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LOCALLY Σ-SPACES

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Introduction. Σ-spaces and reduced Σ-spaces were introduced by 0. Loos [**11**]; they include reflexion spaces [**7**, **8**, **9**] and symmetric spaces where the group Σ is just $\mathbf{Z}_2$, also s-manifolds [**1**, **5**, **6**] where Σ is cyclic. The main purpose of this paper is to define local analogues of (reduced) Σ-spaces and to show that certain desirable properties are then satisfied. Thus, we would expect that, in addition to the above, such spaces should include locally symmetric spaces. Also, with suitable restrictions on Σ, there should be an extension to the theorem that a locally symmetric space admits a connection invariant by local symmetries [**10**].

Theorem 1 in §3 contains the essential results with respect to the above remarks, where Σ is assumed to be cyclic or compact. It also indicates the importance of those cases where an affine connection related to the local structure exists. Such spaces are introduced in §4, and Theorem 2 provides an alternative characterisation of them. This leads, in Theorem 3, to a further simplification of affine locally Σ-spaces under assumptions of analyticity. In particular, as shown in the corollary to Theorem 3, such analytic conditions always hold when Σ is cyclic or compact, Finally, we obtain alternative forms of Theorem 3 in which only tensor properties are required. Further results on these local structures will appear in a later paper.

1. Notation. Manifolds are always finite dimensional and smooth unless otherwise stated. For any smooth map ϕ between manifolds we again write ϕ (or perhaps $T\phi$) for its differential. For any manifold M, $\mathscr{X}(M)$ is the Lie algebra of smooth vector fields on M, TM (resp. T_xM) is the tangent bundle over M (resp. the tangent space to M at x), and $T_1^1(M)$ is the $(1, 1)$ tensor bundle over M. The tangent bundle over a product $M \times N$ is identified with $TM \times TN$ in the usual way.

For manifolds M, N we write ϕ: $M \rightarrowtail N$ for any smooth map ϕ with domain some subset U of M, and write $\boldsymbol{\phi}_x$ for the germ of ϕ at x. Let $G(M, N)$ denote the set of all germs $\boldsymbol{\phi}_x$, $x \in M$. The projection π: $G(M, N) \to M$ is defined by $\boldsymbol{\phi}_x \to x$, and the evaluation map ν: $G(M, N) \to N$ by $\nu(\boldsymbol{\phi}_x) = \phi(x)$. To each ϕ there corresponds a section $\boldsymbol{\phi}$; $x \to \boldsymbol{\phi}_x$, and a topology is defined on $G(M, N)$ by the condition that sets $\operatorname{im}\boldsymbol{\phi}$ should

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form a basis of open sets; π then becomes a local homeomorphism.

Finally, we remark that ϕ: $M \rightarrowtail N$ is called a local diffeomorphism (resp. a local affine transformation) at p if ϕ is a diffeomorphism (resp. an affine transformation) of some open neighbourhood of p onto its image.

2. Locally Σ-spaces. We model the following on the definitions of a locally symmetric space in terms of germs of smooth maps [**10**, Ch.II] and of a Σ-space.

DEFINITION 1. Let M be a manifold, Σ a Lie group, and define maps f, g, h, k as follows:

$$f\colon M \times \Sigma \times \Sigma \times M \to M \times \Sigma \times M \times \Sigma \times M;$$
$$(x, \sigma, \tau, y) \to (x, \sigma, x, \tau, y),$$
$$g\colon M \times \Sigma \times \Sigma \times M \to M \times \Sigma \times M;$$
$$(x, \sigma, \tau, y) \to (x, \sigma\tau, y),$$
$$h\colon M \times \Sigma \times M \times \Sigma \times M \to M \times \Sigma \times M \times \Sigma \times M \times \Sigma \times M;$$
$$(x, \sigma, y, \tau, z) \to (x, \sigma, y, \sigma\tau\sigma^{-1}, x, \sigma, z),$$
$$k\colon M \times M \to M;\ (x, y) \to y.$$

Suppose $\boldsymbol{\mu}$: $M \times \Sigma \to G(M \times \Sigma \times M, M)$ is a continuous map such that for each $p \in M$ and $\sigma, \tau \in \Sigma$

(i) $\pi(\boldsymbol{\mu}(p, \sigma)) = (p, \sigma, p)$, $\nu(\boldsymbol{\mu}(p, \sigma)) = p$;

(ii) $\boldsymbol{\omega}(\boldsymbol{\mu}(p, e)) = \mathbf{k}_{(p,p)}$;

(iii) $\boldsymbol{\mu}(\mathrm{p}, \sigma) \circ (\mathbf{id}_{(p,\sigma)} \times \boldsymbol{\mu}(p, \tau)) \circ \mathbf{f}_{(p,\sigma,\tau,p)} = \boldsymbol{\mu}(p, \sigma\tau) \circ \mathbf{g}_{(p,\sigma,\tau,p)}$;

(iv) $\boldsymbol{\mu}(p, \sigma\tau\sigma^{-1}) \circ (\boldsymbol{\mu}(p, \sigma) \times \mathbf{id}_{\sigma\tau\sigma^{-1}} \times \boldsymbol{\mu}(p, \sigma)) \circ \mathbf{h}_{(p,\sigma,p,\tau,p)}$
$$= \boldsymbol{\mu}(p, \sigma) \circ (\mathbf{id}_p \times \mathbf{id}_\sigma \times \boldsymbol{\mu}(p, \tau));$$

here e is the identity element of Σ and $\boldsymbol{\omega}$: $G(M \times \Sigma \times M, M) \to G(M \times M, M)$ is defined by $\boldsymbol{\omega}(\boldsymbol{\phi}_{(x,\sigma,y)}) = \boldsymbol{\psi}_{(x,y)}$ where $\phi(u, \sigma, v) = \psi(u, v)$. Then we call the triple $(M, \Sigma, \boldsymbol{\mu})$, or perhaps just M, a locally Σ-space.

We next describe the above in terms of local maps. Let the smooth map ϕ: $M \times \Sigma \times M \rightarrowtail M$ represent $\boldsymbol{\mu}(p, \sigma)$, where dom $\phi = U' \times V' \times U'$. Now im $\boldsymbol{\phi}$ is a neighbourhood of $\boldsymbol{\mu}(p, \sigma)$ in $G(M \times \Sigma \times M, M)$ so, by the continuity of $\boldsymbol{\mu}$, there is a neighbourhood $U \times V$ of (p, σ) such that $\boldsymbol{\mu}(U \times V) \subset$ im $\boldsymbol{\phi}$. Hence, for each $(x, \tau) \in U \times V$, ϕ represents $\boldsymbol{\mu}(x, \tau)$, and it follows from (i) that $U \times V \times U \subset$ dom ϕ. Write $U \times V$ as $U_\sigma \times V_\sigma$, and $\phi | U_\sigma \times V_\sigma \times U_\sigma$ as μ_σ. Then a straightforward interpretation of the conditions in Definition 1 leads to the following equivalent formulation.

