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PROPERTIES OF SHEPARD'S SURFACES

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ABSTRACT. Shepard's formula is an interpolation method for arbitrarily placed bivariate data. In this paper the continuity class and interpolation properties are proved. This is followed by generalizations which make the original method more useful. Graphical illustrations of the various methods conclude the paper.

1. Introduction. In this paper we consider the problem of interpolation to arbitrarily spaced data. Typically the problem arises when a surface model is required to interpolate scattered spatial measurements. This problem is encountered in such areas as geology, cartography, earth sciences and many others. One example would be to generate a surface model for a mineral deposit from data gathered at exploratory drillings.

The interpolation problem is given $\{(x_i, y_i, F_i)\}_{i=1}^n$ find a surface function G so that $G(x_i, y_i) = F_i, \quad i = 1, 2, \ldots, n$. Different methods for solving this problem are the following.

(1) Triangulation of the domain followed by the appropriate triangular interpolant [1].

(2) Preprocess the data so that procedures requiring rectangularly gridded data are applicable [7].

(3) Shepard type methods [1, 3, 5, 8].

The Shepard type method is the focal point of this paper. What we propose are methods that not only interpolate to positional information but allow the interpolation to specified derivatives at the scattered points (x_i, y_i). The methods we propose do not necessarily require higher order derivatives, but we make the option of supplying more general interpolation data available to a user. Figure 1, 2, 4, and 5 are four surfaces which interpolate the same (arbitrarily specified) positional information. These four surfaces are quite different because of the preprocessor and the derivatives used in the interpolation.

In the next section we discuss the mathematics that leads us to the

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general interpolation theorem. How this method may be combined with other techniques is taken up in §3.

2. Continuity and interpolation properties of Shepard's method. Shepard's surface given in [**8**] is

$$(2.1)\qquad S_0F(x, y) = \begin{cases} F_i, & (x, y) = (x_i, y_i) \text{ for some } i. \\ \left(\sum_{i=1}^{n} F_i/d_i^\mu\right)\Big/\left(\sum_{i=1}^{n} 1/d_i^\mu\right), & \text{otherwise.} \end{cases}$$

where $\mu > 0$ and $d_i = [(x - x_i)^2 + (y - y_i)^2]^{1/2}$. The function S_0F is a weighted average of the data values where closer points have greater weight. This function may be rewritten in cardinal form as

$$(2.2)\qquad S_0F(x, y) = \sum_{i=1}^{n} F_i w_i(x, y)$$

where

$$(2.3)\qquad w_i(x, y) = \begin{cases} \delta_{ij}, & (x, y) = (x_j, y_j) \text{ for some } j. \\ (1/d_i^\mu)\Big/\left(\sum_{j=1}^{n} 1/d_j^\mu\right), & \text{otherwise.} \end{cases}$$

If $(x, y) \neq (x_j, y_j)$ for all j, then $\prod_{j=1}^{n} d_j^\mu \neq 0$ and so $w_i(x, y)$ may be rewritten as

$$(2.4)\qquad w_i(x, y) = \prod_{\substack{j=1 \\ j\neq i}}^{n} d_j^\mu \Big/ \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j\neq k}}^{n} d_j^\mu .$$

If the (x_i, y_i) are distinct, then the denominator of (2.4) is never zero and so $w_i(x, y)$ is a globally defined continuous function. If μ is an even integer, then each w_i is the quotient of polynomials with denominators that do not vanish and so S_0 is in C^∞.

The exponent μ has a definite effect on the surface. For $0 < \mu < 1$, Shepard's interpolant has cusps at the point (x_i, y_i) and, for $\mu = 1$, it has corners. For $\mu > 1$, the tangent plane at each (x_i, y_i) is parallel to the xy-plane, which produces flat spots at the (x_i, y_i). The cusps and corners are unsatisfactory. The flat spots, though not as severe a problem, have discouraged the use of this method. One of this paper's contributions is to construct extensions of Shepard's method which do not have these flat spots. The effect of μ on the surface is illustrated in [**4, 6**].

The continuity class of Shepard's Formula depends upon μ and, for $\mu > 0$, is as follows:

(i) if μ is an even integer, then $S_0F \in C^\infty$,

(ii) if μ is an odd integer, then $S_0F \in C^{\mu-1}$; and

(iii) if μ is not an integer, then $S_0F \in C^{[\mu]}$ where $[\mu] \equiv$ the largest integer $\leqq \mu$.

We make use of the following notation:

$$D^{i,j}G(x, y) = \frac{\partial^i}{\partial x^j}\frac{\partial^i}{\partial y^j}G(x, y)$$

where G is sufficiently smooth to allow the derivatives to be taken in any order.

THEOREM 2.1. *If* $0 \leqq p + q < \mu$, *then*

$$D^{p,q}w_i(x_j, y_j) = \begin{cases}\delta_{ij}, & p = q = 0\\ 0, & 0 < p + q < \mu .\end{cases} \tag{2.5}$$

Theorem 2.1 is proved in the Appendix. The interpolation properties of Shepard's method and its generalizations depend upon equation (2.5). Let

$$S_mF(x, y) = \sum_{i=1}^{n} w_i(x, y)T_i^mF(x, y) \tag{2.6}$$

where $T_i^mF(x, y)$ is the truncated Taylor expansion of F up to derivatives of order m about the point (x_i, y_i).

