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DIFFERENTIAL-BOUNDARY OPERATORS AND ASSOCIATED NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction. It has been known for some time that differential-boundary operators play an important role in the adjoint theory of linear differential operators with general boundary conditions. In addition to the classical work of Feller [10] and Phillips [21], the theory of differential-boundary operators has been applied to such diverse fields as spline analysis (Brown [2], Brown and Krall [3]), variational and oscillation theory (Reid [22, 24]), boundary control of parabolic (Seidman [27]) and hyperbolic (Russell [25, 26]) partial differential equations. However, it is interesting to note that although the early work of Feller and Phillips was concerned with the well-posedness of Cauchy problems associated with these operators most of the current literature on differential-boundary operators does not consider this problem. Since 1960 the theory has generally been devoted to the study of adjoint operators, derivation of Green's matrices and eigenfunction expansions (see [14–19] and the survey paper by Krall [20]). In this paper we study a general class of 1st order differential-boundary operators and derive necessary conditions and sufficient conditions for these operators to generate C_0-semigroups. Moreover, we show that there exists a fundamental relationship between these operators and Cauchy problems for neutral functional differential equations. Although we shall not pursue the point, similar observations have recently been used to develop numerical methods for approximating solutions and optimal controls for certain integro-differential systems (see [7]).

Notation used in the paper is fairly standard. For example, $L_2 = L_2([0, 1]; \mathbf{R}^n)$ denotes the usual Lebesgue space of $\mathbf{R}^n$-valued "functions" on [0, 1] whose components are square integrable. We shall also make use of the Sobolev space $H^1 = H^1([0, 1]; \mathbf{R}^n)$ and the Banach space of

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continuous functions $C = C([0, 1], \mathbf{R}^n)$. If X and Y are Banach spaces, then the space of bounded linear operators from X into Y will be denoted by $\mathscr{B}(X, Y)$. The symbol $|\cdot|$ is used to denote the norm in a Banach space, the precise space being clear from the context. Given a function $x\colon [0, +\infty) \to \mathbf{R}^n$ and $t \geqq 0$, the function $x_t\colon [0, 1] \to \mathbf{R}^n$ is defined by $x_t(s) = x(t + s)$.

The remainder of this section is devoted to the statement of the basic problem and preliminary results needed in §2 and §3. Proofs can be found in the cited references. §2 contains the statement of our main results along with illustrative examples. The proofs of these results are given in §3.

Let L and D be linear $\mathbf{R}^n$-valued functions with domains $\mathscr{D}(L)$ and $\mathscr{D}(D)$ satisfying $H^1 \subseteqq \mathscr{D}(L) \cap \mathscr{D}(D) \subseteqq L_2$. At this point we make no continuity assumptions on L and D. Let $H(\cdot)$ be an $n \times n$ matrix-valued function whose columns belong to H^1, and define the differential-boundary operator T by

$$\mathscr{D}(T) = \{\phi \in L_2 \mid \phi \in H^1,\ D\phi = 0\} \tag{1.1}$$

$$[T\phi](t) = \dot{\phi}(t) - H(t)L\phi. \tag{1.2}$$

Our primary concern will be the question of whether or not T generates a C_0-semigroup on L_2. However, we shall see that this question is closely related to the "well-posedness" of an associated neutral functional differential equation. In order to make this statement more precise, we define the operator A in $\mathbf{R}^n \times L_2$ by

$$\mathscr{D}(A) = \{(\eta, \phi) \in \mathbf{R}^n \times L_2 \mid \phi \in H^1,\ D\phi = \eta\} \tag{1.3}$$

$$A(\eta, \phi) = (L\phi, \dot{\phi}). \tag{1.4}$$

The next two theorems may be found in [4] and [5].