LEMMA 1. *Let M be a locally Σ-space and let $p \in M$. Then to each $\sigma \in \Sigma$ there can be assigned an open neighbourhood $U_\sigma \times V_\sigma \times U_\sigma$ of (p, σ, p)*

in $M \times \Sigma \times M$ *together with a smooth map* μ_σ: $U_\sigma \times V_\sigma \times U_\sigma \to M$ *such that*

(Σ1) $\mu_\sigma(p, \sigma, p) = p$;

(Σ2) $\mu_e(x, e, y) = y$ *for* $x, y \in U_e$;

(Σ3) *for each* $\sigma, \tau \in \Sigma$ *there exist neighbourhoods* U *of* p, V *of* σ, *and* W *of* τ *such that if* $x, y, z \in U$, $\bar{\sigma} \in V$, $\bar{\tau} \in W$, *then*

(a) $\mu_\sigma(x, \bar{\sigma}, \mu_\tau(x, \bar{\tau}, y)) = \mu_{\sigma\tau}(x, \bar{\sigma}\bar{\tau}, y)$, *and*

(b) $\mu_\sigma(x, \bar{\sigma}, \mu_\tau(y, \bar{\tau}, z)) = \mu_{\sigma\tau\sigma^{-1}}(\mu_\sigma(x, \bar{\sigma}, y), \bar{\sigma}\bar{\tau}\bar{\sigma}^{-1}, \mu_\sigma(x, \bar{\sigma}, z))$; *and*

(Σ4) *if* $p, p' \in M$ *and* $\{\mu_\sigma | \sigma \in \Sigma\}$, $\{\mu'_\sigma | \sigma \in \Sigma\}$ *denote the corresponding sets of maps then for* $\sigma, \sigma' \in \Sigma$ *and* $(x, \tau, x) \in \operatorname{dom} \mu_\sigma \cap \operatorname{dom} \mu'_{\sigma'}$ *there is a neighbourhood of* (x, τ, x) *on which* $\mu_\sigma = \mu'_{\sigma'}$.

Conversely, suppose M *is a manifold,* Σ *is a Lie group, and for each* $p \in M$ *there is a set of smooth maps* $\mathscr{F}_p = \{\mu_\sigma: U_\sigma \times V_\sigma \times U_\sigma \to M | \sigma \in \Sigma\}$, *where each* $U_\sigma \times V_\sigma \times U_\sigma$ *is an open neighbourhood of* (p, σ, p) *in* $M \times \Sigma \times M$. *If* $\mathscr{F}_M = \bigcup_{p \in M} \mathscr{F}_p$ *satisfies* (Σ1)–(Σ4), *then* M *is a locally* Σ*-space such that, for each* $(p, \sigma) \in M \times \Sigma$ *and* $(x, \tau) \in U_\sigma \times V_\sigma$, μ_σ *represents* $\boldsymbol{\mu}(x, \tau)$.

We shall call $\mathscr{F}_M$, as above, a locally Σ-structure on M, and $\mathscr{F}_p$ a locally Σ-structure at p. Of course $\mathscr{F}_M$ is not unique for a given locally Σ-space; in particular, for each $p \in M$ and $\mathscr{F}_p$ as above, the domain of any μ_σ can be restricted to some neighbourhood $\bar{U}_\sigma \times \bar{V}_\sigma \times \bar{U}_\sigma$ of (p, σ, p); we call any $\bar{\mathscr{F}}_p(\bar{\mathscr{F}}_M)$ obtained in this way a refinement of $\mathscr{F}_p(\mathscr{F}_M)$. It will always be clear that any proof using one $\mathscr{F}_M$ or $\mathscr{F}_p$ is independent of such a choice and we omit such details. We also remark that Definition 1 allows us to avoid the tedious introduction of equivalence relations on locally Σ-structures.

The above notation will be used for $\mathscr{F}_p$. Also, for fixed $(x, \tau) \in U_\sigma \times V_\sigma$ we denote by $\mu_\sigma(x, \tau, -)$ the map $U_\sigma \to M$; $y \to \mu_\sigma(x, \tau, y)$. Then, in particular, we may write $\mu_\sigma(x, \sigma, -)$ more simply as σ_x. We also note from (Σ1) and (Σ4) that $\mu_\sigma(x, \tau, x) = x$ for $(x, \tau, x) \in \operatorname{dom} \mu_\sigma$.

Now, as a consequence of (Σ1), for each $\sigma \in \Sigma$, a (1, 1) tensor field S^σ is defined on M by $S^\sigma_p X_p = \sigma_{p*} X_p$ for $p \in M$, $X_p \in T_p M$. This definition is clearly independent of the choice of $\mathscr{F}_M$. We call the induced map S: $\Sigma \times M \to T^1_1(M)$; $(\sigma, x) \to S^\sigma_x$ the *symmetry map*; it has the following properties.

LEMMA 2. (i) *The map* $\sigma \to S^\sigma_p$ *is a representation of* Σ *on* $T_p M$;

(ii) $\sigma^p_* X_p = (I - S^\sigma) X_p$, *where* σ^p: $U_\sigma \to M$ *is defined by* $\sigma^p(x) = \sigma_x(p)$;

(iii) *for each* $\sigma, \tau, \in \Sigma$ *there exist open neighbourhoods* U *of* p, V *of* σ, *and* W *of* τ *such that for* $x \in U$, $\bar{\sigma} \in V$, $\bar{\tau} \in W$, *and* $X \in \mathscr{X}(U)$

$$\mu_\sigma(x, \bar{\sigma}, -)_*(S^{\bar{\tau}} X) = S^{\bar{\sigma}\bar{\tau}\bar{\sigma}^{-1}} \mu_\sigma(x, \bar{\sigma}, -)_* X.$$

(iv) *the symmetry map is smooth.*

PROOF. (i) This it immediate from (Σ1), (Σ2) and (Σ3). In particular, we note that S_p^σ has inverse $S_p^{\sigma^{-1}}$ and so σ_p is a local diffeomorphism at p.

(ii) Using Leibniz's formula on the map $U_\sigma \to M$; $x \to \mu_\sigma(x, \sigma, x) = x$ we have $X_p = \mu_\sigma(X_p, 0_\sigma, X_p) = \sigma^p X_p + \sigma_p X_p$ and (ii) follows.

(iii) In the notation described after Lemma 1, we obtain from (Σ3)(b)

$$(\mu_\sigma(x, \bar{\sigma}, -) \circ \mu_\tau(y, \bar{\tau}, -))\, X_y$$
$$= \mu_{\sigma\tau\sigma^{-1}}(\mu_\sigma(x, \bar{\sigma}, y), \bar{\sigma}\bar{\tau}\bar{\sigma}^{-1}, -) \circ \mu_\sigma(x, \bar{\sigma}, -)X_y.$$

Then by (Σ4) and the definition of S

$$\mu_\sigma(x, \bar{\sigma}, -)\,(S^{\bar{\tau}}X_y) = S^{\bar{\sigma}\bar{\tau}\bar{\sigma}^{-1}}\mu_\sigma(x, \bar{\sigma}, -)\, X_y,$$

which gives (iii).

(iv) Let $X \in \mathscr{X}(U_\sigma)$. The composition of smooth maps

$$U_\sigma \times V_\sigma \to TU_\sigma \times TV_\sigma \times TU_\sigma \xrightarrow{T\mu_\sigma} TM$$
$$(y, \tau) \to (0_y, 0_\tau, X_y) \to S^\tau X_y$$

shows that S is smooth on $U_\sigma \times V_\sigma$ and hence on $M \times \Sigma$ as required.

3. Reduced locally Σ-spaces.

DEFINITION 2. A locally Σ-space M is reduced if for each $p \in M$

(Σ5) T_pM is generated by the set of all $(I - S^\sigma)(T_pM)$, that is $T_pM = \text{gen}\{(I - S^\sigma)X_p | X_p \in T_pM, \sigma \in \Sigma\}$;

(Σ6) if $X_p \in T_pM$ and $(I - S^\sigma)X_p = 0$ for all $\sigma \in \Sigma$, then $X_p = 0$, thus no non-zero vector at p is fixed by all S^σ.

We note that a Σ-space arises as a special case of Lemma 1 when a smooth map μ: $M \times \Sigma \times M \to M$ is prescribed with the properties that maps σ_x: $M \to M$; $y \to \sigma_x(y) = \mu(x, \sigma, y)$ satisfy (Σ1)–(Σ3) globally. Thus Σ-spaces are also locally Σ-spaces. The same is true for the reduced case since (Σ5) and (Σ6) are global conditions.