THEOREM 2.2. *If* $0 \leqq p + q < \mu$ *and*

$$D^{p,q}T_j^mF(x, y)\big|_{(x_j, y_j)} = D^{p,q}F(x, y)\big|_{(x_j, y_j)},$$

then

$$D^{p,q}S_mF(x, y)|_{(x_j, y_j)} = D^{p,q}F(x, y)|_{(x_j, y_j)}$$

for $j = 1, 2 \ldots, n$.

PROOF. From equation (2.5)

$$\begin{aligned} D^{p,q}S_mF(x_j, y_j) &= \sum_{i=1}^{n} D^{p,q}\{w_i(x, y)T_i^mF(x, y)\}\Big|_{(x_j, y_j)}\\ &= \sum_{i=1}^{n}\sum_{k=0}^{p}\sum_{\ell=0}^{q}\binom{p}{k}\binom{q}{\ell}D^{k,\ell}w_iD^{p-k,q-\ell}T_i^mF\Big|_{(x_j, y_j)}\\ &= \sum_{i=1}^{n}\delta_{ij}D^{p,q}T_i^mF(x_i, y_i) = D^{p,q}F(x_j, y_j). \end{aligned}$$

For example, with $m = 1$ and $\mu = 2$ we have

$$T_i^1F = F(x_i, y_i) + (x - x_i)F_{1,0}(x_i, y_i) + (y - y_i)F_{0,1}(x_i, y_i)$$

with $i = 1, 2, \ldots, n$ and $D^{1,0}F(x, y)|_{(x_j, y_j)} \equiv F_{1,0}(x_i, y_i)$. The surface function is

$$S_1F = \sum_{i=1}^{n} w_i(x, y)T_i^1F.$$

S_1F interpolates to function and first derivatives at the points (x_i, y_i), $i = 1, 2, \ldots, n$. See Figures 1, 2 and 3 for examples of S_0F and S_1F.

Let $m_i \in \{0, 1, \ldots, m\}$, $i = 1, 2, \ldots, n$. Then S_mF may be written in the following more general form:

$$S_mF = \sum_{i=1}^{n} w_i(x, y)T_i^{m_i}F. \tag{2.7}$$

Assume $\mu > m$, so that S_mF takes on the positional and derivative values of $T_i^{m_i}F$ specified at the data points. For those values not specified, that is, $\{m_i + 1, \ldots, m\}$, the Shepard function has the corresponding derivative value zero at (x_i, y_i). See Figure 4 for an example of the more general S_mF given in equation (2.7).

S_m is said to be precise for a function P if $S_mP \equiv P$. Since $\sum_{i=1}^{n} w_i(x, y) \equiv 1$ and $w_i(x, y) \geqq 0$ for all i, S_mF is a convex combination of the functions $T_i^{m_i}F$. Hence the function precision set of S_m contains all polynomials $x^r y^s$ with $r + s \leqq m^* = \min_i\{m_i\}$.

3. Shepard's formula and Boolean sums. In the representation (2.6) a truncated Taylor series is used. However, more general linear operators G_i may be used. Let

$$SF = \sum_{i=1}^{N} w_i(x, y)G_iF. \tag{3.1}$$

The derivative values that SF recaptures from the G_iF at (x_i, y_i) depend on the exponent μ. For example, if $\mu = 2$ and G_iF is polynomial interpolation, then

$$\begin{aligned} SF(x_i, y_i) &= G_iF\big|_{(x_i, y_i)} \\ D^{1,0}SF(x_i, y_i) &= D^{1,0}G_iF\big|_{(x_i, y_i)} \\ D^{0,1}SF(x_i, y_i) &= D^{0,1}G_iF\big|_{(x_i, y_i)} \end{aligned}$$

Higher order derivatives will not be reproduced unless a larger μ is picked. The only exception to this occurs if $G_iF = F$ for all i; in which case $SF = F$.

One algorithm that has been successful is to let G_iF be a weighted or constrained quadratic least squares scheme. The operator G_i has quadratic precision. With $\mu = 2$, SF is in C^∞, has quadratic precision and, depending on the least squares algorithm used, will either smooth the data or interpolate to it.

Barnhill and Gregory [1] have shown that the Boolean sum of two interpolants, $P_1 \oplus P_2 \equiv P_1 + P_2 - P_1P_2$, has at least the interpolation properties of P_1 and the function precision of P_2. The Shepard function SF is normally used with positional and first derivative data and so has linear precision. In order to gain greater precision without requiring higher order derivatives we replace P_1 by S and replace P_2 with some other operator Q that has a high degree of precision. Thus our new formula is $S \oplus Q$. S may be taken to be S_0 or S_1 of (2.6) and Q the least squares quadratic. $S_0 \oplus Q$ yields interpolation to positional information and has quadratic precision, while $S_1 \oplus Q$ gives, in addition, interpolation to the specified first derivatives in S_1.

