \to W(\Gamma^\infty(X_A))$ has 2-torsion kernel and cokernel.*

Surjectivity is implied by Proposition 6.8 (and 6.6), since any element of $W(\Gamma^\infty(X_A))$ clearly comes from some iterated formal localization and square root extension of A. Injectivity is a little harder. Suppose $j_*(q) = 0$, $q \in W(A)$, j: $A \to \Gamma^\infty$. By 6.4, we may as well assume q is represented by a matrix over A. Also, by adding a hyperbolic matrix if necessary, we may as well assume q is conjugate to a hyperbolic matrix over Γ^∞, that is, ${}^t mqm = h$, $h = \oplus\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, m a matrix over Γ^∞. Let $A^{(1)} = A_{S(1)}[\sqrt[2^k]{S(1)}, k \geqq 1]$, meaning formally invert and adjoin 2^k-th roots of all $1 + \sum x_i^2$, $x_i \in A$. Note $X_{A^{(1)}} \cong X_A$. Also, kernel(i_*: $W(A) \to W(A^{(1)})$ is 2-torsion, essentially by the formula $\mathrm{tr}_*(i_*(q)) = \langle 1, d\rangle q$ of Proposition 6.7, along with 6.6 and 6.5, which implies $2^m\langle d\rangle = 2^m\langle 1\rangle$, some m, if $d \in A^*$, $d > 0$ on X_A. Let $A^{(n)} = (A^{(n-1)})^{(1)}$ if $n > 1$. Our matrix m over Γ^∞ comes from an invertible matrix m' over some $A^{(n)}$. But, over $A^{(n)}$ we can only conclude ${}^t m'\ i_*(q)m' \equiv h$ modulo the real radical of $A^{(n)}$, that is, modulo kernel $(A^{(n)} \to \Gamma^\infty)$. But then a straightforward Gauss diagonalization over $A^{(n+1)}$ shows $i_*(q) = 0 \in W(A^{(n+1)})$, so q is 2-torsion in $W(A)$.

7. An Abstract Stone-Weierstrass Theorem. The following examples show that polynomials are not in any good sense dense among semi-algebraic functions.

EXAMPLE 1. If k is a real closed field which contains an infinitesimal ε relative to $\mathbf{Q}$, then the function $|x|$ on $[-1, 1] \subset k^1$ cannot be approximated within ε by a polynomial.

EXAMPLE 2. There exist pairs of disjoint, closed semi-algebraic sets in affine space over any real closed field k which cannot be separated by a polynomial. Thus the semi-algebraic function which is $+1$ on one set and -1 on the other is not even homotopic to a polynomial, as functions with values in $k^* = k - (0)$. If k is non-Archimedean, the disjoint closed sets can actually be chosen bounded, **[16]**.

Both these examples can be eliminated if square roots are allowed. For example, $0 < \sqrt{x^2 + \varepsilon^2} - |x| < \varepsilon$. Also, a result of Bochnak, Efroymson and Mostowski is that if the polynomial ring A is replaced by the extension A^∞ of §3, then disjoint closed sets can be separated. However, square roots will not do everything, as the following example of Ron Brown shows.

EXAMPLE 3. Let k be the real closure of $\mathbf{Q}(\varepsilon)$ (or of $\mathbf{R}(\varepsilon)$) with $\varepsilon > 0$ infinitesimal. Then $\sqrt[3]{x}$ on $[0, 1] \subset k^1$ cannot be approximated within ε by an element of A^∞, where $A = k[x]$.

On the other hand, Efroymson **[5]** has proved roughly that arbitrary continuous semi-algebraic functions can be approximated by Nash functions, within any semi-algebraic tolerance or error. Our purpose in this section is to give an abstract version of Efroymson's results. We first state our Stone-Weierstrass theorem for closed constructibles, which is all we need for the main theorem. However, below we will give a version for arbitrary constructibles.

PROPOSITION 7.1. *Suppose A is a ring, $X \subseteq X_A$ a closed constructible* *subset. Suppose $\Lambda \subseteq C(X)$ is a subring which contains (the image of) A and which satisfies the condition: $\lambda \in \Lambda$, $\lambda > 0$ on X implies $\sqrt{\lambda} \in \Lambda$. Suppose $e \in C(X)$, $e > 0$ on X and consider any $g \in C(X)$. Then there exists $f \in \Lambda$ with $|f - g| < e$ on X if and only if for each $\alpha \in X$ there exists $f_\alpha \in \Lambda$ with $|f_\alpha(\alpha) - g(\alpha)| < e(\alpha)$ in $k(\alpha)$.*

Before discussing the proof, we make several remarks. First, a variant of 7.1 for all constructibles.

PROPOSITION 7.2. *If $X \subset X_A$ is an arbitrary constructible, the same statement holds if $\Lambda \subseteq C(X)$ satisfies: $\lambda \in \Lambda$, $\lambda > 0$ on X implies $\sqrt{\lambda}$ and $\lambda^{-1} \in \Lambda$.*

REMARK 7.3. In either version of the result, one needs only to approximate g at (relatively) maximal points $\alpha \in X$, because of Propositions 2.2 and 3.3.

