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ON MODULI OF REAL CURVES

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Dedicated to the memory of Gus Efroymson

The space $\bar{M}^g$ of isomorphism classes of stable complex curves of genus g, $g \geqq 2$, is a projective variety. It is usually referred to as the moduli variety. It is a compactification of the moduli space M^g of smooth complex curves of genus g which is a quasiprojective variety. $\bar{M}^g - M^g$ is a divisor.

The real moduli problem is concerned with the real moduli spaces $\bar{M}^g(\mathbf{R}) = \{[C] \in \bar{M}^g | C$ can be defined over $\mathbf{R}\}$ and $M^g(\mathbf{R}) = \bar{M}^g(\mathbf{R}) \cap M^g$, which are suprisingly mysterious.

The obvious thing to do is to observe that the complex conjugation induces real structures on $\bar{M}^g$ and on M^g. More precisely, let $\bar{C}$ denote the complex conjugate of a stable curve C. Then $\sigma: \bar{M}^g \to \bar{M}^g$, $[C] \to [\bar{C}]$ is a well-defined antiholomorphic involution of $\bar{M}^g$. Such an involution is called a real structure. It maps M^g onto itself and hence defines a real structure of M^g as well.

The set $\bar{M}^g_\sigma$ of fixed-points of $\sigma: \bar{M}^g \to \bar{M}^g$, is the real part of $\bar{M}^g$. It is immediate that $\bar{M}^g(\mathbf{R})$ is contained in the real part of $\bar{M}^g$. Examples show, alas, that $\bar{M}^g(\mathbf{R}) \neq \bar{M}^g_\sigma$.

To state a positive result another definition is required. The quasiregular real part of $(\bar{M}^g, \sigma)$ is

$$(\bar{M}^g)^\wedge_\sigma = \{p \in \bar{M}^g_\sigma | \dim_{\mathbf{R}}(\bar{M}^g_\sigma, p) = \dim_{\mathbf{C}}(\bar{M}^g, p)\}.$$

THEOREM 1 ([1]). *$M^g(\mathbf{R})$ is a real analytic subset of M^g. For $g \geqq 4$, $M^g(\mathbf{R}) = (M^g)^\wedge_\sigma$. For $g \geqq 3$, the irreducible components of $M^g(\mathbf{R})$ correspond to real curves of a given topological type. Consequently, $M^g(\mathbf{R})$ has $2[g/2] + [(g+1)/2] + 2$ irreducible components.*

This can be proved considering the Teichmüller space T^g which is the universal covering space of M^g ([1], see also [2]).

Theorem 1 can be partly extended for stable real curves by the following lemma.

LEMMA ([3]). *$\bar{M}^g(\mathbf{R})$ is the closure of $M^g(\mathbf{R})$ in the strong topology of $\bar{M}^g$.*

Then we get the following theorem.

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THEOREM 2 [3]. *For* $g \geqq 4$, $\bar{M}^g(\mathbf{R}) = (\bar{M}^g)^{\hat{}}_{\sigma}$.

Examples show [3] that for $g = 2$ the only statement of theorems 1 and 2 which survives is that $M^g(\mathbf{R})$ is a real analytic set. All other statements are false if $g = 2$.

In order to study $\bar{M}^g$, Bers invented strong deformation spaces of stable Riemann surfaces with nodes. They can be applied also to our purposes and we can show [3] the following theorem.

THEOREM 3. $\bar{M}^g(\mathbf{R})$ *is connected in the strong topology of* $\bar{M}^g$.

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