The Boolean sum $(S \oplus Q)F$ reproduces the derivative data specified in S if μ is larger than the order of the derivatives. If these derivative values are not specified, then the zero properties of the $w_i(x, y)$ guarantee that these derivative values come from QF. For example, if $S \leftarrow S_0$, Q is a least squares quadratic and $\mu = 2$, then $S_0 \oplus Q$ interpolates to positional values specified in S_0 but picks up its first derivative from Q. This is easily verified since $S_0 \oplus Q = S_0 + Q - S_0Q$ and S_0Q has a zero first derivative at the data points. See Figure 5 for an example of $(S_0 \oplus Q)F$ with $\mu = 2$.

The Boolean sum of two interpolants may be written as

$$(P_1 \oplus P_2)F = (P_1 + P_2 - P_1P_2)F = (P_2 + P_1\{I - P_2\})F.$$

where I is the identity operator, $IF = F$. In other words the Boolean sum of P_1 and P_2 acting on F is P_2 acting on F plus P_1 acting on the remainder $(F - P_2F)$. Let P_1 be the interpolant S of (3.1) and Q be an interpolation operator such that $G_iQF = G_iF$ for $i \in L$, where L is a proper subset of $\{1, 2 \ldots, n\}$. Then

$$(S \oplus Q)F = QF + S(I - Q)F = QF + \sum_{i \in L'} w_iG_i\{(I - Q)F\}$$

where $L' = \{1, 2 \ldots, n\} - L$.

This idea may be used to extend the interpolation set of an existing interpolant. For example, let Q be a linear Courant interpolant at the points $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$, and (x_4, y_4) be interior to this triangle. Then

$$\begin{aligned}(S_0 \oplus Q)F &= QF + S_0(I - Q)F \\ &= QF + w_4(x, y)(F(x_4, y_4) - QF(x_4, y_4)).\end{aligned}$$

$S_0 \oplus Q$ interpolates to the first three points and in addition captures the fourth value. This idea is considered in more detail and implemented graphically in Poeppelmeier's thesis [6].

4. A recursive Shepard formula. The Shepard formula S_0 has the defect that if another point of interpolation is added, then all the w_i's must be reformulated. In this section we seek a Shepard type formula which has the property that an additional point may be added to the interpolation set by simply adding an extra term to the original formula, i.e., a "permanence principle" [2]. The remainder form of the Boolean sum $P_1 \oplus P_2 = P_2 + P_1(I - P_2)$, is used to generate this method. We first motivate this procedure by considering a univariate example.

Let L_n be the Lagrange interpolation operator to $n + 1$ points $\{x_0, x_1, \ldots, x_n\}$. Then

$$L_n F = \sum_{j=0}^{n} F(x_j)\phi_{n,j}(x)$$

where

$$\phi_{n,j}(x) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{x - x_k}{x_j - x_k}$$

$j = 0, 1, \ldots, n$. We define recursively the interpolation operator R_n:

$$R_n F = \begin{cases} L_0 F, & n = 0 \\ (L_n \oplus R_{n-1})F, & n > 0 \end{cases}$$

From the remainder form of the Boolean sum we have

$$\begin{aligned} (L_n \oplus R_{n-1})F &= R_{n-1}F + L_n(I - R_{n-1})F \\ &= R_{n-1}F + \phi_{n,n}(x)[(I - R_{n-1})F](x_n) \end{aligned} \tag{4.1}$$

This follows from the fact that $(I - R_{n-1})F$ is zero for $x = x_i$, $i = 0, 1, \ldots, n - 1$. Since R_{n-1} is interpolated at the x_i up through $n - 1$, we have, by the addition of one term, a new interpolation function which captures the correct values at all the x_i, $i = 0, 1, \ldots, n$. Representation (4.1) is the Newton form of the interpolating polynomial.

We now return to the bivariate case. As in the previous section we have scattered data (x_i, y_i) and corresponding F_i, $i = 1, 2, \ldots, n$. Let $S_0^n F$ be the Shepard function $S_0 F$ of (2.1) interpolating at n data points $\{(x_i, y_i)\}_{i=1}^{n}$. Define

$$Q_n F = \begin{cases} S_0^1 F, & n = 1 \\ (S_0^n \oplus Q_{n-1})F, & n > 1\,. \end{cases}$$

$Q_n F$ is a recursive Shepard type function. If $Q_{n-1}F$ interpolates at $n - 1$ points (x_i, y_i), $i = 1, 2, \ldots, n - 1$, then Q_n interpolates at n points by simply adding one term to Q_{n-1}. To see this, consider

$$\begin{aligned} Q_n F &= (S_0^n \oplus Q_{n-1})F \\ &= Q_{n-1}F + S_0^n(I - Q_{n-1})F \\ &= Q_{n-1}F + w_n(x, y)(F_n - Q_{n-1}F(x_n, y_n))\,. \end{aligned}$$

As in the univariate case $(I - Q_{n-1})F$ is zero at (x_i, y_i), $i = 1, 2, \cdots,$

$n - 1$, so the only contribution of S_0^n comes at the point (x_n, y_n). The final form Q_nF interpolates to all n data.