THEOREM 1.1. *If A defined by* (1.3)–(1.4) *is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t \geqq 0}$ on $\mathbf{R}^n \times L_2$, then*

(i) *both L and D belong to $\mathscr{B}(H^1, \mathbf{R}^n)$; and*

(ii) *if $(\eta, \phi) \in \mathscr{D}(A)$, there is a unique $x\colon [0, +\infty) \to \mathbf{R}^n$ such that for each $t \geqq 0$, $x_t \in H^1$, DX_t is continuously differentiable and*

$$\frac{d}{dt} Dx_t = Lx_t \tag{1.5}$$

with

$$x_0 = \phi. \tag{1.6}$$

Moreover,

$$S(t)(\eta, \phi) = (Dx_t, x_t). \tag{1.7}$$

Part (ii) of this theorem shows that in a generalized sense, the Cauchy problem associated with the neutral functional differential equation (1.5) is well-posed with initial data in $\mathbf{R}^n \times L_2$ whenever A is the generator of a C_0-semigroup on $\mathbf{R}^n \times L_2$. Under additional hypotheses on D, the converse is true as well. In particular, we assume $D \in \mathscr{B}(C, \mathbf{R}^n)$. Then standard representation theorems imply that there is a matrix-valued function $\mu: \mathbf{R} \to \mathbf{R}^{n \times n}$ whose entries are of bounded variation, continuous from the right on $(0, 1)$, $\mu(s) = \mu(0)$ for $s \leqq 0$, $\mu(s) = \mu(1)$ for $s \geqq 1$ and such that for each $\phi \in C$

$$D\phi = \int_0^1 [d\mu(s)]\phi(s). \tag{1.8}$$

The operator D is said to be atomic at $s \in [0, 1]$ if the jump $J(s) = \mu(s) - \mu(s^-)$ is nonsingular.

A crucial step in the proof of part (i) of the following theorem is showing that for $(\eta, \phi) \in \mathscr{D}(A)$ the problem (1.5)–(1.6) has a unique H^1 solution for $t \geqq 0$.

THEOREM 1.2. *Let $L \in \mathscr{B}(H^1, \mathbf{R}^n)$, $D \in \mathscr{B}(C, \mathbf{R}^n)$ and A be defined by* (1.3)–(1.4).

(i) *If D is atomic at* 1, *then A generates a C_0-semigroup on $\mathbf{R}^n \times L_2$.*

(ii) *If D is atomic at* 0 *and* 1, *then $-A$ generates a C_0-semigroup on $\mathbf{R}^n \times L_2$. Hence, A generates a C_0-group on $\mathbf{R}^n \times L_2$.*

Neither the necessary conditions of Theorem 1.1 nor the sufficient conditions of Theorem 1.2 are sharp. In fact, a condition (on D) that is both necessary and sufficient is currently unknown (see [4]).

In order to relate the neutral functional differential equation (1.5)–(1.6) to the differential-boundary operator T defined by (1.1)–(1.2) we shall make use of a result due to R. Vinter (see [28, 29]). Let X and Y be Banach spaces and A, B, F be linear operators satisfying

$$A: \mathscr{D}(A) \to Y, \ \mathscr{D}(A) \subseteqq Y, \ \mathscr{D}(A) \text{ dense in } Y,$$
$$B: \mathscr{D}(B) \to X, \mathscr{D}(B) = \mathscr{D}(A),$$
$$F: \mathscr{D}(F) \to X, \mathscr{D}(F) \supseteqq \mathscr{D}(A),$$

and denote by A_0 the restriction of A to Ker B. Define the operator $\hat{A}$ in $X \times Y$ by

$$\mathscr{D}(\hat{A}) = \{(x, y) \in X \times Y \mid y \in \mathscr{D}(A), By = x\} \tag{1.9}$$

$$\hat{A}(x, y) = (Fy, Ay). \tag{1.10}$$

THEOREM 1.3. *Assume that B has a bounded right inverse B^+ such that AB^+, FB^+ are bounded and F is $(A_0 - B^+F)$-bounded (i.e., there are constants c_1 and c_2 such that $|Fy| \leqq c_1 |y| + c_2 |(A_0 - B^+F)y|$ for all $y \in$* Ker B).

Then, $A_0 - B^+F$ (with domain $\mathscr{D}(A_0)$) generates a C_0-semigroup on Y if and only if $\hat{A}$ generates a C_0-semigroup on $X \times Y$.