It is also clear from Definitions 1 and 2 that any open submanifold of a (reduced) locally Σ-space is a (reduced) locally Σ-space.

We now state the theorem referred to in the introduction.

THEOREM 1. *Let (M, Σ, μ) be a reduced locally Σ-space where Σ is cyclic or compact. Then there exists a unique connection on M with respect to which*

(i) *the curvature and torsion tensor fields R and T associated with the connection are covariant constant;*

(ii) *each S^τ is covariant constant;*

(iii) *for any $\mathscr{F}_p$ as in Lemma 1, $\sigma \in \Sigma$, and $(x, \tau) \in U_\sigma \times V_\sigma$, the map $\mu_\sigma(x, \tau, -)$ is an affine transformation of U_σ into M.*

In proving the theorem, the two cases will be treated separately except for (i) and the uniqueness of the connection which are proved jointly at the end.

First we prove the following lemma.

LEMMA 3. *Let (M, Σ, μ) be a locally Σ-space and $(p, \sigma) \in M \times \Sigma$. Then for all (x, τ) in some open neighbourhood $U \times V$ of (p, σ), $\mu_\sigma(x, \tau, -)$ is a diffeomorphism of U into M.*

PROOF. We see from (Σ2) and (a) of (Σ3) that there is an open neighbourhood $U \times V$ of (p, σ) such that for $(x, \tau) \in U \times V$

$$\mu_\sigma^{-1}(x, \tau^{-1}, \mu_\sigma(x, \tau, -)) = \mathrm{id}_U.$$

Thus $\mu_\sigma(x, \tau, -)|U : U \to M$ is injective and has an invertible differential at each point, hence $\mu_\sigma(x, \tau, -)$ is a diffeomorphism of U into M as required.

Case 1. Suppose (M, Σ, μ) is a reduced locally Σ-space with Σ cyclic and generated by some element σ. In this case one easily proves that (Σ5) and (Σ6) are equivalent. Furthermore, since Σ is cyclic it has the discrete topology and we may choose $\mathscr{F}_p = \{U_\tau \times \{\tau\} \times U_\tau | \tau \in \Sigma\}$ Write $\mu_\tau(x, \tau, -) = \tau_x$ as usual. By Lemma 3, we may assume that, for each $x \in U_\tau$, τ_x is a diffeomorphism of U_τ into M. Now for $X, Y \in \mathscr{X}(U_\sigma)$ define a vector field $\nabla_X Y$ on U_σ by

$$(\nabla_X Y)_x = [(I - S^{\sigma^{-1}})^{-1} X,\ Y - S^{\sigma^{-1}}\sigma_x Y]_x.$$

Note that $(I - S^{\sigma^{-1}})^{-1}$ exists because σ generates Σ and (Σ6) is satisfied. One can easily verify that a connection ∇ is defined on U by the above relation. Moreover, $\sigma_x(y)$ is smooth in x and y, from which ∇ is smooth. Now $p \in M$ is arbitrary and, as a consequence of (Σ4), we see that a connection, again denoted by ∇, is defined on M by the above construction.

From (iii) of Lemma 2 we have

$$\begin{aligned}(\nabla_X(S^\sigma Y))_p &= [(I - S^{\sigma^{-1}})^{-1}X,\ S^\sigma Y - S^{\sigma^{-1}}\sigma_p S^\sigma Y]_p \\ &= [(I - S^{\sigma^{-1}})^{-1}X,\ S^\sigma(Y - S^{\sigma^{-1}}\sigma_p Y)]_p \\ &= S^\sigma[(I - S^{\sigma^{-1}})^{-1}X,\ Y - S^{\sigma^{-1}}\sigma_p Y]_p,\end{aligned}$$

where the last equality is a consequence of

$$(Y - S^{\sigma^{-1}}\sigma_p Y)_p = 0.$$

Thus $\nabla S^\sigma = 0$ at p and hence on M. Since S^σ is invertible, $\nabla S^\tau = 0$ for each $\tau \in \Sigma$.

Next, from (b) of (Σ3), there is a neighbourhood U of p such that for $y, z \in U$,

$$\sigma_p \sigma_y(z) = \sigma_{\sigma_p(y)}\sigma_p(z).$$

This implies, in particular, that $\sigma_p(y) \in U_\sigma$. Then, for $y \in U$,

$$\begin{aligned}\sigma_p(\nabla_X Y)_y &= [(I - S^{\sigma^{-1}})^{-1}\sigma_p X, \sigma_p Y - S^{\sigma^{-1}}\sigma_p\sigma_y Y]_{\sigma_p(y)} \\ &= [(I - S^{\sigma^{-1}})^{-1}\sigma_p X, \sigma_p Y - S^{\sigma^{-1}}\sigma_{\sigma_p(y)}\sigma_p]_{\sigma_p(y)} \\ &= (\nabla_{\sigma_p X}\sigma_p Y)_{\sigma_p(y)}.\end{aligned}$$

Hence σ_p is a local affine transformation at p; clearly, the same is true for each τ_p, $\tau \in \Sigma$. Thus we have proved (ii) and (iii).

Before proving the theorem for compact Σ we require the following lemma.

LEMMA 4. *Let $(M, \Sigma, \boldsymbol{\mu})$ be a locally Σ-space with Σ compact, and let $p \in M$. Then there exists an open neighbourhood $\tilde{U}$ of p in M and a smooth map μ_p: $\tilde{U} \times \Sigma \times \tilde{U} \to M$ which represents $\boldsymbol{\mu}(x, \sigma)$ for each $(x, \sigma) \in \tilde{U} \times \Sigma$. Write $\mu_p(x, \sigma, y) = \sigma_x(y)$. Then p has an open neighbourhood $U \subset \tilde{U}$ such that for all $x, y, z \in U$ and $\sigma, \tau \in \Sigma$,*

(i) *$\sigma_x(\tau_x(y)) = (\sigma\tau)_x(y)$, $\sigma_x(\tau_y(z)) = (\sigma\tau\sigma^{-1})_{\sigma_x(y)}(\sigma_x(z))$, and*
(ii) *σ_x is a diffeomorphism of U into M.*

PROOF. Let $\mathscr{F}_p = \{\mu_\sigma\colon U_\sigma \times V_\sigma \times U_\sigma \to M | \sigma \in \Sigma\}$ be any locally Σ-structure at p, and for each σ choose a closed neighbourhood $\bar{V}_\sigma$ of σ with $\bar{V}_\sigma \subset V_\sigma$. Now $\{\bar{V}_\sigma | \sigma \in \Sigma\}$ is a cover of Σ by neighbourhoods and so has a finite subcover $\{\bar{V}_{\sigma_i} | i \in 1, 2, \ldots r\}$ of Σ.

Write $W = U_{\sigma_1} \cap U_{\sigma_2} \cap \cdots \cap U_{\sigma_r}$, and let $E \subset W$ be a Euclidean neighbourhood of p. Let $W_1 \supset W_2 \supset \cdots$ be a sequence of open balls in E with intersection $\{p\}$, and suppose for each n there exists $\tau_n \in \Sigma$ such that, for some i, j, we have $\tau_n \in \bar{V}_{\sigma_i} \cap \bar{V}_{\sigma_j}$ and for some $x_n, y_n \in W_n$, $\mu_{\sigma_i}(x_n, \tau_n, y_n) \neq \mu_{\sigma_j}(x_n, \tau_n, y_n)$. We know that the sequence (τ_n) has a subsequence converging to a point $\tau \in \Sigma$.