We now make this formula more explicit. Let

$$B_1(x, y) = 1$$

$$B_k(x, y) = \prod_{i=1}^{k-1} d_i^\mu \Big/ \sum_{j=1}^{k} \prod_{\substack{i=1\\ i\neq j}}^{k} d_i^\mu .$$

Note that $B_k(x_i, y_i) = 0$ for $i < k$, as is necessary. The new "expandable" Shepard formula is defined by

$$Q_nF = \sum_{k=1}^{n} B_k(x, y)C_k$$

where

$$C_1 = F_1$$

$$C_2 = F_2 - B_1(x_2, y_2)C_1$$

$$C_i = F_i - \sum_{k=1}^{i-1} B_k(x_i, y_i)C_k = F_i - Q_{i-1}(x_i, y_i) .$$

This scheme allows data points to be added without recomputing the basis functions, unlike the regular Shepard formula. For example, if $Q_{n-1}F(x_i, y_i) = F_i$, $i = 1, 2, \ldots, n - 1$, and we wish to add another value F_n at (x_n, y_n) to the interpolation set, then

$$Q_nF(x, y) = Q_{n-1}F(x, y) + B_n(x, y)C_n$$

is the required function.

For computational efficiency, we could define the function

$$\hat{B}_k(x, y) = \prod_{i=1}^{k-1} d_i^\mu(x, y) / \sum_{j=1}^{k} \prod_{\substack{i=1\\ i\neq j}}^{k} d_i^\mu(x_k, y_k)$$

and combine the (constant) denominator of $\hat{B}_k(x, y)$ with C_k. This scheme is simpler than original Shepard's formula to evaluate: although each contains n terms, the *k-th* term of the new formula involves k^2 operations and the original formula involves n^2 operations.

Appendix: Proof of Theorem 2.1. The interpolation properties of Shepard's method and its generalizations follow from the fact that, for $\mu > 0$,

$$D^{p,q}w_i(x_j, y_j) = \begin{cases} \delta_{ij}, & p = q = 0 \\ 0, & \mu > p + q > 0 \end{cases}$$

which we now prove.

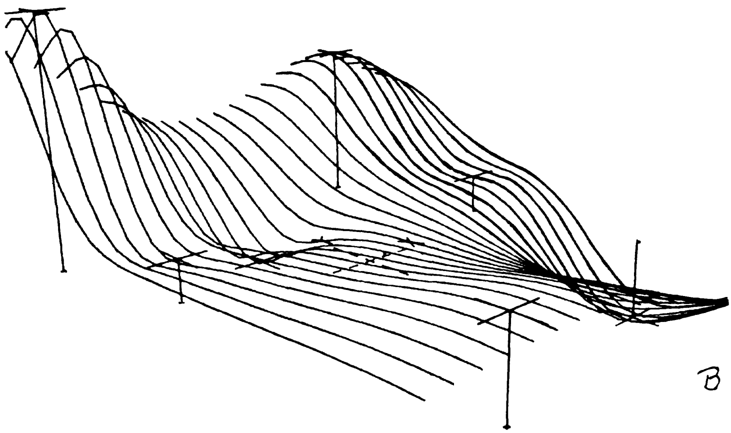

Figure 1. Shepard's interpolation function $S_0F(x, y)$ to scattered spatial measurements with no derivative data and $\mu = 2$.

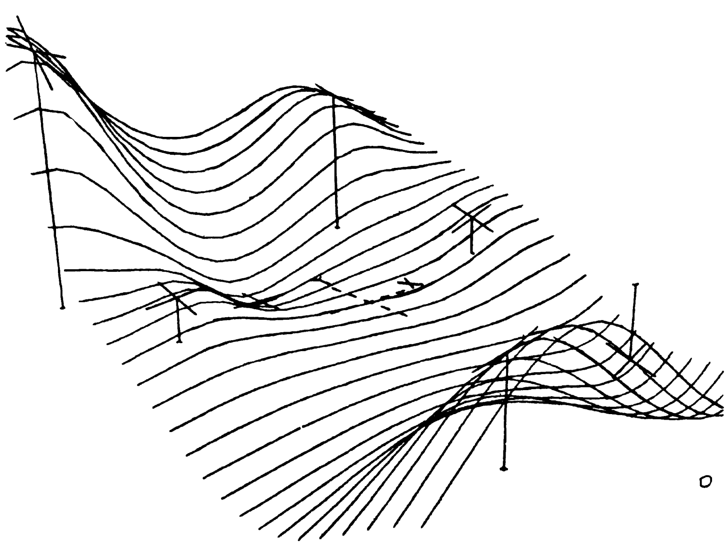

Figure 2. Interpolation to position and derivative values with the function $S_1F(x, y)$ and $\mu = 2$.

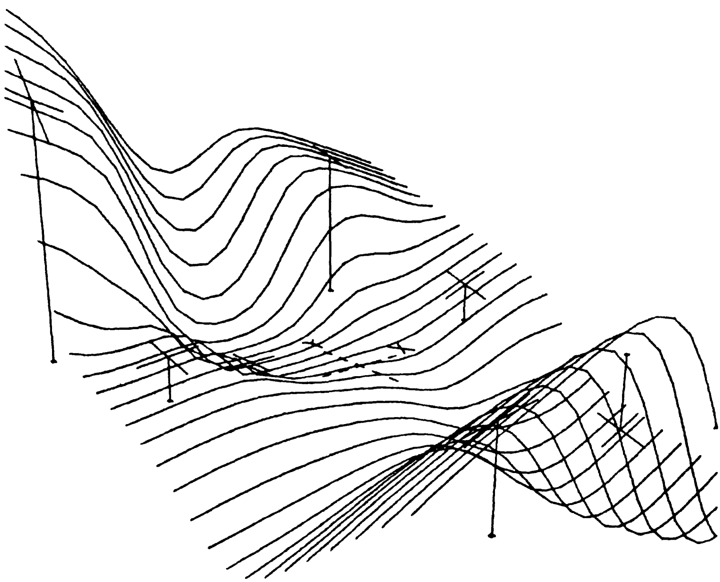

Figure 3. Interpolation to position and derivative values of Figure 2 with the function $S_1F(x, y)$ and $\mu = 4$.