REMARK 7.4. For general X, the rings $N(X)$ and $\Gamma^\infty(X)$ satisfy the hypothesis on Λ. If X is closed, so does $A^\infty(X)$. (In fact, if the Tietze extension theorem is used, the proof below even applies to $\Lambda = \text{image}(A^\infty(X_A) \to C(X))$.) Density of $N(X)$ in $C(X)$ follows once one argues that an arbitrary element of $k(\alpha)$, for $\alpha \in X$, α maximal, can be approximated by the value of a Nash function defined on all of X. This is pretty hard, but Efroymson's arguments for the case of affine semi-algebraic sets over a real closed field essentially carry over. One does not expect actual density for any other subrings of $C(X)$, precisely because in the real closed fields $k(\alpha)$ it is difficult to find dense subfields. Instead our application will be that piecewise $-A^\infty$ or Γ^∞ functions are A^∞ or Γ^∞ approximable.

We now discuss the proofs of propositions 7.1 and 7.2. The idea is to first show how to find $f_{\alpha,\beta} \in \Lambda$, for each pair of maximal points α, β, so that $|f_{\alpha,\beta}(\alpha) - g(\alpha)| < e(\alpha)$ and $|f_{\alpha,\beta}(\beta) - g(\beta)| < e(\beta)$. Then the standard proof of Stone-Weierstrass goes through. That is, fixing α, finitely many neighborhoods $U_{\alpha,\beta_j} = \{\beta | f_{\alpha,\beta_j}(\beta) < g(\beta) + e(\beta)\}$ cover X. Let $F_\alpha = \min_j(f_{\alpha,\beta_j})$, so $F_\alpha < g + e$ on X and $g(\alpha) - e(\alpha) < F_\alpha(\alpha)$. Then finitely many $V_{\alpha_i} = \{\alpha | g(\alpha) - e(\alpha) < F_{\alpha_i}(\alpha)\}$ cover X. Take $F = \max_i(F_{\alpha_i})$, so $|F - g| < e$ on X. Now, F_α and F are not necessarily in Λ, but we have the formulas $\min(f_1, f_2) = (1/2)(f_1 + f_2 - |f_1 - f_2|)$ and $\max(f_1, f_2) = (1/2)(f_1 + f_2 + |f_1 - f_2|)$, and the approximation $|f_1 - f_2| < \sqrt{(f_1 - f_2)^2 + (e_0)^2} < |f_1 - f_2| + e_0$, so it is only necessary to find very small positive $e_0 \in \Lambda$ to get $f \in \Lambda$ with $|f - F| < e - |F - g|$, hence $|f - g| < e$. (See Proposition 7.7 below for the construction of arbitrarily small positive $e_0 \in A^\infty(X) \subset \Lambda$.)

How do we find $f_{\alpha,\beta}$, given approximations f_α, f_β of g at α and β separately? We take $f_{\alpha,\beta} = \varphi_\alpha f_\alpha + \varphi_\beta f_\beta$, where $0 \leqq \varphi_\alpha, \varphi_\beta \in A^\infty = A^\infty(X_A)$ if X is closed, $\varphi_\alpha + \varphi_\beta = 1$, and $\varphi_\alpha(\beta)$ and $\varphi_\beta(\alpha)$ are very small. Then $|f_{\alpha,\beta} - g| =$

$|\varphi_\alpha(f_\alpha - g) + \varphi_\beta(f_\beta - g)| \leqq \varphi_\alpha|f_\alpha - g| + \varphi_\beta|f_\beta - g|$, which is close to $|f_\alpha - g|$ at α and close to $|f_\beta - g|$ at β.

REMARK 7.5. If X is arbitrary, we can only construct $\varphi_\alpha, \varphi_\beta \in \Gamma^\infty(X)$.

Here is the construction of $\varphi_\alpha, \varphi_\beta \in A^\infty$ if X is closed. First, choose $h_1 \in A$, with $h_1(\alpha) > 0$, $h_1(\beta) < 0$. Since X closed implies α and β are maximal in X_A, by Remaık 2.3 find

$$h_2 \in A \text{ with } h_2(\alpha) > \frac{1 + (f_\beta(\alpha) - g(\alpha))^2}{(e(\alpha) - |f_\alpha(\alpha) - g(\alpha)|)h_1(\alpha)}$$

and

$$h_3 \in A \text{ with } h_3(\beta) > \frac{1 + (f_\alpha(\beta) - g(\beta))^2}{(e(\beta) - |f_\beta(\beta) - g(\beta)|)\,|h_1(\beta)|}.$$