REMARK 1.4. The proof of Theorem 1.3 is essentially contained in Vinter's proof of Theorem 2.1 in **[29]**, although his statement of the result differs from ours. In fact, Theorem 1.3 appears in **[29]** with the $(A_0 - B^+F)$-boundedness of F replaced by the conditions that $|B^+| < 1$ and F is A_0-bounded. It is asserted in **[29]** that these two conditions imply the $(A_0 - B^+F)$-boundedness of F. However, this conclusion is unwarranted as the following example shows.

Let Y be the Banach space c_0 (the set of real sequences $\{e_m\}_{m=1}^{\infty}$ such that $e_m \to 0$ as $m \to +\infty$, with the sup norm). Let $\dot{X} = \mathbf{R}$ and define $\mathscr{D}(A) = \mathscr{D}(B) = \mathscr{D}(F)$ to be the set of all finite sequences in c_0. The operators A, B and F are defined by $A\{e_m\} = \{\sum_{m=1}^{\infty} e_m, 0, 0, \ldots\}$ $B\{e_m\} = 2e_1$ and $F\{e_m\} = 2\sum_{m=1}^{\infty} e_m$, respectively. A right inverse B^+ is given by $B^+\eta = \{\eta/2, 0, 0, \ldots\}$ and clearly, B^+, AB^+ and FB^+ are all bounded. It is easy to check that $|B^+| < 1$ and F is A_0-bounded. Since $A_0 - B^+F = 0$ and F is not bounded, F can not be $(A_0 - B^+F)$-bounded.

We now direct our attention to the study of the differential-boundary operator T defined by (1.1)–(1.2).

2. Statement of results and some examples. We consider the operator T of §1 defined by

$$\mathscr{D}(T) = \{\phi \in L_2 \mid \phi \in H^1, D\phi = 0\} \tag{2.1}$$

$$(T\phi](t) = \dot{\phi}(t) - H(t)L\phi, \tag{2.2}$$

with the same conditions on L, D and $H(\cdot)$ as given there (see (1.1)–(1.2)). We have the following necessary conditions on L and D analogous to those stated in Theorem 1.1 (part (i)).

THEOREM 2.1. *Let L, D, $H(\cdot)$ and T be as above and assume that the column vectors of $H(\cdot)$ are linearly independent in L_2. If T defined by* (2.1)–(2.2) *generates a C_0-semigroup on L_2, then L and D belong to $\mathscr{B}(H^1, \mathbf{R}^n)$. Moreover, $D \notin \mathscr{B}(L_2, \mathbf{R}^n)$.*

The need for the "nondegeneracy" assumption in $H(\cdot)$ is clear. Indeed, if $V = \{b \in \mathbf{R}^n \mid H(\cdot)b = 0 \text{ (in } L_2)\}$ is nontrivial and T generates a C_0-semigroup on L_2, then we may perturb L by any linear map of H into V without altering the form of T. In light of this theorem, we will proceed under the assumption that $L \in \mathscr{B}(H^1, \mathbf{R}^n)$. Consequently, standard representation theorems (see **[1]**) imply that there exist $n \times n$ matrix-valued functions $A(\cdot)$, $B(\cdot)$ whose columns belong to L_2 and such that if $\phi \in H^1$, then

$$L\phi = \int_0^1 \{A(s)\phi(s) + B(s)\dot{\phi}(s)\}ds. \tag{2.3}$$

Concerning sufficient conditions on L and D so that T generates a C_0-semigroup on L_2, we first state a lemma important for our analysis. It allows us to convert the problem to a more manageable one concerning the generation of semigroups on product spaces. Let $K = D(H(\cdot))$ and define the operator A_k in $\mathbf{R}^n \times L_2$ by

$$\mathscr{D}(A_k) = \{(\eta, \phi) \in \mathbf{R}^n \times L_2 \mid \phi \in H^1, D\phi = \eta\} \tag{2.4}$$

$$A_k(\eta, \phi) = (KL\phi, \dot{\phi}). \tag{2.5}$$

LEMMA 2.2. *Assume that* $L \in \mathscr{B}(H^1, \mathbf{R}^n)$. *If* $K = D(H(\cdot))$ *is nonsingular, then* T $(-T)$ *generates a* C_0*-semigroup on* L_2 *if and only if* A_k $(-A_k)$ *generates a* C_0*-semigroup on* $\mathbf{R}^n \times L_2$.