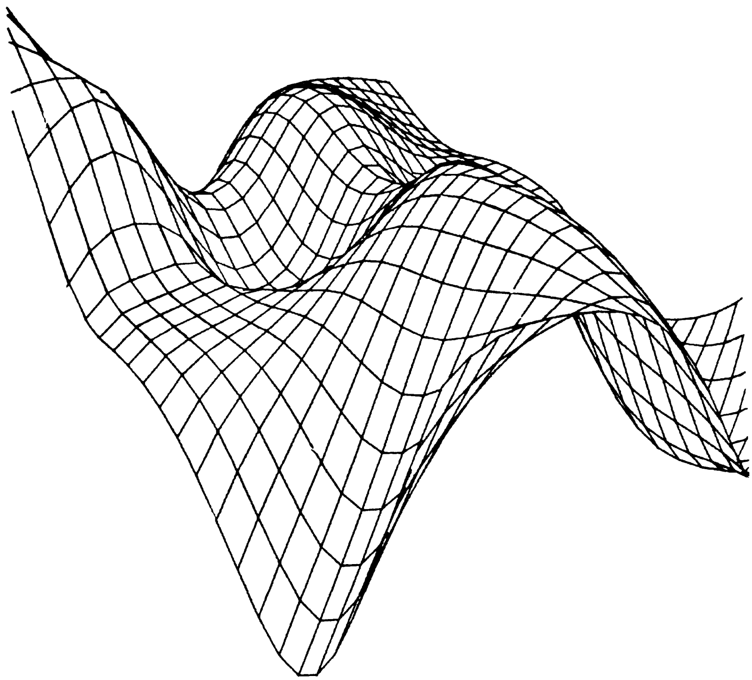

Figure 4. Interpolation function to scattered data with x-partials, y-partials and cross partials. The interpolation function is $S_2F(x, y)$ with $\mu = 3$ and $F_{0,2} = F_{2,0} = 0$ at the data points.

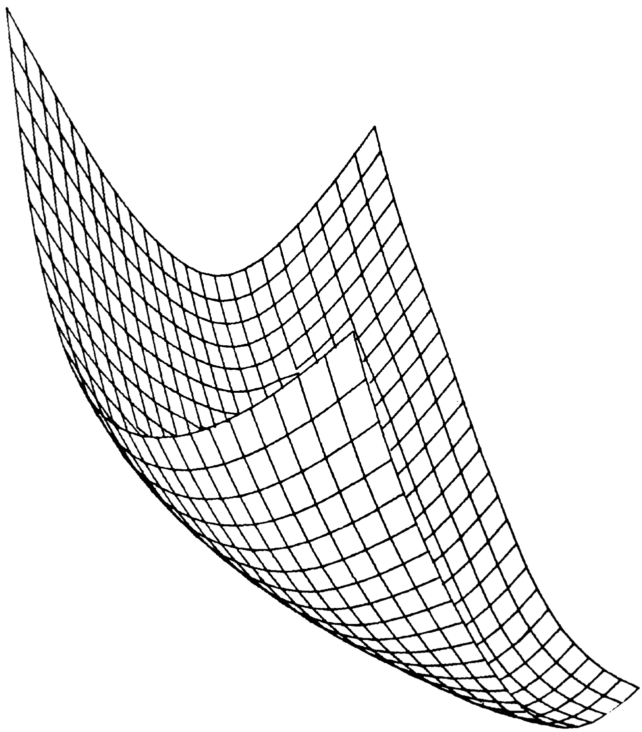

Figure 5. The Boolean sum of the Shepard function $S_0F(x, y)$ with $\mu = 2$ and weighted least squares quadratic. This function interpolates to the data in Figure 1.