Now it follows from (Σ4) that there are neighbourhoods U' of p and V' of τ such that, for $i, j \in \{1, 2, \ldots, r\}$ and $\tau \in \bar{V}_{\sigma_i} \cap \bar{V}_{\sigma_j}$, $\mu_{\sigma_i} = \mu_{\sigma_j}$ on $U' \times V' \times U'$. Moreover, we may assume from the closed property of each $\bar{V}_{\sigma_i}$ that $V' \cap \bar{V}_{\sigma_i}$ is empty if $\tau \notin \bar{V}_{\sigma_i}$. Therefore, by choosing $W_n \subset U'$, we obtain a contradiction to the earlier assumption that $\mu_{\sigma_i}(x_n, \tau_n, y_n) \neq \mu_{\sigma_j}(x_n, \tau_n, y_n)$. Thus p has an open neighbourhood $\tilde{U}$ such that $\tilde{U} \subset W$ and, for $i, j \in \{1, 2, \ldots, r\}$, μ_{σ_i} and μ_{σ_j} agree on $(\tilde{U} \times \bar{V}_{\sigma_i} \times \tilde{U}) \cap (\tilde{U} \times \bar{V}_{\sigma_j} \times \tilde{U})$. Hence we obtain a unique map μ_p: $\tilde{U} \times \Sigma \times \tilde{U} \to M$ satisfying for each i, $\mu_p = \mu_{\sigma_i}$ on $\tilde{U} \times \bar{V}_{\sigma_i} \times \tilde{U}$. Clearly μ_p represents $\boldsymbol{\mu}(x, \sigma)$ for $(x, \sigma) \in \tilde{U} \times \Sigma$.

Next, choose $W_1 \supset W_2 \supset \cdots$ as before, and suppose for each n there exists $(\sigma_n, \tau_n) \in \Sigma \times \Sigma$ such that the equations

$$\mu_p(x, \sigma_n, \mu_p(x, \tau_n, y)) = \mu_p(x, \sigma_n\tau_n, y),$$

$$\mu_p(x, \sigma_n, \mu_p(y, \tau_n, z)) = \mu_p(\mu_p(x, \sigma_n, y), \sigma_n\tau_n\sigma_n^{-1}, \mu_p(x, \sigma_n\mu, z))$$

are not valid everywhere on W_n. Then, by the compactness of $\Sigma \times \Sigma$ there is a subsequence of $((\sigma_n, \tau_n))$ converging to some point $(\bar{\sigma}, \bar{\tau})$. But μ_p represents $\mu(p, \bar{\sigma})$ and $\mu(p, \bar{\tau})$ so, by (iv) of Definition 1, there are neighbourhoods U_1 of p, V_1 of σ, and V_2 of τ such that for $x, y, z \in U_1$, $\sigma' \in V_1$, and $\tau' \in V_2$,

$$\mu_p(x, \sigma', \mu_p(x, \tau', y)) = \mu_p(x, \sigma'\tau', y),$$

$$\mu_p(x, \sigma', \mu_p(y, \tau', z)) = \mu_p(\mu_p(x, \sigma', y), \sigma'\tau'\sigma'^{-1}, \mu_p(x, \sigma', z)).$$

This shows that the sequence $((\sigma_n, \tau_n))$ cannot exist, hence p has an open neighbourhood $U' \subset \tilde{U}$ for which (i) is true.

To prove (ii), we again consider $W_1 \supset W_2 \supset \cdots$ where $W_1 \subset \tilde{U}$, and suppose that for each n there exists $(x_n, \tau_n) \in W_n \times \Sigma$ for which the map $W_n \to M$; $y \to \mu_p(x_n, \tau_n, y)$ is not a diffeomorphism into M. Then there exists $\tau \in \Sigma$ and a subsequence of $((x_n, \tau_n))$ converging to (p, τ). This is clearly impossible by Lemma 3. Thus p has an open neighbourhood $U \subset U'$ satisfying the two conditions.

Case 2. Let $(M, \Sigma\ \mu)$ be a reduced locally Σ-space with Σ-compact. Let $p \in M$ and use the notation and properties of Lemma 4. For fixed X, $Y \in \mathscr{X}(M)$, and $\sigma \in \Sigma$, write

$$Z_{(p,\sigma)}(X, Y) = [X, S^{\sigma^{-1}}(\sigma_p Y)]_p.$$

Clearly, $Z_{(p,\sigma)}$ can be defined similarly for each $x \in M$ so a map $Z(X, Y)$: $M \times \Sigma \to TM$; $(x, \sigma) \mapsto Z_{(x,\sigma)}(X, Y)$ is defined.

Now for $(x, \sigma) \in \tilde{U} \times \Sigma$ we have $Z_{(x,\sigma)}(X, Y) = [X, S^{\sigma^{-1}}(\sigma_x Y)]_x$, and it follows from the smoothness of μ_p in Lemma 4 that $Z(X, Y)$ is smooth on $\tilde{U} \times \Sigma$, hence on $M \times \Sigma$. Next, write $\int_\Sigma F(\sigma) d\sigma$ for the normalised bi-invariant Haar integral on Σ, where F may be vector valued, and define a vector field $\nabla_X Y$ on M by

$$(\nabla_X Y)_x = [X, Y]_x - \int_\Sigma Z_{x,\sigma}(X, Y) d\sigma.$$

Since $Z(X, Y)$ is smooth the integral exists and $\nabla_X Y$ is smooth. We show that ∇ is a connection on M.

Firstly, $(X, Y) \to \nabla_X Y$ is clearly additive in X and Y. Secondly, for any smooth function f on M

$$\begin{aligned} Z_{p,\sigma}(fX, Y) &= [fX, S^{\sigma^{-1}}\sigma_p Y]_p \\ &= -Y_p(f)X_p + f(p)Z_{p,\sigma}(X, Y). \end{aligned}$$

Hence, using $\int_\Sigma d\sigma = 1$, we have

$$\begin{aligned} (\nabla_{fX} Y)_p &= [fX, Y]_p + Y_p(f)X_p - f(p)\int_\Sigma Z_{p,\sigma}(X, Y) d\sigma \\ &= f(p)(\nabla_X Y)_p. \end{aligned}$$

Thirdly,

$$\text{(2)} \qquad \begin{aligned} Z_{p,\sigma}(X, fY) &= [X, S^{\sigma^{-1}}(f \circ \sigma_p^{-1})\sigma_p Y]_p \\ &= ((S^{\sigma^{-1}} X_p)f) Y_p + f(p) Z_{p,\sigma}(X, Y). \end{aligned}$$

Furthermore, it is clear from the linearity and left invariance of the integral that for all $\tau \in \Sigma$

$$S^\tau \int_\Sigma S^\sigma X_p d\sigma = \int_\Sigma S^{\tau\sigma} X_p d\sigma = \int_\Sigma S^\sigma X_p d\sigma,$$

and so $\int_\Sigma S^\sigma X_p d\sigma = 0$ by (Σ6). Hence $\int_\Sigma (S^{\sigma^{-1}} X_p) f d\sigma = 0$, and from (2) $(\nabla_X(fY))_p = [X, fY]_p - f(p) \int_\Sigma Z_{p,\sigma}(X, Y) d\sigma = X_p(f) Y_p + f(p)(\nabla_X Y)_p$. Thus ∇ is a connection on M.

Next, from (iii) of Lemma 2,

$$\begin{aligned} Z_{p,\sigma}(X, S^\tau Y) &= [X, S^{\sigma^{-1}} \sigma_p(S^\tau Y)]_p \\ &= [X, S^{\sigma^{-1}} S^{\sigma\tau\sigma^{-1}} \sigma_p Y]_p \\ &= [X, S^\tau S^{\sigma^{-1}} \sigma_p Y]_p \\ &= (\mathscr{L}_X S^\tau)_p Y_p + S^\tau Z_{p,\sigma}(X, Y) \end{aligned}$$

where $\mathscr{L}_X$ denotes Lie derivation. Hence

$$\begin{aligned} (\nabla_X(S^\tau Y))_p &= [X, S^\tau Y]_p - (\mathscr{L}_X S^\tau)_p Y_p - S^\tau \int_\Sigma Z_{p,\sigma}(X, Y) d\sigma \\ &= S^\tau (\nabla_X Y)_p, \end{aligned}$$

and it follows that each S^τ is covariant constant at p, hence on M.