Note that K being nonsingular implies that the column vectors of $H(\cdot)$ are linearly independent in L_2. As a consequence of this lemma, we have the following sufficient condition for T to generate a C_0-semigroup on L_2.

THEOREM 2.3. *Assume that* $L \in \mathscr{B}(H^1, \mathbf{R}^n)$, $D \in \mathscr{B}(C, \mathbf{R}^n)$ *and* $K = D(H(\cdot))$ *is nonsingular.*

(i) *If D is atomic at* 1, *then T generates a C_0-semigroup on L_2.*

(ii) *If D is atomic at* 0 *and* 1, *then T generates a C_0-group on L_2.*

Observe that Theorem 2.3 and Theorem 1.2 are very similar. Indeed, the relationship between the differential-boundary operator T and the neutral functional differential equation (1.5)–(1.6) is clear. Under the assumption that $K = D(H(\cdot))$ is nonsingular, T is a generator if and only if (1.5)–(1.6) is well posed. In order to illustrate the theorem, we consider a few examples.

EXAMPLE 2.4. Let $D \in \mathscr{B}(C, \mathbf{R}^n)$ be atomic at 1 and $L \in \mathscr{B}(L_2, \mathbf{R}^n)$. Then T generates a C_0-semigroup on L_2 under less restrictive assumptions on $H(\cdot)$. In fact, if $H(\cdot)$ is any $n \times n$ matrix-valued function whose columns belong to L_2, then $[T\phi](t) = \dot{\phi}(t) - H(t)L\phi$ can be viewed as a bounded perturbation of the operator

$$[T_\lambda\phi](t) = \dot{\phi}(t) = \dot{\phi}(t) - e^{\lambda(t-1)}\cdot 0$$

with domain $\mathscr{D}(T_\lambda) = \mathscr{D}(T) = \{\phi \in L_2 \mid \phi \in H^1, D\phi = 0\}$. Since $K_\lambda = D(e^{\lambda(\cdot-1)}I) \to J(1)$ as $\lambda \to +\infty$, K_λ is nonsingular for large λ. If $L \in \mathscr{B}(L_2, \mathbf{R}^n)$, then $B(\cdot) \equiv 0$ in (2.3) and Theorem 2.3 implies that T_λ generates a C_0-semigroup on L_2. Since T is a bounded perturbation of T_λ, T also generates a C_0-semigroup. Note also that if $H(\cdot)L\phi \equiv 0$, then the operator $[T\phi](t) = \dot{\phi}(t)$ lies withing the scope of this example.

Example 2.4, Example 2.7 below and others not presented here lead us to conjecture that both the regularity conditions on H and the nonsingularity of $K = D(H(\cdot))$ (while important for our approach) can, in general be relaxed.

REMARK 2.5. We note that a similar perturbation argument implies that Theorem 2.3 remains valid if T is replaced by

$$[T\phi](t) = \dot{\phi}(t) - H(t)L\phi + \mathscr{K}\phi,$$

where $\mathscr{K} \in \mathscr{B}(L_2, L_2)$. Such operators include a large class of integro-differential-boundary operators.

EXAMPLE 2.6. Let $P(\cdot)$ be an $n \times n$ matrix-valued function whose entries are essentially bounded on $[0, 1]$ and let $Q(\cdot)$ be an $n \times n$ matrix-valued function whose columns belong to L_1. We assume that the columns of $H(\cdot)$ belong to H^1 and that C, G, E and F are $n \times n$ matrices. For $\phi \in H^1$ let

$$D\phi = E\phi(0) + F\phi(1) + \int_0^1 Q(s)\phi(s)ds,$$

$$L\phi = C\phi(0) + G\phi(1),$$

and consider the differential-boundary operator T defined on

$$\mathscr{D}(T) = \left\{\phi \in L_2 \,|\, \phi \in H^1,\ E\phi(0) + F\phi(1) + \int_0^1 Q(s)\phi(s)ds = 0\right\}$$