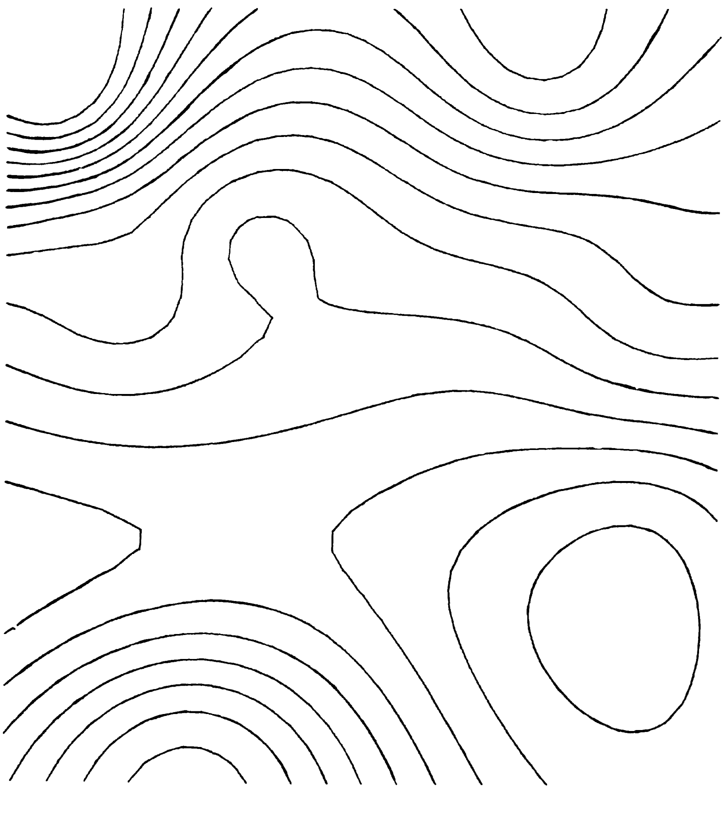

Figure 6. Contours of Figure 2.

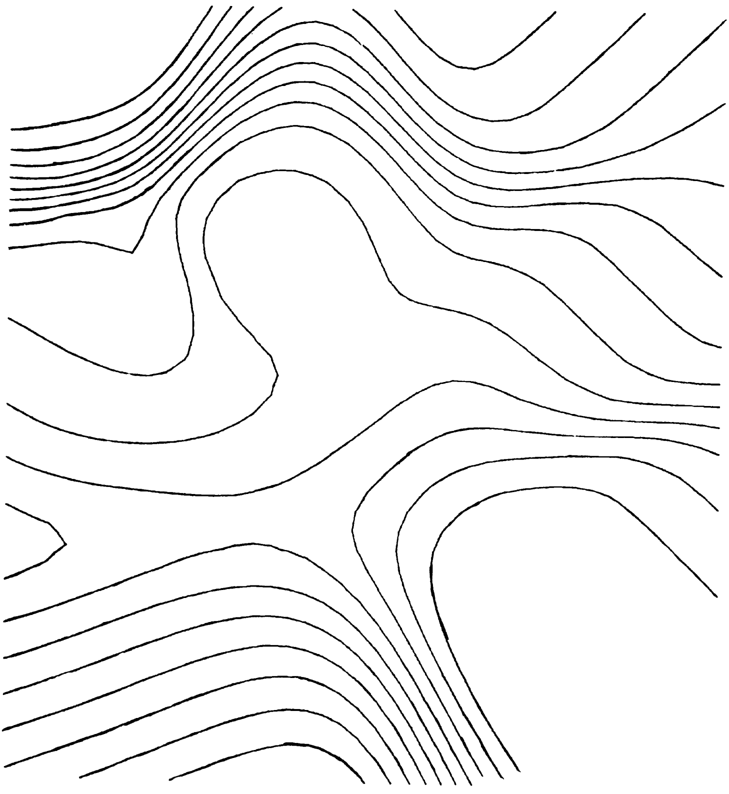

Figure 7. Contours of Figure 3.

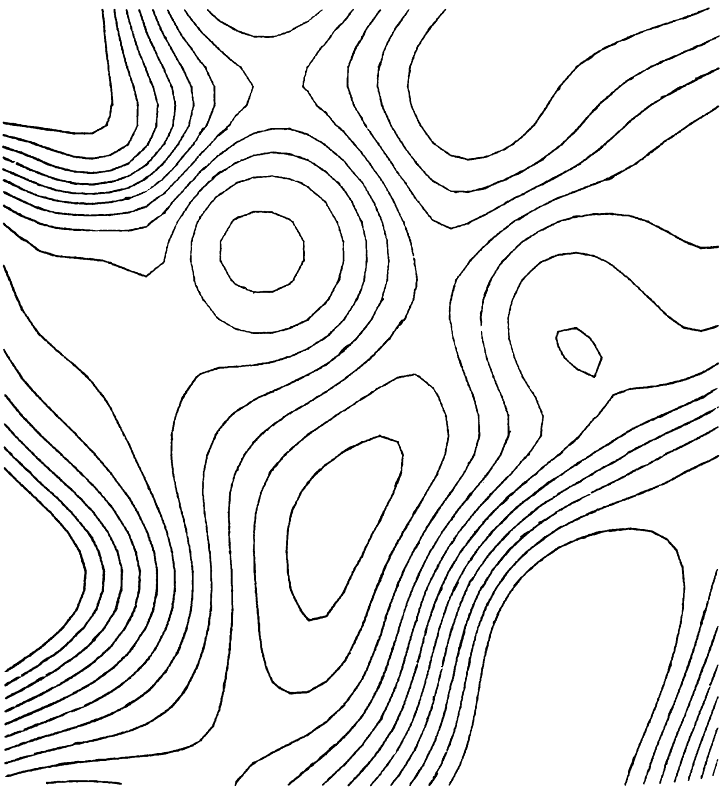

Figure 8. Contours of Figure 4.