Set $h_4 = (1 + h_2^2)(1 + h_3^2)h_1 \in A$. The point is that h_4 is now very large and positive at α and very large and negative at β. Set $h_5 = \sqrt{h_4^2 + 1} - h_4$, so h_5 is very large at β and very small at α. Set $\varphi_\alpha = \sqrt{h_5^2 + 1} - h_5$ and $\varphi_\beta = 1 - \varphi_\alpha$. Then $h_5 \in A^\infty$, hence also $\varphi_\alpha, \varphi_\beta \in A^\infty$ and

$$\varphi_\alpha(\beta) < \frac{1}{h_5(\beta)} < \frac{1}{|h_4(\beta)|} < \frac{e(\beta) - |f_\beta(\beta) - g(\beta)|}{1 + (f_\alpha(\beta) - g(\beta))^2},$$

$$\varphi_\beta(\alpha) < h_5(\alpha) < \frac{1}{h_4(\alpha)} < \frac{e(\alpha) - |f_\alpha(\alpha) - g(\alpha)|}{1 + (f_\beta(\alpha) - g(\alpha))^2}.$$

Thus, $|f_{\alpha,\beta} - g| \leqq \varphi_\alpha|f_\alpha - g| + \varphi_\beta|f_\beta - g| < e$ at both α and β.

To complete the proof of 7.1 and 7.2, we need to construct arbitrarily small positive $e_0 \in A^\infty(X)$ (at the step in the proof where we approximate $|f_1 - f_2|$ by $\sqrt{(f_1 - f_2)^2 + e_0^2} \in \Lambda$) and, for non-closed X, we need to construct $\varphi_\alpha, \varphi_\beta \in \Gamma^\infty(X)$. Here is really the key lemma.

PROPOSITION 7.6. *Let $X \subseteq X_A$ be a constructible subset, $\alpha \in X$ a maximal point of X, $0 < e_\alpha \in k(\alpha)$. Then there exists $d \in A^\infty(X)$ with $0 < d < 1$ on X and $d(\alpha) < e_\alpha$. If X is closed, there exists $d' \in A_{S(1)}$ and also $d'' \in A^\infty(X_A)$ with $0 < d', d'' < 1$ on X_A and $d'(\alpha), d''(\alpha) < e_\alpha$.*

PROPOSITION 7.7. *Suppose $e \in C(X)$ with $0 < e$ on X. Then there exists $d \in A^\infty(X)$ with $0 < d < e$ on X. If X is closed, there exists $d \in A^\infty(X_A)$ with $0 < d$ on X_A and $d < e$ on X.*

We deduce 7.7 from 7.6 by choosing d_α with $0 < d_\alpha < 1$, $d_\alpha(\alpha) < e(\alpha)$, for each maximal $\alpha \in X$. Finitely many neighborhoods $U_{\alpha_i} = \{\beta \in X | d_{\alpha_i}(\beta) < e(\beta)\}$ cover X. Then take $d = \prod_i d_{\alpha_i}$.

Proposition 7.6 for X closed is really pretty easy, since then α is actually maximal in X_A. By Remark 2.3, find $p \in A$ with $p^2(\alpha) > 1/e_\alpha$. Then $d' = 1/(1 + p^2) \in A_{S(1)}$ satisfies $0 < d' < 1$ on X_A and $d'(\alpha) < e_\alpha$. Also,

$d'' = \sqrt{p^4 + 1} - p^2 \in A^\infty$ has these same properties. This then completes the proof of Proposition 7.1.

For general X, Proposition 7.6 seems quite a bit harder. It does follow from

PROPOSITION 7.8. *For any X, let $S^\infty(X) = \{q \in A^\infty(X) | q > 0$ on $X\}$. Then the real spectrum of $\Gamma^\infty(X) = A^\infty(X)_{S^\infty(X)}$ coincides with $\{\beta \in X_A | \beta$ has a specialization in $X\}$.*

Assuming 7.8, any maximal $\alpha \in X$ is also maximal in the real spectrum of $A^\infty(X)_{S^\infty(X)}$. Thus, by the argument just above, there exists $p/q \in A^\infty(X)_{S^\infty(X)}$ with $p(\alpha)/q(\alpha) > 1/e^\alpha$. Then $q(\sqrt{p^2+1} - p) = d_0 \in A^\infty(X)$ satisfies $0 < d_0$ on X, $d_0(\alpha) < e_\alpha$. Finally, take $d = d_0(\sqrt{d_0^2 + 1} - d_0) < \min(d_0, 1)$ on X (or replace p by $1 + p^2 + q$, so that $p/q > 1$ on X and $d_0 < 1$).

To prove 7.8, suppose $\beta \in X_A$ does not have a specialization in X. If $\gamma \in X$, choose $g_\gamma \in A$ with $g_\gamma(\beta) \leqq 0$, $g_\gamma(\gamma) > 0$. Finitely many $U(g_{\gamma_i})$ cover X and $\beta \in W\{-g_{\gamma_i}\}$. Let $U = \bigcup_i U(x_i) \subset x_{\mathbf{Z}[x_1 \cdots x_n]}$. The point now is that [3, Theorem 5.2] constructs an explicit element q in some iterated square root extension B of $\mathbf{Z}[x_1 \cdots x_n]$, obtained by adjoining square roots of nowhere negative functions strictly positive on U, such that $q > 0$ on U and $q = 0$ on $W\{-x_i\} \subset X_B$. Then $\varphi\colon \mathbf{Z}[x_1 \cdots x_n] \to A$, $\varphi(x_i) = g_{\gamma_i}$ extends to $\varphi\colon B \to A^\infty(X)$ and $1/(\varphi(q)) \in A^\infty(X)_{S^\infty(X)}$. Thus $\beta\colon A \to k(\beta)$ with $g_{\gamma_i}(\beta) \leqq 0$ cannot possibly be extended to $A^\infty(X)_{S^\infty(X)} \to k(\beta)$.