by

$$[T\phi(t) = \dot{\phi}(t) + P(t)\phi(t) - H(t)[C\phi(0) + G\phi(1)].$$

Operators of this form have been considered by a number of investigators during the past twenty years (see **[20]** and the references therein). Observe that if the matrices F and $EH(0) + FH(1) + \int_0^1 Q(s)H(s)ds$ are nonsingular, then T generates a C_0-semigroup on L_2. If in addition E is nonsingular, then T generates a C_0-group. By Theorem 2.3 the results of this example remain valid if L is replaced by any operator belonging to $\mathscr{B}(H^1, \mathbf{R}^n)$. Moreover, the measure defining D could have additionally an infinite number of jumps on $(0, 1)$. Thus, this example includes differential-boundary operators with multipoint and Stieltjes boundary conditions (see **[19, 20]**).

The underlying spaces here are reflexive. Thus, $T(A_k)$ generates a C_0-semigroup if and only its adjoint $T^*(A_k^*)$ is a generator. Exploitation of this fact can, in some cases, allow one to consider operators T not satisfying the hypotheses of Theorem 2.3. The use of T^* is illustrated in the following example; the derivation and an application of A_k^* is given later.

EXAMPLE 2.7. Let C, G, E, F be as in Example 2.6, $Q(\cdot)$ an $n \times n$ matrix-valued function whose columns belong to H^1 and assume that $H(\cdot)$ is an $n \times n$ matrix-valued function whose columns belong to L_2. Define the operator T in L_2 on

$$\mathscr{D}(T) = \{\phi \in L_2 \,|\, \phi \in H^1,\ D\phi = 0\}$$

by

$$[T\phi](t) = \dot{\phi}(t) - H(t)[C\phi(0) + G\phi(1)],$$

where

$$D\phi = E\phi(0) + F\phi(1) + \int_0^1 Q(s)\phi(s)ds.$$

We require that F is nonsingular; without loss of generality we set $F = I$, the $n \times n$ identity. It follows that (see [20]) T is densely defined and has adjoint T^* defined on

$$\mathscr{D}(T^*) = \{\phi \in L_2 \,|\, \phi \in H^1,\ D^+\phi = 0\}$$

by

$$[T^*\phi](t) = -\dot{\phi}(t) - Q^*(t)\left[\phi(1) - G^* \int_0^1 H^*(s)\phi(s)ds\right],$$

where D^+ is defined by

$$D^+\phi = \phi(0) + E^*\phi(1) + [C^* - E^*G^*] \int_0^1 H^*(s)\phi(s)ds.$$

Define the isometry U: $L_2 \to L_2$ by $[U\phi](t) = \phi(1 - t)$, $t \in [0, 1]$. It follows that $U^{-1} = U$ and T^* generates a C_0-semigroup on L_2 if and only if $\hat{T} = UT^*U$ (with domain $\mathscr{D}(\hat{T}) = U\mathscr{D}(T^*)$) is a generator.

Easy calculations show

$$[\hat{T}\phi](t) = \dot{\phi}(t) - Q^*(1 - t)\left[\phi(0) - G^* \int_0^1 H^*(1 - s)\phi(s)ds\right] \tag{2.8}$$

with

$$\mathscr{D}(\hat{T}) = \{\phi \in L_2 \,|\, \phi \in H^1,\ \hat{D}\phi = 0\} \tag{2.9}$$

where

$$\hat{D}\phi = \phi(1) + E^*\phi(0) + [C^* - E^*G^*] \int_0^1 H^*(1 - s)\phi(s)ds. \tag{2.10}$$

The operator $\hat{T}$ defined by (2.8)–(2.10) is of the form treated by Theorem 2.3. Consequently, $\hat{T}$ (and hence T) generates a C_0-semigroup on L_2 if the matrix

$$Q^*(0) + E^*Q^*(1) + [C^* - E^*G^*]\int_0^1 H^*(s)Q^*(s)ds$$

is nonsingular. If $Q(\cdot) \equiv 0$, then Example 2.4 can be applied to show that $\hat{T}$ is a generator.