We start by examining the derivatives of $d_k = d_k(x, y) \equiv ((x - x_k)^2 + (y - y_k)^2)^{1/2}$. The first partial derivatives of d_k are given by

$$\tag{A.1} \begin{aligned} \frac{\partial}{\partial x} d_k(x, y) &= \frac{x - x_k}{d_k} \equiv A_k \\ \frac{\partial}{\partial y} d_k(x, y) &= \frac{y - y_k}{d_k} \equiv B_k \end{aligned}$$

We note that $|A_k| \leqq 1$ and $|B_k| \leqq 1$. The second derivatives of d_k are

$$\tag{A.2} \begin{aligned} \frac{\partial^2}{\partial x^2} d_k(x, y) &= \frac{1}{d_k}(1 - A_k^2) \\ \frac{\partial^2}{\partial x \partial y} d_k(x, y) &= \frac{1}{d_k}(-A_k B_k) \\ \frac{\partial^2}{\partial y^2} d_k(x, y) &= \frac{1}{d_k}(1 - B_k^2)\,. \end{aligned}$$

In the following discussion we omit the fixed subscript k.

LEMMA A. 1. *If $p + q \geqq 0$, then $D^{p,q}d(x, y) = d^{1-p-q}P_{p,q}(A, B)$ where $P_{p,q}(A, B)$ is a polynomial of degree p in A and degree q in B.*

PROOF. Let $m = p + q$. The conclusion is trivially true for $m = 0$ and is true for $m = 1, 2$ from equations (A.1) and (A.2), respectively. We proceed by induction on m. Without loss of generality, assume that $m \geqq 2$ and $p \geqq 1$. Then

$$\begin{aligned} D^{p,q}d(x, y) &= D^{1,0}D^{p-1,q}d(x, y) \\ &= D^{1,0}(d^{2-p-q}P_{p-1,q}(A, B)) \\ &= P_{p-1,q}(A, B)(2 - p - q)d^{1-p-q}A \\ &\quad + d^{2-p-q}\left[\frac{\partial P_{p-1,q}(A, B)}{\partial A}\frac{\partial A}{\partial x} + \frac{\partial P_{p-1,q}(A, B)}{\partial B}\frac{\partial B}{\partial x}\right] \end{aligned}$$

Since $\partial A/\partial x = (1 - A^2)/d$ and $\partial B/\partial x = (-AB)/d$, we have

$$\begin{aligned} D^{p,q}d(x, y) &= P_{p-1,q}(A, B)(2 - p - q)Ad^{1-p-q} \\ &\quad + d^{2-p-q}\left[\frac{\partial P_{p-1,q}(A, B)}{\partial A}(1 - A^2)\frac{1}{d} + \frac{\partial P_{p-1,q}(A, B)}{\partial B}(-AB)\frac{1}{d}\right] \\ &= d^{1-p-q}\left[P_{p-1,q}(A, B)(2 - p - q)A + \frac{\partial P_{p-1,q}(A, B)}{\partial A}(1 - A^2)\right. \\ &\quad \left. + \frac{\partial P_{p-1,q}(A, B)}{\partial B}(-AB)\right] = d^{1-p-q}P_{p,q}(A, B)\,. \end{aligned}$$

LEMMA A.2. *Let $\mu > 1$ and $p + q = m < \mu$. Then $D^{p,q}d^\mu = d^{\mu-p-q}P_{p,q}(A, B)$ where $P_{p,q}(A, B)$ is a polynomial in A and B.*

PROOF. We use induction on m. For $m = 1$, we have $D^{1,0}d^{\mu} = \mu d^{\mu-1}A$ and $D^{0,1}d^{\mu} = \mu d^{\mu-1}B$. Consider

$$\begin{aligned} D^{p,q}d^{\mu} &= D^{p-1,q}(\mu d^{\mu-1}D^{1,0}d(x, y)) \\ &= \mu \sum_{i=0}^{p-1} \sum_{j=0}^{q} \binom{p-1}{i}\binom{q}{j} D^{i,j}d^{\mu-1}\, D^{p-1-i,q-j}\, D^{1,0}d(x, y)\,. \end{aligned}$$

By the induction hypothesis we have $D^{i,j}d^{\mu-1} = d^{\mu-1-i-j}P_{i,j}(A, B)$ and, by Lemma A.1,

$$D^{p-1-i,q-j}D^{1,0}d(x, y) = d^{1-(p-i)-(q-j)}P_{p-i,q-j}(A, B)\,.$$

Hence

$$\begin{aligned} &D^{p,q}d^{\mu} \\ &= \sum_{i=0}^{p-1} \sum_{j=0}^{q} \binom{p-1}{i}\binom{q}{j}[d^{\mu-1-i-j}P_{i,j}(A, B)\ d^{1-(p-i)-(q-j)}P_{p-i,q-j}(A, B)] \\ &= d^{\mu-p-q}\, P_{p,q}(A, B)\,. \end{aligned}$$

We observe that Lemma A.2. implies that $D^{p,q}d_i^{\mu}(x_i, y_i) = 0$ for $0 < p + q < \mu$.

Let B be the set of all positive functions $G(x, y)$ defined on some domain Ω whose derivatives of order less than μ are bounded. That is, $G \in B$ means that there exist numbers $a_E > 0$ and $M_E < \infty$ so that $G(x, y) \geqq a_E$ and $|D^{p,q}G(x, y)| < M_E$ for all p and q such that $0 \leqq p + q < \mu$.

LEMMA A.3. *If* $G(x, y) \in B$, *then* $[G(x, y)]^2 \in B$.