Finally, to prove (iii) of the theorem, it is sufficient to show that each $\tau_p|U$ is an affine transformation of U into M. Now, from Lemma 4, if $x \in U$, then

$$\begin{aligned} \tau_p Z_{x,\sigma}(X, Y) &= \tau_p [X, S^{\sigma^{-1}}(\sigma_x Y)]_x \\ &= [\tau_p X, \tau_p S^{\sigma^{-1}}(\sigma_x Y)]_{\tau_p(x)} \\ &= [\tau_p X, S^{\tau\sigma^{-1}\tau^{-1}}(\tau_p \sigma_x Y)]_{\tau_p(x)} \\ &= [\tau_p X, S^{\tau\sigma^{-1}\tau^{-1}}(\tau\sigma\tau^{-1})_{\tau_p(x)} \tau_p Y]_{\tau_p(x)} \\ &= Z_{\tau_p(x), \tau\sigma\tau^{-1}}(\tau_p X, \tau_p Y), \end{aligned}$$

where we note that $\tau_p(U) \subset \tilde{U}$.
Hence for $x \in U$,

$$\begin{aligned} \tau_p \int_\Sigma Z_{x,\sigma}(X, Y) d\sigma &= \int_\Sigma \tau_p Z_{x,\sigma}(X, Y) d\sigma \\ &= \int_\Sigma Z_{\tau_p(x), \tau\sigma\tau^{-1}}(\tau_p X, \tau_p Y) d\sigma \\ &= \int_\Sigma Z_{\tau_p(x),\sigma}(\tau_p X, \tau_p Y) d\sigma. \end{aligned}$$

It follows immediately from the definition of $\nabla_X Y$ that τ_p is an affine transformation of U into M, as required.

It remains only to prove (i) and the uniqueness of the connection; the next lemma does this for both cases.

LEMMA 5. *Let (M, Σ, μ) be a reduced locally Σ-space which admits connections ∇, ∇'. Suppose, with respect to each connection, we have that, for all p, σ, the tensor field S^σ is covariant constant and, with the usual notation, each σ_p is a local affine transformation at p. Then $\nabla = \nabla'$. Moreover, the curvature and torsion tensor fields R and T associated with ∇ are covariant constant.*

PROOF. Define a tensor field D of type $(1, 2)$ on M by $D(X, Y) = \nabla_X Y - \nabla'_X Y$. Let $\mathscr{F}_p$ be any locally Σ-structure at p, and use the notation of Lemma 1. Then

$$\begin{aligned} S^\sigma(D(X_p, Y_p)) &= \sigma_p D(X_p, Y_p) = D(\sigma_p X_p, \sigma_p Y_p) \\ &= D(S^\sigma X_p, S^\sigma Y_p) = S^\sigma D(S^\sigma X_p, Y_p). \end{aligned}$$

Hence, for all $\sigma \in \Sigma$, $S^\sigma D_p((I - S^\sigma)X, Y) = 0$, and so $D_p((I - S^\sigma)X, Y) = 0$. But X, Y are any vector fields and by applying $(\Sigma 5)$ we obtain $D_p = 0$; hence $D = 0$ on M and $\nabla = \nabla'$ as required.

Next, with the usual notation, let γ be a geodesic through p in U_σ with tangent vector field V, and let X, Y, Z be parallel vector fields along γ. Since σ_p is a local affine transformation and S^σ is covariant constant we have at p

$$S^\sigma R(X, Y)Z = R(S^\sigma X, S^\sigma Y)S^\sigma Z \tag{3}$$

$$S^\sigma(\nabla_V R)(X, Y)Z = (\nabla_{S^\sigma V} R)(S^\sigma X, S^\sigma Y)S^\sigma Z. \tag{4}$$

But, clearly, these equations are valid at every point of γ. Covariant differentiation of (3) along γ gives

$$S^\sigma(\nabla_V R)(X, Y)Z = (\nabla_V R)(S^\sigma X, S^\sigma Y)S^\sigma Z. \tag{5}$$

Then (4) and (5) implies

$$(\nabla_{(I-S^\sigma)V} R)(S^\sigma X, S^\sigma Y)S^\sigma Z = 0$$

at p, and it follows from $(\Sigma 5)$ that $\nabla R = 0$ at p, hence on M. The same method of proof given $\nabla T = 0$, as required.

REMARK 1. If Σ is finite of order k, then Case 2 still applies with the usual interpretation that the integral should be replaced by an average value. Thus, the formula for $\nabla_X Y$ becomes

$$(\nabla_X Y)_p = [X, Y]_p - (1/k) \sum_{\sigma \in \Sigma} Z_{p,\sigma}(X, Y).$$

REMARK 2. It is not immediately clear that the definition of a locally symmetric space given in [**10**, Ch. II] is equivalent to the present definition of a reduced locally Σ-space with $\Sigma = \mathbf{Z}_2$ because ($\Sigma 6$) is replaced in [**10**, Ch. II] by the condition that p should be an isolated fixed point of σ_p. However, as a consequence of either definition, there exists a connection with the two properties described in Lemma 5; in particular σ_p has differential S_p^σ at p. Equivalence of the two definitions then follows easily. By Remark 1, the common connection is given by $(\nabla_X Y)_p = (1/2)[X, Y + \sigma_p(Y)]_p$.

4. Affine locally Σ-spaces. A locally Σ-space $(M, \Sigma, \boldsymbol{\mu})$ will be called affine if a connection ∇ is defined on it satisfying (iii) of Theorem 1. The space is then denoted by $((M, \nabla), \Sigma, \boldsymbol{\mu})$. Thus such spaces contain those of Theorem 1, and, as the following lemma shows, can be largely characterised by properties of the symmetry map, thereby providing in some cases a simplification of Definition 1.

THEOREM 2. *Let Σ be a Lie group and M a manifold with a connection ∇ satisfying the following properties*:

($S1$) *there exists a map*

$$S\colon \Sigma \times M \to T_1^1(M);\ (\sigma, x) \to S_x^\sigma;$$

($S2$) *for each $p \in M$, the map $\sigma \to S_p^\sigma$ is a representation of Σ on T_pM*;

($S3$) *for each $(p, \sigma) \in M \times \Sigma$ there exist open neighbourhoods U_σ of p, V_σ of σ and a smooth amp $\mu_\sigma\colon U_\sigma \times V_\sigma \times U_\sigma \to M$ such that for each $(x, \tau) \in U_\sigma \times V_\sigma$, $\mu_\sigma(x, \tau, -)$ is an affine transformation of U_σ into M which fixes x and has differential S_x^τ at x*; and

($S4$) *for each $p \in M$, and $\sigma, \tau \in \Sigma$ there exist open neighbourhoods U of p, V of σ, and W of τ such that for $x \in U$, $\bar\sigma \in V$, $\bar\tau \in W$ and $X \in \mathscr{X}(U)$*

$$\mu_\sigma(x, \bar\sigma, -)(S^{\bar\tau}X) = S^{\bar\sigma\bar\tau\bar\sigma^{-1}}\,\mu_\sigma(x, \bar\sigma, -)X.$$

Then there is a unique affine locally Σ-space $((M, \nabla), \Sigma, \boldsymbol{\mu})$ with symmetry map S. Conversely, every affine locally Σ-space has the above properties.

PROOF. For $p \in M$ write $\mathscr{F}_p$ for the set of maps μ_σ, $\sigma \in \Sigma$ in ($S3$); clearly it may be assumed that each U_σ is connected. We then show that $\mathscr{F}_M = \bigcup_{p \in M}\mathscr{F}_p$ is the required locally Σ-structure on M. As usual, it is sufficient to consider only $\mathscr{F}_p$.