The scalar case with $Q(\cdot) \equiv 0, E = G = 0, F = C = 1$ and $\|H(\cdot)\|^2 \leqq 2$ was used by Phillips **[21]** to illustrate that maximal dissipative extensions of dissipative differential operators do not have to be contractions of maximal operators (see **[20]**).

We close this section by considering the adjoint of the operator A_k defined by (2.4)–(2.5) and giving an example to illustrate how this can be applied to a larger class of differential-boundary operators. Moreover, the adjoint itself is of interest in the theory of neutral functional differential equations since it generates the adjoint semigroup. Having the form of A_k^* is a also important in state space decompositions for neutral equations (see **[12, 13]**) and in the study of certain optimal control problems (see **[9, 11]**).

If (η, ϕ) and (ξ, ψ) belong to $\mathbf{R}^n \times L_2$, then we denote by $\langle\ ,\ \rangle$ the usual inner product on $\mathbf{R}^n \times L_2$ defined by

$$\langle(\eta, \phi), (\xi, \psi)\rangle = \langle\eta, \phi\rangle + \int_0^1 \langle\phi(s), \psi(s)\rangle ds,$$

where $\langle\eta, \phi\rangle = \eta^*\xi$. If $L \in \mathscr{B}(H^1, \mathbf{R}^n)$ has the representation (2.3) and $(\xi, \psi) \in \mathbf{R}^n \times L_2$, then it is convenient to define the function $\Psi(\cdot) = \Psi(\xi, \psi)(\cdot) \in L_2$ by

$$\Psi(t) = [B^*K^*\xi + \psi](t). \tag{2.11}$$

Concerning the adjoint operator A_k^* of A_k defined by (2.4)–(2.5), we have the following result.

THEOREM 2.8. *Let* $L \in \mathscr{B}(H^1, \mathbf{R}^n)$ *have the representation* (2.3), $D \in \mathscr{B}(C, \mathbf{R}^n)$ *have the representation* $D\phi = \int_0^1 d\mu(s)\phi(s)$ *and assume D is atomic at* 1. *The adjoint* A_k^* *of* A_k *is defined by*

$$\mathscr{D}(A_k^*) = \{(\xi, \psi) \in \mathbf{R}^n \times L_2 \mid \Psi(\cdot) \text{ is of bounded variation}, \tag{2.12}$$
$$[\Psi(\cdot) + \mu^*(\cdot)[J^*(1)]^{-1}\Psi(1^-)] \in H^1 \text{ and } \Psi(0) = J^*(0)[j^*(1)]^{-1}\Psi(1^-)\},$$

$$A_k^*(\xi, \psi) = ([J^*(1)]^{-1}\Psi(1^-), A^*(\cdot)K^*\xi \tag{2.13}$$
$$-[\Psi(\cdot) + \mu^*(\cdot)[J^*(1)]^{-1}\Psi(1^-)]'),$$

where $\Psi(\cdot)$ *is defined by* (2.11) (*here prime* ′ *denotes differentiation*).

REMARK 2.9. Note that if $L \in \mathscr{B}(H^1, \mathbf{R}^n)$, then the representation (2.3) is not unique. Therefore, it might appear that A_k^* depends on the particular representation of L. To see that this is not the case, assume that $A_1(\cdot)$,

$A_2(\cdot)$, $B_1(\cdot)$ and $B_2(\cdot)$ are $n \times n$ matrix-valued functions whose columns belong to L_2 and for each $\phi \in H^1$

$$L\phi = \int_0^1 \{A_i(s)\phi(s) + B_i(s)\dot{\phi}(s)\}ds, \qquad i = 1, 2.$$

It follows that for $\phi \in H^1$

$$\int_0^1 \{A_1(s) - A_2(s))\phi(s) + (B_1(s) - B_2(s)\dot{\phi}(s)\}ds = 0.$$

Hence, the Fundamental Lemma of the Calculus of Variations (see [**23**, page 112]) implies that $(B_1 - B_2)(\cdot)$ is absolutely continuous on [0, 1],

$$(2.14) \qquad (B_1 - B_2)(t) = \int_0^t (A_1(s) - A_2(s))ds$$

and

$$(2.15) \qquad (B_1 - B_2)(0) = (B_1 - B_2)(1) = 0.$$

Conversely, if $A_i(\cdot)$, $B_i(\cdot)$ are any matrices satisfying (2.14)–(2.15), then they determine the same operator $L \in \mathscr{B}(H^1, \mathbf{R}^n)$. The definition of Ψ, the form of A_k^* and the expressions (2.14)–(2.15) together imply that A_k^* is defined independently of the particular representation of L.