REMARK. The arguments of [3] use special properties of affine spaces over real closed fields. Our argument above shows how certain abstract results are easily reduced to the affine case. On the other hand, a closer look at [3, Theorem 5.2] shows that the formula for q is constructed over any ring, by induction on n, the number of g_{γ_i}, once one has available the following abstract Mostowski separation theorem.

PROPOSITION 7.9. *Let A be a ring, W_1, $W_2 \subset X_A$ disjoint, closed constructibles. Then there exists $\theta \in A^\infty(X_A)$ with $\theta(W_1) > 0$ and $\theta(W_2) < 0$.*

I know a direct proof of 7.9 which uses the Positivstellensatz and some simple constructions with square roots. Also, 7.9 can be deduced from Proposition 7.1 for closed X, namely, construct a simple piecewise-A function on X_A which is positive on W_1 and negative on W_2, similar to the partition of unity construction of Proposition 3.6. Then approximate by an $A^\infty(X_A)$ function. Essentially combining 7.9 with the Remark above yields also the relative version of Mostowski separation.

PROPOSITION 7.10. *Let A be a ring, $U \subset X_A$ an open constructible, and W_1, $W_2 \subset U$ disjoint relatively closed constructibles. Then there exists $\theta \in A^\infty(U)$ with $\theta(W_1) > 0$ and $\theta(W_2) < 0$.*

Now back to the construction of $f_{\alpha,\beta} = \varphi_\alpha f_\alpha + \varphi_\beta f_\beta$, with $\varphi_\alpha, \varphi_\beta \in \Gamma^\infty(X)$, for general X. Since α and β are both maximal in X, they are not comparable. Hence, there is $h \in A$ with $h(\alpha) > 0$ in $k(\alpha)$ and $h(\beta) < 0$ in $k(\beta)$. Find $d \in A^\infty(X)$ with $0 < d < 1$ on X, $d(\alpha) < h(\alpha)$, $d(\beta) < |h(\beta)|$. Find $e' \in A^\infty(X)$ with $0 < e' < 1$ and $e'(\alpha) < e(\alpha) - |f_\alpha(\alpha) - g(\alpha)|$, $e'(\beta) < e(\beta) - |f_\beta(\beta) - g(\beta)|$. Find $e'' \in A^\infty(X)$ with $0 < e'' < de'/(1 + (f_\alpha - g)^2 + (f_\beta - g)^2)$. Then take $\varphi_\alpha = m_\alpha/m_\alpha + m_\beta$, $\varphi_\beta = m_\beta/m_\alpha + m_\beta$, where

$$m_\alpha = \sqrt{(h + d)^2 + (e'')^2} + h + d$$

$$m_\beta = \sqrt{(h - d)^2 + (e'')^2} - (h - d).$$

The point is that

$$m_\alpha(\alpha) > 2d(\alpha),\ m_\alpha(\beta) < e''(\beta),$$

$$m_\beta(\beta) > 2d(\beta),\ m_\beta(\alpha) < e''(\alpha).$$

Thus,

$$\varphi_\beta(\alpha) < \frac{e''(\alpha)}{2d(\alpha)} < \frac{e'(\alpha)}{1 + (f_\beta(\alpha) - g(\alpha))^2}$$

and

$$\varphi_\alpha(\beta) < \frac{e''(\beta)}{2d(\beta)} < \frac{e'(\beta)}{1 + (f_\alpha(\beta) - g(\beta))^2},$$

so $|f_{\alpha,\beta} - g| \leqq \varphi_\alpha|f_\alpha - g| + \varphi_\beta|f_\beta - g| < e$ at both α and β, completing the proof of Proposition 7.2.

REMARK. These formulas for $\varphi_\alpha, \varphi_\beta$ were stolen from Efroymson's paper [5].

We also need a relative version of the Stone-Weierstrass theorem in the next section.

PROPOSITION 7.11. *Let* $A \subset C(X)$, *with hypotheses of* 7.1 *or* 7.2 *Suppose* $Y \subset X$ *is a subspace defined as the zeros of some* $h \in A$, $Y = Z(h)$. *Suppose* $g \in C(X)$ *with* $g|_Y = p|_Y$, *some* $p \in A$. *Then there exists* $f \in A$ *with* $|f - g| < e$ *and* $f|_Y = g|_Y$ *if for each* $\alpha \in X$ *there exists* $f_\alpha \in A$ *with* $|f_\alpha(\alpha) - g(\alpha)| < e(\alpha)$.