PROOF. Since $G(x, y) \in B$ there exists $a_G > 0$ so that $G(x, y) \geqq a_G$; hence $[G(x, y)]^2 \geqq a_G^2$. Now consider

$$\begin{aligned} \left|D^{p,q}[G(x, y)]^2\right| &= \left|D^{p,q}[G(x, y)\ G(x, y)]\right| \\ &= \left|\sum_{i=0}^{p} \sum_{j=0}^{q} \binom{p}{i}\binom{q}{j} D^{i,j}G(x, y)D^{p-i,q-j}G(x, y)\right| \\ &\leqq \sum_{i=0}^{p} \sum_{j=0}^{q} \binom{p}{i}\binom{q}{j} M_G M_G = M_G^2\,. \end{aligned}$$

LEMMA A.4. *If* $G \in B$, *then* $D^{p,q}[G(x, y)]^{-1}$ *is bounded.*

PROOF. We proceed by induction on $m = p + q$. With $p \geqq 1$,

$$\begin{aligned} \left|D^{1,0}[G(x, y)]^{-1}\right| &= \left|\frac{-1}{[G(x, y)]^2}\, D^{1,0}G(x, y)\right| \\ &\leqq \left(\frac{1}{a_G}\right)^2 M_G\,. \end{aligned}$$

Next,

$$\begin{aligned}|D^{p,q}[G(x,y)]^{-1}| &= |D^{p-1,q}[(D^{1,0}G)(-G^2)^{-1}]| \\ &= \left|\sum_{i=0}^{p}\sum_{j=0}^{q}\binom{p-1}{i}\binom{q}{j}D^{i+1,j}GD^{p-1-i,q-j}(-G^2)^{-1}\right| \\ &\leqq \sum_{i=0}^{p}\sum_{j=0}^{q}\binom{p-1}{i}\binom{q}{j}|D^{i+1,j}G||D^{p-1-i,q-j}(G^2)^{-1}|\end{aligned}$$

By Lemma A.3 and the induction hypothesis each term of the above is bounded and so the result follows.

COROLLARY A.5. *Let*

$$G(x,y) = \sum_{i=1}^{n}\prod_{\substack{k=1\\k\neq i}}^{n} d_k^{\mu}(x,y)$$

in any bounded region Ω. *Then* $G(x,y) \in B$ *and so* $|D^{p,q}[G(x,y)]^{-1}| \leqq M < \infty$ *for* $0 \leqq p+q < \mu$ *and* $(x,y) \in \Omega$.

PROOF OF THEOREM 2.1. The case $p = q = 0$ was shown earlier in this section. Consider $0 < p + q < \mu$.

$$D^{p,q}w_i(x_j,y_j) = \sum_{\ell=0}^{p}\sum_{m=0}^{q}\binom{p}{\ell}\binom{q}{m}\left[D^{\ell,m}\prod_{\substack{k=1\\k\neq i}}^{n} d_k^{\mu}(x_j,y_j)\right] \cdot\left[D^{p-\ell,q-m}\left(\sum_{r=1}^{n}\prod_{\substack{k=1\\k\neq r}}^{n} d_k^{\mu}(x_j,y_j)\right)^{-1}\right]$$

The general term in the sum contains the product of two factors involving derivatives. The second factor is bounded from Corollary A.5, whereas the first factor is

$$\begin{aligned}&D^{\ell,m}\prod_{\substack{k=1\\k\neq i}}^{n} d_k^{\mu}(x_j,y_j) \\ &= \sum_{r=0}^{\ell}\sum_{s=0}^{m}\binom{\ell}{r}\binom{m}{s}D^{r,s}d_j^{\mu}(x_j,y_j)D^{\ell-r,m-s}\prod_{\substack{k=1\\k\neq i,j}}^{n} d_k^{\mu}(x_j,y_j) \\ &= 0\end{aligned}$$

since $D^{r,s}d_j^{\mu}(x_j,y_j) = 0$ for all j and $0 \leqq p + q < \mu$, as we observed after Lemma A.2.

REFERENCES

1. R.E. Barnhill, *Representation and approximation of surfaces*, Mathematical Software III, ed. J.R. Rice, Academic Press, New York, 1977, 69–120.

2. P.J. Davis, *Interpolation and Approximation*, Blaisdell Publ., 1963; republished by Dover Publ., New York, 1975.

3. R.H. Franke, *Locally determined smooth interpolation at irregularly space points in several variables*, Naval Postgraduate School Technical Report, Monterey, CA, 1975.

4. W.J. Gordon and J.A. Wixom, *Shepard's method of metric interpolation to bivariate and multivariate data*, Mathematics of Computation **32** (1978), 253–264.

5. D.H. McLain, *Drawing contours from arbitrary data points*, The Computer Journal **17** (1974), 318–324.

6. C. Poeppelmeier, *A Boolean sum interpolation scheme to random data for computer aided geometric design*, M.S. thesis, Computer Science Department, University of Utah, Salt Lake City, UT, 1975.

7. L.L. Schumaker, *Fitting surfaces to scattered data*, Approximation Theory II, eds. G.G. Lorentz, C.K. Chui and L.L. Schumaker, Academic Press, New York, 1976, 203–268.

8. D. Shepard, *A two dimensional interpolation function for irregularly spaced data*, Proc. 23rd Nat. Conf. ACM (1968), 517–523.

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