EXAMPLE 2.10. Let $x_1 \in (0, 1)$, $F(\cdot)$ be an $n \times n$ matrix-valued function whose columns belong to L_2 and C, $\hat{C}$ be $n \times n$ matrices. We consider the system of partial differential equations

$$(2.16) \qquad \frac{\partial}{\partial t} y(t, x) = -\frac{\partial}{\partial x} y(t, x) - F(x)y(t, 1); \qquad t > 0, 0 < x < 1,$$

with boundary conditions

$$(2.17) \qquad Cy(t, 1) + y(t, 0) = 0; \qquad t > 0$$

$$(2.18) \qquad \hat{C}y(t, 1) + y(t, x_1^+) - y(t, x_1^-) = 0; \qquad t > 0$$

and initial condition

$$(2.19) \qquad y(0, \cdot) = \phi(\cdot) \in L_2.$$

The term $F(x)y(t, 1)$ can be viewed as a boundary control term in feedback form. Equation (2.16) is a regular mixed boundary condition while (2.17) represents an interface condition at $x = x_1$. The problem (2.15)–(2.18) can be written in the abstract form

$$(2.20) \qquad \frac{d}{dt} z(t) = Tz(t), \qquad t > 0$$

$$(2.21) \qquad z(0) = z_0,$$

where T is the operator defined on

$$\mathscr{D}(T) = \{\psi \in L_2 \mid \psi \text{ is a.c. on } (0, x_1) \text{ and } (x_1, 1), \dot{\psi} \in L_2, \tag{2.22}$$
$$C\psi(1) + \psi(0) = 0, \hat{C}\psi(1) + \psi(x_1^+) - \psi(x_1^-) = 0\}$$

by

$$[T\psi](x) = -\dot{\psi}(x) - F(x)\psi(1). \tag{2.23}$$

We will show that the system (2.16)–(2.19) is well-posed in the sense that the corresponding abstract Cauchy problem (2.20)–(2.21) is well-posed for $z_0 \in \mathscr{D}(T)$ (i.e., the operator T with domain $\mathscr{D}(T)$ generates a C_0-semigroup on L_2).

The general schme is to apply Theorem 1.3 to convert the problem to one involving the generation of semigroups on a product space. To this new problem we will apply Theorem 2.8 and finally Theorem 1.2. (Our presentation is in reverse order, however.) A number of operators must be defined. In particular, the operators L and D (with $\mathscr{D}(L) \cap \mathscr{D}(D) \supseteqq H^1$) are defined by

$$L\phi = \phi(0) \tag{2.24}$$

and

$$D\phi = \phi(1) + \hat{C}^*\phi(x_1) + C^*\phi(0) + \int_0^1 (F(s) - H(s))^*\phi(s)ds, \tag{2.25}$$

where $H(\cdot)$ is any matrix-valued function satisfying the conditions that its columns belong to $H^1([0, x_1]; \mathbf{R}^n)$ and $H^1([x_1, 1]; \mathbf{R}^n)$, $\hat{C}\mathrm{H}(1) + H(x_1^+) - H(x_1^-) = 0$ and $\mathrm{CH}(1) + \mathrm{H}(0) = \mathrm{I}$. Clearly, $L \in \mathscr{B}(H^1, \mathbf{R}^n)$, $D \in \mathscr{B}(C, \mathbf{R}^n)$ and D is atomic at 1.