One proof is to take some $f_1 \in A$ close to g. Then take $f = \varphi_1 f_1 + (1 - \varphi_1)p$, where $\varphi_1|_Y = 0$, $0 \leqq \varphi_1 \leqq 1$, with φ_1 very close to 1 outside the neighborhood of Y defined by $|g - p| < e/2$. First, by multiplying h by a suitable product $\prod(1 + h_j^2)$, one can assume h is absolutely huge on $|g - p| \geqq e/2$. Then take $\varphi_1 = h(\sqrt{h^2 + 2} - h)$, which satisfies $1 - (1/2h^2) < \varphi_1 < 1$ if $h > 1$.

8. Proof of the Main Theorem. If A is any ring, we now have homomorphisms $W(A) \to W(\Gamma^\infty(X_A)) \to W(C(X_A)) \to KO(X_A)$. The first is an isomorphism modulo 2-torsion by Proposition 6.9 and the last is an isomorphism by Proposition 6.3. Let $W'(A)$ be the free Witt ring, defined as the Grothendieck ring based on forms on free A-modules, modulo the ideal of metabolic forms on free modules. (When $1/2 \in A$, metabolic forms are hyperbolic.) Then $W'(A) \to W(A)$ is an isomorphism modulo 2-torsion. In fact, the kernel and cokernel consist of elements of order 2, essentially by the argument proving our Proposition 6.4 in [15]. The main theorem of the Introduction then follows from

PROPOSITION 8.1. *For any constructible* $X \subseteq X_A$, $W'(\Gamma^\infty(X)) \simeq W'(C(X))$ *is an isomorphism.*

There are two key ideas in the proof of 8.1. We first give homotopy theoretic interpretations of both $W'(\Gamma^\infty(X))$ and $W'(C(X))$. Then we prove that these homotopy theoretic interpretations coincide.

Suppose $U \subset X_{\mathbf{Z}[T_1 \cdots T_n]}$ is an open constructible. For any A, let $U(A) \subset X_{A[T_1 \cdots T_n]}$ be the constructible defined by the same formulas a U. ($U(A) = \varphi_*^{-1}(U)$, φ: $\mathbf{Z}[T_1 \cdots T_n] \to A[T_1 \cdots T_n]$.) If $X \subseteq X_A$, we have $[X, U]$, homotopy classes of $U(A)$-valued sections of $X_{A[T_1 \cdots T_n]} \to X_A$ over X, as in §5. Let $[X, U]_{\Gamma^\infty}$ denote the Γ^∞-homotopy classes of $U(A)$-valued sections whose coordinates are in $\Gamma^\infty(X)$, where two sections are Γ^∞-homotopic if there is a $U(A[T])$-valued section over $X \times I$ with entries in $\Gamma^\infty(X \times I)$ which restricts to the given sections over $X \times \{0\}$ and $X \times \{1\}$. A similar definition of $[X, U]_B$ can be given for any universally defined ring B between A and C, for example, $A^\infty(X)$ and $N(X)$. It seems to me that many nice questions in "algebraic topology" may take the form of a comparison $[X, U]_B \to [X, U]$. Roughly, which functions do you really need to understand homotopy theory? Topologists use the Stone-Weierstrass theorem rather casually. Obviously, sufficiently close maps to U are homotopic, by a canonical linear homotopy, because of local convexity of U. In our abstract setting, this gives $[X, U]_N \cong [X, U]$, by Efroymson's Nash function version of Stone-Weierstrass, Remark 7.4. But other subrings $B \subset C$ are not actually dense, so more subtle geometric properties of U and algebraic properties of B will be required to find homotopies when studying $[X, U]_B \to [X, U]$. Note the simplest proper open set in an affine space is $k^* = k - (0)$ (the complement of the simplest closed set). The question of surjectivity of $[X, k^*]_B \to [X, k^*]$ is the question of which functions are needed to separate components.

We will work with $U = \text{Symm}(n)$, the symmetric, non-singular $n \times n$ matrices. So U is defined in affine $n(n+1)/2$ space, by a single non-equality $\det(s_{ij}) \neq 0$, where $s_{ij} = s_{ji}$. Stabilize $\text{Symm}(n) \to \text{Symm}(n+2)$ by $s \to \left(\begin{smallmatrix} s & 0 \\ 0 & h \end{smallmatrix}\right)$, $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

PROPOSITION 8.2. *There is a natural bijection* $\lim_{n\to\infty}[X, \mathrm{Symm}(n)] \cong W'(C(X))$.

This result is a consequence of the bundle covering homotopy theorem, Proposition 5.1. A Symm(n)-valued section over X defines an element of $W'(C(X))$, and this correspondence is surjective in the limit $n \to \infty$. A homotopy defines a form on the trivial bundle over $X \times I$. The form amounts to an interpretation of the structure group as various orthogonal groups preserving the standard indefinite forms $\begin{pmatrix} I^k & 0 \\ 0 & -I^{n-k} \end{pmatrix}$. Proposition 5.1 then gives isometries of the forms over $X \times \{0\}$ and $X \times \{1\}$, so $[X, \mathrm{Symm}(n)] \to W'(C(X))$ is well defined. Stable injectivity is standard. Choose a homotopy from $\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ to $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, say $\begin{pmatrix} I-tI, & tI \\ -tI, & I-tI \end{pmatrix}$, $0 \leqq t \leqq 1$. Since $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & m \end{pmatrix}$, we see $\begin{pmatrix} {}^tm & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} {}^tmqm & 0 \\ 0 & I \end{pmatrix}$ is homotopic through symmetric matrices to $\begin{pmatrix} I & 0 \\ 0 & {}^tm \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & m \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & {}^tmm \end{pmatrix}$, then to $\begin{pmatrix} q & 0 \\ 0 & I \end{pmatrix}$, since ${}^tmm$ is positive definite. Thus $\begin{pmatrix} q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{pmatrix}$ is homotopic to $\begin{pmatrix} {}^tmqm & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{pmatrix}$.