Now ($\Sigma 1$) is immediate from ($S3$). Also ($\Sigma 2$) follows by noting that $\mu_e(x, e, -)$ fixes x and has differential $S_x^e = \mathrm{id}$ at x; since $\mu_e(x, e, -)$ is an affine transformation on the connected neighbourhood U_e, it is the identity map [**3**, Ch. VI].

For ($\Sigma 3$), it follows by continuity that each side of (a) and (b) is defined for some connected neighbourhood U of p and neighbourhoods V, W

of σ, τ respectively. We may also assume ($S4$) for such neighbourhoods. Consider y as a variable in (a). Using ($S2$), we see that each side of (a) defines an affine transformation of U into M which fixes x and has differential $S_x^{\bar{\sigma}\bar{\tau}}$ at x. Similarly, considering z as a variable in (b), each side defines an affine transformation of U into M sending y to $\mu_\sigma(x, \bar{\sigma}, y)$. The differentials of the two maps at y are $X_y \to \mu_\sigma(x, \bar{\sigma}, -)\, S^{\bar{\tau}}X_y$ and $S^{\bar{\sigma}\bar{\tau}\bar{\sigma}^{-1}}\mu_\sigma(x, \bar{\sigma}, -)X_y$. It follows, using ($S4$), that (a) and (b) are satisfied.

A similar proof establishes ($\Sigma 4$). Thus, with the notation of ($\Sigma 4$), each $(x, \tau, x) \in \operatorname{dom} \mu_\sigma \cap \operatorname{dom} \mu'_{\sigma'}$ has a neighbourhood $U \times V \times U$ in $\operatorname{dom}\mu_\sigma \cap \operatorname{dom}\mu'_{\sigma'}$ with U connected. Then $\mu_\sigma(x, \tau, -)$ and $\mu'_{\sigma'}(x, \sigma, -)$ agree at x and have the same differential there; hence they agree on U. The existence of $(M, \Sigma, \boldsymbol{\mu})$ is now a consequence of Lemma 1. Its uniqueness, for a given ∇ and S, follows from (iii) of Theorem 1 since each $\mu_\sigma(x, \tau, -)$ is determined on U_σ by S_x^τ (assuming, as we may, that U_σ is connected). The converse is immediate from earlier results.

We shall call $(M, \Sigma, \boldsymbol{\mu})$ an *analytic locally Σ-space* when M is a (real) analytic manifold and in Definition 1 $G(M \times \Sigma \times M, M)$ is the set of germs of local analytic maps. Then Lemma 1 still applies with the obvious modification that each μ_σ should be analytic. Similarly, for such $(M, \Sigma, \boldsymbol{\mu})$ to be affine it is understood that the connection is required to be analytic. We now prove the following theorem for this case.

THEOREM 3. *Let Σ be a Lie group and M an analytic manifold with an analytic connection ∇ such that to each $(x, \sigma) \in M \times \Sigma$ there corresponds a local affine transformation σ_x at x with $\sigma_x(x) = x$. Define*

$$S\colon \Sigma \times M \to T_1^1(M);\ (\sigma, x) \to S_x^\sigma \text{ by } S_x^\sigma X_x = \sigma_x X_x.$$

Suppose

(i) *for each $x \in M$, the map $\sigma \to S_x^\sigma$ is a representation of Σ on T_xM;*
(ii) *S is an analytic map; and*
(iii) *for each $(x, \sigma) \in M \times \Sigma$ there is a neighbourhood N_x of x on which*

$$\sigma_x(S^\tau X) = S^{\sigma\tau\sigma^{-1}}(\sigma_x X) \text{ for } \tau \in \Sigma,\ X \in \mathscr{X}(N_x).$$

Then there is a unique analytic affine locally Σ-space $((M, \nabla), \Sigma, \boldsymbol{\mu})$ with symmetry map S. Conversely, every analytic affine locally Σ-space has the above properties.

PROOF. We use some standard properties of the exponential map denoted by Exp: $TM \rightarrowtail M$. First choose a neighbourhood N of the zero section in TM for which the map exp: $N \to M \times M$; $X_x \to (x, \operatorname{Exp} X_x)$ is an analytic diffeomorphism onto a neighbourhood of the diagonal in $M \times M$. Note that the map $\bar{S}\colon \Sigma \times TM \to TM$; $(\sigma, X_x) \to S^\sigma X_x$ is analytic since S is analytic. Now let $p \in M$ and, for each $\sigma \in \Sigma$, choose open neighbourhoods V_σ of σ and W_σ of 0_p such that $\bar{S}(V_\sigma \times W_\sigma) \subset N$. We may assume $W_\sigma \subset N$ and $\exp W_\sigma = U_\sigma \times U_\sigma$ where U_σ is a connected neighbourhood of p. It follows that the map

$$\mu_\sigma\colon U_\sigma \times V_\sigma \times U_\sigma \to M;\ (x, \tau, y) \to (\mathrm{Exp} \circ S^\tau \circ \exp^{-1})(x, y)$$

is defined and analytic. We show that the conditions in Theorem 2 are satisfied by maps μ_σ as above.

For $(x, \tau) \in U_\sigma \times V_\sigma$, choose a normal neighbourhood $U = \mathrm{Exp}\ U_0$ of x on which τ_x acts as an affine transformation. Then for $X_x \in U_0$, $\mathrm{Exp}\ S^\tau X_x = \tau_x \mathrm{Exp}\ X_x$. Hence, for $y \in U$, $(\mathrm{Exp} \circ S^\tau \circ \exp^{-1})(x, y) = \tau_x(y)$, and so $\mu_\sigma(x, \tau, -)$ is an affine transformation (equal to τ_x) on a neighbourhood of x. Since U_σ is connected and μ_σ analytic, $\mu_\sigma(x, \tau, -)$ is an affine transformation of U_σ into M. The rest of ($S3$) is easily verified.

To prove ($S4$), let $(x, \bar{\sigma}, \tau) \in U_\sigma \times V_\sigma \times \Sigma$. We know that $\mu_\sigma(x, \bar{\sigma}, -) = \bar{\sigma}_x$ on some neighbourhood of x, so from (iii), the two maps

$$y \mapsto \mu_\sigma(x, \bar{\sigma}, -)(S^\tau X_y),$$
$$y \mapsto S^{\bar{\sigma}\tau\bar{\sigma}^{-1}}\mu_\sigma(x, \bar{\sigma}, -)X_y$$

agree on a neighbourhood of x, X being an analytic vector field on U_σ. Since each map is analytic with domain U_σ they agree on U_σ. Now ($S4$) follows as a special case. Thus the first part of Theorem 3, including uniqueness, follows using Theorem 2.

Conversely, suppose given an analytic affine locally Σ-space with symmetry map S. Then (ii) follows just as (iv) of Lemma 2. Also, as a special case of (iii) of Lemma 2,

$$\sigma_p(S^\tau X) = S^{\sigma\tau\sigma^{-1}}(\sigma_p X)$$

on some neighbourhood of p, where σ_p is now analytic. As before, by choosing U_σ connected and using the analyticity of S it results that the above equation is valid on U_σ (independent of the choice of τ), and (iii) is an immediate consequence. This completes the proof.

COROLLARY. *Let (M, Σ, μ) be a reduced locally Σ-space with Σ cyclic or compact. Then (M, Σ, μ) admits the structure of an analytic affine reduced locally Σ-space.*

PROOF. With respect to the connection ∇ obtained in Theorem 1, we know that (M, Σ, μ) is affine. Also R and T are covariant constant, hence, by [3, Ch. VI], M is an analytic manifold with respect to the atlas of normal coordinate neighbourhoods, and ∇ is analytic. The corollary will now follow from Theorem 2 provided we prove that S is analytic.