A more subtle result is the following.

PROPOSITION 8.3. *There is a natural bijection* $\lim_{n\to\infty}[X, \mathrm{Symm}(n)]_{\Gamma^\infty} \cong W'(\Gamma^\infty(X))$.

We need to examine the above proof more closely, specifically, to understand why Γ^∞-homotopic forms are Γ^∞-conjugate, at least stably. Basically, this is because the Gram-Schmidt process, even for indefinite forms, is a Γ^∞-process. If q is a symmetric $n \times n$ matrix, let q_p denote the upper left $p \times p$ submatrix. Suppose q and q' are such that $\mathrm{sign}(\det(q_p)) = \mathrm{sign}(\det(q'_p)) \neq 0$, for $1 \leqq p \leqq n$. Then q and q' are canonically conjugate by an upper triangular matrix m, ${}^tmqm = q'$. This is just Gram-Schmidt. The diagonal entries of m are determined from the equation $({}^tmqm)_p = {}^tm_pq_pm_p$ (m upper triangular), giving $(\det(m_p))^2\det(q_p) = \det(q'_p)$. The above diagonal entries of m are solutions of simple linear equations. So if q and q' are Γ^∞-matrices, so is m.

Of course, the hypothesis $\det(q_p) \neq 0$ is restrictive. However, one can conjugate any q by a matrix z defined over $\mathbf{Z}$ so that $\det(({}^tzqz)_p) \neq 0$, $1 \leqq p \leqq n$. In fact, one only needs to iterate the two operations, switching a pair of basis vectors and adding or subtracting one basis vector from another.

Now consider $q(t)$, a symmetric matrix with entries in $\Gamma^\infty(X \times I)$. We would like an isometry between the forms defined by $q(t)$ and $\pi^*(q(0))$, $\pi\colon X \times I \to X$, $q(0)$ defined over $\Gamma^\infty(X)$. By the discussion in the above two paragraphs, this is trivial locally, that is, near any point $\beta \in X \times I$. The method of proof of the bundle covering homotopy theorem, as discussed briefly following Proposition 5.2, then constructs a global isometry, piece-by-piece. However, the final matrix $m(t)$ which conjugates $\pi^*q(0)$ to $q(t)$ is not necessarily a $\Gamma^\infty(X \times I)$ matrix. Instead, it is defined over pieces of $X \times I$, say Y_{ij}, with entries in $\Gamma^\infty(Y_{ij})$ over each piece. In local diagonalization we need formulas like $m_{11} = \sqrt[4]{(q'_{11})^2/q_{11}^2}$, and the denominators introduced, though defined globally, may go to 0 outside the subsets Y_{ij}.

We are saved by the Stone-Weierstrass theorem. Such an m_{11} can be approximated at any one point by a globally defined $\Gamma^\infty(X \times I)$ function, like $m'_{11} = \sqrt[4]{(q'_{11})^2/q_{11}^2 + e^2}$, where $0 < e \in A^\infty(X \times I)$, e very small. Thus, the matrix $m(t)$ can be arbitrarily closely approximated by $m'(t) \in \Gamma^\infty(X \times I)$. The conclusion is that ${}^t m'(1)q(0)m'(1)$ is very close to $q(1)$, as matrices over $\Gamma^\infty(X)$.

Finally, we can at least conclude $q(0)$ and $q(1)$ are stably conjugate as $\Gamma^\infty(X)$-forms. Namely, $\begin{pmatrix} q(0) & 0 \\ 0 & -q(0) \end{pmatrix}$ is conjugate to a hyperbolic matrix h, so $\begin{pmatrix} q(1) & 0 \\ 0 & -q(0) \end{pmatrix}$ is conjugate to a matrix h' as close to h as we want. But now diagonalization shows h' and h are Γ^∞-conjugate, so $q(1)$ and $q(0)$ represent the same element of $W'(\Gamma^\infty(X))$, completing the only hard part of the proof of Proposition 8.3.

It remains to study $[X, \mathrm{Symm}(n)]_{\Gamma^\infty} \to [X, \mathrm{Symm}(n)]$. If $U \subset X_{\mathbf{Z}[T_1 \cdots T_n]}$ is an open affine constructible which is a finite union of convex, open constructibles, then it is rather easy to prove that $[X, U]_{\Gamma^\infty} \to [X, U]$ is a bijection, using Γ^∞ near partitions of unity, that is, Γ^∞ approximations of piecewise Γ^∞ partitions of unity. However, this essentially never happens if U is defined by (non-linear) non-equalities $d(T_i) \neq 0$. So we need a weaker kind of decomposition of U, which still allows certain linear combinations of functions and homotopies to be defined.

Consider the diagonal $\Delta \subset X_{\mathbf{Z}[S_1 \cdots S_n, T_1 \cdots T_n]}$, defined by $S_i = T_i$, $1 \leqq i \leqq n$. Let $\Delta(U) = \Delta \cap U \times U$ where $U \times U$ is the open constructible defined by the defining equations for U in both the S and T variables. If $\beta \in X_{\mathbf{Z}[S_i]}$ and $C \subseteq X_{\mathbf{Z}[S_i, T_i]}$, let $C(\beta) = C \cap X_{k(\beta)[T_i]}$, where $X_{k(\beta)[T_i]} \subset X_{\mathbf{Z}[S_i, T_i]}$ is the fibre of the projection $X_{\mathbf{Z}[S_i, T_i]} \to X_{\mathbf{Z}[S_i]}$ over β. By a rational point in $X_{k(\beta)[T_i]}$ we mean a point ρ so that $T_i(\rho) \in \mathbf{Q}$, $1 \leqq i \leqq n$. We now define the following property of an open set U in affine n-space.

Property C. There exists an open constructible $C \subset X_{\mathbf{Z}[S_i, T_i]}$ with $\Delta(U) \subseteq C \subseteq U \times U$ such that for each $\beta \in U \subset X_{\mathbf{Z}[S_i]}$, $C(\beta) \subset X_{k(\beta)[T_i]}$ is convex and contains a rational point.

For any ring A, one then has $C(A) \subset X_{A[S_i, T_i]}$ with the same properties. Roughly, Property C says that for any real closed field k and $\vec{b} \in U_k \subset k^n$, there is a convex open set $C(\vec{b}) \subset U_k$ large enough to contain both $\vec{b}$ and a rational point $\vec{r} \in \mathbf{Q}^n \subset k^n$.

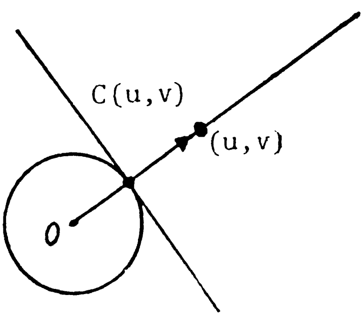
PROPOSITION 8.4. *If U has Property C and $X \subseteq X_A$ is any constructible,* *then* $[X, U]_{\Gamma^\infty} \to [X, U]$ *is bijective. If X is closed, then* $[X, U]_{A^\infty} \to [X, U]$ *is bijective.*

First, if $\vec{r} \in \mathbf{Q}^n$ is a fixed rational vector, then $D(\vec{r}) = \{\beta \in U(A) | \vec{r} \in C(\beta)\}$ is an open constructible in $U(A) \subset X_{A[S_1 \cdots S_n]}$, and the $D(\vec{r})$ cover $U(A)$. Thus, there is a finite subcover $U(A) = \bigcup_j D(\vec{r}_j)$. If $\beta \in X_{k(\alpha)[S_1 \cdots S_n]}$ lies over $\alpha \in X_A$, then there is the vector called $\vec{s}(\beta) \in k(\beta)^n$, and the line segment $[\vec{s}(\beta), \vec{r}]$ is in the convex open set $C(\beta) \subset X_{k(\beta)[T_i]}$ whenever $\vec{r} \in C(\beta)$.

If $s \in [X, U]$ is a $U(A)$-valued section over $X \subseteq X_A$, let $V_j = s^{-1}(D(\vec{r}_j)) \subset X$ and let let $\sum p_j = 1$, $0 \leqq p_j \leqq 1$, be a piecewise-$A_{S(X)}$ partition of unity subordinate to the cover $X = \bigcup_j V_j$. Then $r = \sum p_j \vec{r}_j \in [X, U]$, since if $p_j(\alpha) \neq 0$, then $\vec{r}_j \in C(s(\alpha))$, which is convex, so $r(\alpha) \in U$. Moreover, tr $+ (1 - t)s \in [X \times I, U]$ is a homotopy from s to r. Since U is open and r is piecewise $-\Gamma^\infty$, there are arbitrarily close approximations to r in $[X, U]_{\Gamma^\infty}$ by the Stone-Weierstrass theorem, and of course, these close approximations are homotopic to r. Thus $[X, U]_{\Gamma^\infty} \to [X, U]$ is surjective.