Let U be a normal neighbourhood of p in M. Now each S^σ is covariant constant and so has constant components with respect to adapted frames and coframes of vector fields and 1-forms parallel along geodesics through

p. Furthermore, as a particular case of Lemma 2, the map $\Sigma \to T_pM$; $\sigma \to S_p^\sigma$ defines a continuous homomorphism of Σ into $\mathrm{GL}(T_pM)$, and hence is analytic. Thus, S is analytic on $\Sigma \times U$ and therefore on $\Sigma \times M$. This completes the proof.

We next give an alternative version of Theorem 3 where the assumptions that particular local affine transformations should exist are replaced by assumptions on R, T and S. Thus, analytic affine locally Σ-spaces can be characterised entirely by tensor properties, albeit infinite in number.

First, a few remarks on notation. A tensor field P of type $(1, r)$ will be called *S^Σ-invariant* if, for all $X_1, \dots X_r \in \mathscr{X}(M)$ and $\sigma \in \Sigma$, $S^\sigma P(X_1, \dots, X_r) = P(S^\sigma X_1, \dots, S^\sigma X_r)$. Also higher order covariant differentials are defined inductively by $(\nabla^{m+1}P)(X_1, \dots, X_{m+r}, Y) = (\nabla_Y(\nabla^m P))(X_1, \dots, X_{m+r})$ and $(\nabla^{m+1}P)_x(X_1, \dots, X_{m+r}, Y)$ denotes evaluation at x.

THEOREM 4. *Let Σ be a Lie group, M an analytic manifold with an analytic connection ∇, and S: $\Sigma \times M \to T_1^1(M)$; $(\sigma, x) \to S_x^\sigma$ a map. Suppose*

(i) *for each $p \in M$ the map $\sigma \to S_p^\sigma$ is a representation of Σ on T_pM;*

(ii) *S is an analytic map;*

(iii) *R, T and all their covariant differentials are S^Σ-invariant; and*

(iv) *$S^\sigma(\nabla^r S^\tau)(X_1, \dots, X_{r+1}) = (\nabla^r S^{\sigma\tau\sigma^{-1}})(S^\sigma X_1, \dots, S^\sigma X_{r+1})$ for all $\sigma, \tau \in \Sigma$, $X_1, \dots, X_{r+1} \in \mathscr{X}(M)$, $r = 1, 2, 3, \dots$.*

Then there is a unique analytic affine locally Σ-space $((M, \nabla), \Sigma, \mu)$ *with symmetry map S. Conversely, every analytic affine locally Σ-space has the above properties.*

PROOF. Assume (i)–(iv) above. Now it is immediate from (iii) and [3, Ch. VI] that, for each $p \in M$, there exists a local affine transformation σ_p with differential S_p^σ at p. Next let γ be a geodesic through p with tangent vector field V, and let X be a parallel vector field along γ. Then, using (iv), we have, for $m \geqq 1$,

$$\begin{aligned}(\nabla^m_{\sigma_p V}(\sigma_p S^\tau X))_p &= \sigma_p(\nabla^m_V(S^\tau X))_p = S^\sigma((\nabla^m_V S^\tau)X)_p \\ &= S^\sigma(\nabla^m S^\tau)_p(X_p, V_p, \dots, V_p) \\ &= (\nabla^m S^{\sigma\tau\sigma^{-1}})_p(S^\sigma X_p, S^\sigma V_p, \dots, S^\sigma V_p) \\ &= (\nabla^m S^{\sigma\tau\sigma^{-1}})_p(\sigma_p X_p, \sigma_p V_p, \dots, \sigma_p V_p) \\ &= (\nabla^m_{\sigma_p V}(S^{\sigma\tau\sigma^{-1}}\sigma_p X))_p.\end{aligned}$$

also

$$\sigma_p S^\tau X_p = S^\sigma S^\tau X_p = S^{\sigma\tau\sigma^{-1}}\sigma_p X_p$$

by (i).

Thus the vector field $\sigma_p S^\tau X - S^{\sigma\tau\sigma^{-1}}\sigma_p X$ and its covariant derivatives with respect to $\sigma_p V$ are zero at p.

Since the vector field is analytic along $\sigma_p \circ \gamma$ it must be zero there, and (iii) of Theorem follows easily.

Conversely, the conditions given in Theorem 3 imply (iii) immediately and (iv) follows by the above method. The proof of the corollary is now complete.

We next obtain analogues of Theorem 4 assuming Σ to be cyclic or compact. The infinite set of equations then reduces to a finite set. We require the following lemma.

LEMMA 6. *Suppose* $((M, \bar{\nabla}), \Sigma, \boldsymbol{\mu})$ *is an affine locally* Σ*-space and* D *is a tensor field of type* $(1, 2)$ *on* M *which is* S^{Σ}*-invariant and satisfies* $\bar{\nabla} D = 0$. *Then* $((M, \nabla), \Sigma, \boldsymbol{\mu})$ *is an affine locally* Σ*-space with respect to the connection* ∇ *defined by* $\nabla_X = \bar{\nabla}_X + D_X$, *where* $D_X Y = D(X, Y)$ *for* $X, Y \in \mathscr{X}(M)$.

PROOF. With the notation of Lemma 1, each $\mu_\sigma(x, \tau, -)$ is an affine transformation, with respect to $\bar{\nabla}$, of U_σ into M. We may assume U_σ to be connected and then it is clear from the above assumptions on D, that $\mu_\sigma(x, \tau, -)$ is an affine transformation with respect to ∇. The existence of $((M, \nabla), \Sigma, \boldsymbol{\mu})$ follows immediately from Theorem 2.

THEOREM 5. *Let* Σ *be a cyclic group generated by* σ, *and* (M, ∇) *an affine manifold with a smooth* $(1, 1)$ *tensor field* S^σ *such that*

(a) S^σ *and* $I - S^\sigma$ *are invertible*; *and*

(b) $R, \nabla R, T, \nabla T, \nabla S^\sigma$, *and* $\nabla^2 S^\sigma$ *are* S^σ*-invariant.*

Then there exists a unique $((M, \nabla), \Sigma, \boldsymbol{\mu})$ *with symmetry map* S *defined* by

$$S_x^{\sigma^k} = (S_x^\sigma)^k,\ k \in \mathbf{Z}.$$

PROOF. Define a tensor field D and a connection $\bar{\nabla}$ on M by

$$D(X, Y) = (\nabla_{(I-S^\sigma)^{-1}X} S^\sigma) S^{\sigma^{-1}} Y$$

and $\bar{\nabla}_X Y = \nabla_X Y - D(X, Y)$, $X, Y \in \mathscr{X}(M)$. Let $\bar{R}$ and $\bar{T}$ denote the curvature and torsion tensor fields associated with $\bar{\nabla}$. By [1], we know that $\bar{\nabla}\bar{R} = \bar{\nabla}\bar{T} = \bar{\nabla}\bar{D} = \bar{\nabla} S^\sigma = 0$, also $\bar{R}$, $\bar{T}$, and D are S^σ-invariant. As remarked earlier, the properties $\bar{\nabla}\bar{R} = \bar{\nabla}\bar{T} = 0$ imply that $(M, \bar{\nabla})$ admits an analytic structure. Then $\bar{\nabla} S^\sigma = 0$ implies that S^σ is analytic, hence S is analytic. We now see from Theorem 4 that there exists a unique $((M, \bar{\nabla}), \Sigma, \boldsymbol{\mu})$ with symmetry map S. Hence, by Lemma 6, there exists a unique $((M, \nabla), \Sigma, \boldsymbol{\mu})$ as above.