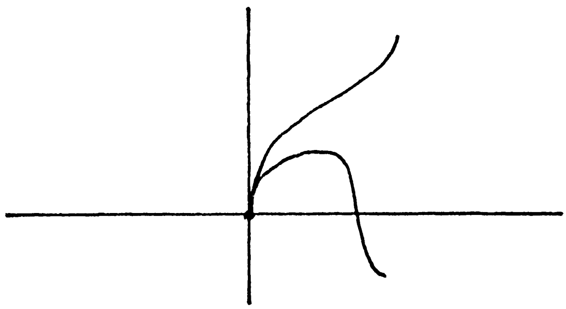
If s_0 and $s_1 \in [X, U]_{\Gamma^\infty}$ are homotopic in $[X, U]$, say by $S \in [X \times I, U]$, the aboveargument produces a piecewise-Γ^∞ $R \in [X \times I, U]$, also a homotopy from s_0 to s_1. (Go around three sides of the box $X \times I \times I$.) By the relative Stone-Weierstrass theorem 7.11, s_0 and s_1 are Γ^∞-homotopic. (If X is closed, use A^∞ near partitions of unity in the argument.)

Here is an example of the above construction. Let U be defined by $x^2 + y^2 > 1$ in the plane. Let C be defined by $xu + yv > \sqrt{u^2 + v^2}$ in $U \times U$. For fixed (u, v), $C(u, v)$ is convex in $x - y$ space. Precisely, $C(u, v)$ is just the half plane not containing the circle $x^2 + y^2 = 1$, determined by the tangent line perpendicular to (u, v). Of course, these half-planes



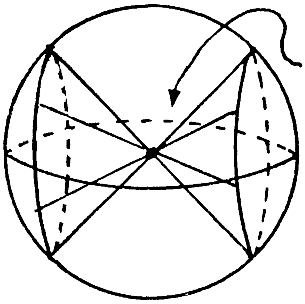
contain rational points, no matter what the ground field. In fact, for any (u, v), one of $(2, 2)$, $(2, -2)$, $(-2, -2)$, $(-2, 2) \in C(u, v)$. (In fact, three suitable points would work.) What is happening in the proof of 8.4 is that we are deforming U itself in a piecewise $-\Gamma^\infty$ manner to the simplicial complex with vertices the $\vec{r}_j$ and structure of the nerve of the cover $U = \bigcup_j D(\vec{r}_j)$.

If U is the complement of a bounded, non-singular variety defined over $\mathbf{Z}$, or the complement of a non-singular (except at the origin) homogeneous variety defined over $\mathbf{Z}$, then U has Property C. A proof exploits the tangent planes and a tubular neighborhood of the variety. However, if the variety has singularities, even at infinity, Property C can fail for the complement. An example is provided by $U = \{(x, y) | y^3 \neq x \pm \sqrt{x^3}\}$.



Over $k = \mathbf{Q}(\varepsilon)$, with $\varepsilon > 0$ infinitesimal, we have $(\varepsilon^3, \varepsilon) \in U$, but no rational point in U is visible from $(\varepsilon^3, \varepsilon)$.

What about $U = \text{Symm}(n)$, the complement of $\det(s_{ij}) = 0$, in $n(n + 1)/2$ space $s_{ij} = s_{ji}$? Here $\det(s_{ij}) = 0$ is homogeneous, but has singularities if $n > 2$. In fact, the singular set is $\Sigma_1 = \{s \mid \det(s) = 0, \text{rank}(s) < n - 1\}$, the singular set of Σ_1 is $\Sigma_2 = \Sigma(\Sigma_1) = \{s | \det(s) = 0, \text{rank}(s) < n - 2\}$, and so on. Let $\Sigma_k(n) = \Sigma(\Sigma_{k-1}(n)) = \{s | \det(s) = 0, \text{rank}(s) < n - k\}$. The dimension of $\Sigma_k(n)$ is $n(n + 1)/2 - (k + 1)(k + 2)/2$, $k \geqq 0$, where $\Sigma_0(n)$ is the variety $\det(s) = 0$, in $n(n + 1)/2$ space. The key is that the structure of $\Sigma_0(n)$ in a tubular neighborhood around the regular set of $\Sigma_k(n)$ is very nice. Specifically, at a regular point of $\Sigma_k(n)$ looke at a small normal $(k + 1)(k + 2)/2$ disc D. Then the pair $\Sigma_0(n) \cap D \subset D$ is smoothly equivalent to the variety $\Sigma_0(k + 1)$ in a disk D_0 around the origin in $(k + 1)(k + 2)/2$ space. In fact, if D' is a small $n(n + 1)/2 - (k + 1)(k + 2)/2$ disc on $\Sigma_k(n)$ around the regular point $s \in \Sigma_k(n) \subset \Sigma_0(n)$, then $\Sigma_0(n) \cap D \times D, \subset D \times D'$ is just like $\Sigma_0(k + 1) \times D' \subset D_0 \times D'$. Thus, one can work inductively with the varieties $\Sigma_0(n)$ and describe big convex neighborhoods of any point in the complement $U = \text{Symm}(n)$, starting with $n = 2$.



So, finally, the main theorem boils down to

PROPOSITION 8.5. *The open affine* $U =$ Symm(n) *of non-singular, symmetric* $n \times n$ *matrices has Property C.*

Thus $[X, U]_{\Gamma^\infty} \simeq [X, U]$ is a bijection and $W'(\Gamma^\infty(X)) \simeq W'(C(X))$ is an isomorphism for any constructible $X \subseteq X_A$.

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