THEOREM 6. *Let* Σ *be a compact Lie group,* (M, ∇) *an affine manifold, and* $S\colon \Sigma \times M \to T_1^1(M); (\sigma, x) \mapsto S_x^\sigma$ *a smooth map. Suppose*

(a) *for each* $p \in M$ *the map* $\sigma \mapsto S_p^\sigma$ *is a representation of* Σ *on* T_pM *satisfying* ($\Sigma 5$) (*or equivalently* ($\Sigma 6$));

(b) *for all* $X, Y, Z \in \mathscr{X}(M)$, *and* $\sigma, \tau \in \Sigma$,

(i) $S^\sigma(\nabla S^\tau)(X, Y) = (\nabla S^{\sigma\tau\sigma^{-1}})(S^\sigma X, S^\sigma Y)$,

(ii) $S^\sigma(\nabla^2 S^\tau)(X, Y) = (\nabla^2 S^{\sigma\tau\sigma^{-1}})(S^\sigma X, S^\sigma Y, S^\sigma X)$; *and*

(c) R , ∇R, T, *and* ∇T *are* S^Σ-*invariant.*

Then there exists a unique $((M, \nabla), \Sigma, \mu)$ *with symmetry map* S.

PROOF. We first construct a connection $\bar\nabla$ satisfying $\bar\nabla\bar R = \bar\nabla\bar T = \bar\nabla S^\sigma = 0$, $\sigma \in \Sigma$, and for which $\bar R$ and $\bar T$ are S^Σ-invariant. Thus, define a tensor field D of type (1, 2) on M by the relation

$$D(X, Y) = D_X Y = \int_\Sigma (\nabla_X S^\sigma) S^{\sigma^{-1}} Y d\sigma.$$

For $\tau \in \Sigma$,

$$\begin{aligned}
D_X(S^\tau Y) &= \int_\Sigma (\nabla_X S^\sigma) S^{\sigma^{-1}}(S^\tau Y) d\sigma \\
&= \nabla_X(S^\tau Y) - \int_\Sigma S^\sigma(\nabla_X(S^{\sigma^{-1}} S^\tau Y) d\sigma \\
&= \nabla_X(S^\tau Y) - S^\tau \int_\Sigma S^{\tau^{-1}} S^\sigma(\nabla_X(S^{\sigma^{-1}} S^\tau Y)) d\sigma \\
&= \nabla_X(S^\tau Y) - S^\tau \int_\Sigma S^{(\sigma^{-1}\tau)^{-1}} (\nabla_X(S^{\sigma^{-1}\tau} Y)) d\sigma \\
&= \nabla_X(S^\tau Y) - S^\tau \int_\Sigma S^\sigma(\nabla_X(S^{\sigma^{-1}} Y)) d\sigma \\
&= \nabla_X(S^\tau Y) + S^\tau \int_\Sigma (\nabla_X S^\sigma) S^{\sigma^{-1}} Y d\sigma - S^\tau \nabla_X Y \\
&= \nabla_X(S^\sigma Y) + S^\tau D_X Y - S^\tau \nabla_X Y.
\end{aligned}$$

Hence, the derivation D satisfies $D_X S^\tau = \nabla_X S^\tau$ for all $\tau \in \Sigma$.

Also, D is S^Σ-invariant. For, using (b)(i),

$$\begin{aligned}
S^\tau(D_X Y) &= \int_\Sigma S^\tau(\nabla_X S^\sigma)(S^{\sigma^{-1}} Y) d\sigma \\
&= \int_\Sigma S^\tau(\nabla S^\sigma)(S^{\sigma^{-1}} Y, X) d\sigma \\
&= \int_\Sigma (\nabla S^{\sigma\tau\sigma^{-1}})(S^{\tau\sigma^{-1}\tau^{-1}}(S^\tau Y), S^\tau X) d\sigma \\
&= \int_\Sigma (\nabla S^\sigma)(S^{\sigma^{-1}} S^\tau Y, S^\tau X) d\sigma \\
&= D_{S^\tau X}(S^\tau Y).
\end{aligned}$$

Next we show that ∇D is S^Σ-invariant. Thus,

$$\begin{aligned}(\nabla D)(X, Y, Z) &= (\nabla_Z D)(X, Y)\\ &= \nabla_Z D(X, Y) - D(\nabla_Z X, Y) - D(X, \nabla_Z Y)\\ &= \int_\Sigma \nabla_Z((\nabla S^\sigma)(S^{\sigma^{-1}} Y, X))d\sigma\\ &\quad - \int_\Sigma (\nabla S^\sigma)(S^{\sigma^{-1}}Y, \nabla_Z X)d\sigma - \int_\Sigma (\nabla_X S^\sigma)S^{\sigma^{-1}}(\nabla_Z Y)d\sigma\\ &= \int_\Sigma (\nabla^2 S^\sigma)(S^{\sigma^{-1}} Y, X, Z)d\sigma\\ &\quad + \int_\Sigma (\nabla S^\sigma)((\nabla_Z S^{\sigma^{-1}})Y, X)d\sigma.\end{aligned}$$

Hence, from (b) (ii),

$$\begin{aligned}S^\tau((\nabla D)(X, Y, Z)) &= \int_\Sigma (\nabla^2 S^{\tau\sigma\tau^{-1}})(S^{\tau\sigma^{-1}\tau^{-1}}(S^\tau Y), S^\tau X, S^\tau Z)d\sigma\\ &\quad + \int_\Sigma (\nabla S^{\tau\sigma\tau^{-1}})((\nabla_{S^\tau Z} S^{\tau\sigma^{-1}\tau^{-1}})S^\tau Y, S^\tau X)d\sigma\\ &= \int_\Sigma (\nabla^2 S^\sigma)(S^{\sigma^{-1}} S^\tau Y, S^\tau X, S^\tau Z)d\sigma\\ &\quad + \int_\Sigma (\nabla S^\sigma)((\nabla_{S^\tau Z} S^{\sigma-1})S^\tau Y, S^\tau X)d\sigma\\ &= (\nabla D)\,(S^\tau X, S^\tau Y, S^\tau Z),\end{aligned}$$

as required.

Define a connection $\bar{\nabla}$ on M by the relation $\bar{\nabla}_X = \nabla_X - D_X$. Then, as shown above, $\bar{\nabla} S^\tau = 0$, $\tau \in \Sigma$. Now let P be any tensor field of type $(1, r)$ on M such that P and ∇P are S^Σ-invariant. Since D is S^Σ-invariant, it follows easily that $\bar{\nabla} P$ is S^Σ-invariant. Then, following the proof of Lemma 5, we obtain $\bar{\nabla} P = 0$. In particular, this givs $\bar{\nabla} D = 0$ and, from (c), $\bar{\nabla} R = \bar{\nabla} T = 0$. As shown in **[1]**, the curvature and torsion tensor fields associated with $\bar{\nabla}$ and ∇ are related by

$$\bar{T}(X, Y) = T(X, Y) + D_Y X - D_X Y$$

and

$$\bar{R}(X, Y) = R(X, Y) - [D_X, D_Y] - D_{\bar{T}(X,Y)}.$$

Consequently, $\bar{R}$ and $\bar{T}$ are S^Σ-invariant and $\bar{\nabla} R = \bar{\nabla} T = 0$. With respect to the corresponding analytic structure on $(M, \bar{\nabla})$ we see that each S^σ is analytic since $\bar{\nabla} S^\sigma = 0$, and it follows easily that S is analytic. As an immediate application of Theorem 4 we see there exists a unique $((M, \bar{\nabla}), \Sigma, \boldsymbol{\mu})$ with symmetry map S. Finally, since D is S^Σ-invariant and $\bar{\nabla} D = 0$ we obtain the required affine locally Σ-space $((M, \nabla), \Sigma, \boldsymbol{\mu})$ by applying Lemma 6, and the proof is complete.

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