With these operators so defined, Theorem 1.2 (part (i)) implies that the operator A defined on

$$\mathscr{D}(A) = \{(\eta, \phi) \in \mathbf{R}^n \times L_2 \mid \phi \in H^1, D\phi = \eta\}$$

by

$$A(\eta, \phi) = (L\phi, \dot{\phi})$$

generates a C_0-semigroup on $\mathbf{R}^n \times L_2$. The adjoint A^* of A (as given by Theorem 2.8 with $K = I$) has domain $\mathscr{D}(A^*) = \{(\xi, \psi) \in \mathbf{R}^n \times L_2 \mid \psi$ is absolutely continuous on $(0, x_1)$ and $(x_1, 1)$, $\dot{\psi} \in L_2$, $\hat{C}\psi(1) + \psi(x_1^+) - \psi(x_1^-) = 0$ and $C\psi(1) + \psi(0) = \xi\}$ and value $A^*(\xi, \psi) = (\psi(1), -\dot{\psi}(\cdot) - [F(\cdot) - H(\cdot)]\psi(1))$. Since $\mathbf{R}^n \times L_2$ is reflexive, A^* generates a C_0-semigroup on $\mathbf{R}^n \times L_2$.

Now, define $A: \mathscr{D}(A) \to L_2$ by $(A\psi)(x) = -\dot{\psi}(x) - [F(x) - H(x)]\psi(1)$ with domain $\mathscr{D}(A) = \{\psi \in L_2 \mid \psi$ is absolutely continuous on $(0, x_1)$ and $(x_1, 1)$, $\dot{\psi} \in L_2$ and $\hat{C}\psi(1) + \psi(x_1^+) - \psi(x_1^\perp) = 0\}$, $B: \mathscr{D}(B) = \mathscr{D}(A) \to L_2$

by $B\psi = C\psi(1) + \psi(0)$, $B^+: \mathbf{R}^n \to \mathscr{D}(A)$ by $B^+\xi = H(\cdot)\xi$ and $F: \mathscr{D}(F) = \mathscr{D}(A) \to \mathbf{R}^n$ by $F\psi = \psi(1)$. By our choice of H, B^+ is a right inverse for B and clearly B^+, AB^+ and FB^+ are bounded.

With these operators A^* may be expressed as $A^*(\xi, \psi) = (F\psi, A\phi)$ with domain $\mathscr{D}(A^*) = \{(\xi, \psi) \in \mathbf{R}^n \times L_2 \,|\, \psi \in \mathscr{D}(A),\ B\psi = \xi\}$. Furthermore, T (defined by (2.22)–(2.23)) can be expressed as $T\psi = A\psi - B^+F\psi$ with domain $\mathscr{D}(T) = \{\psi \in \mathscr{D}(A) \mid B\psi = 0\}$. Theorem 1.3 applies to show that T (with domain $\mathscr{D}(T)$) generates a C_0-semigroup on L_2. Indeed, one need only establish that F is $(A - B^+F)$-bounded on $\{\psi \in \mathscr{D}(A) \mid \dot{B}\psi = 0\}$. To see this, let $0 < \varepsilon < 1 - x_1$. Then

$$\psi(1) = \psi(t) + \int_t^1 \dot{\psi}(s)ds, \qquad 1 - \varepsilon < t < 1;$$

so that

$$\left[I + \int_t^1 F(s)ds\right]\psi(1) = \psi(t) + \int_t^1 [\dot{\psi}(s) + F(s)\psi(1)]ds$$

on $(1 - \varepsilon, 1)$. Integrating both sides over $(1 - \varepsilon, 1)$, one obtains

$$\varepsilon[I + K_\varepsilon]\psi(1) = \int_{1-\varepsilon}^1 \left(\psi(t) + \int_t^1 [\dot{\psi}(s) + F(s)\psi(1)]ds\right)dt,$$

where K_ε is a matrix satisfying $|K_\varepsilon| \to 0$ as $\varepsilon \to 0$. Thus for ε sufficiently small,

$$\begin{aligned} |\psi(t)| &\leqq \varepsilon^{-1} \,|\, [I + K_\varepsilon]^{-1} \,|\, \left\{\left(\int_{1-\varepsilon}^1 |\psi(t)\,|^2\, dt\right)^{1/2} \right. \\ &\quad \left. + \frac{2}{3}\varepsilon^{3/2}\left(\int_{1-\varepsilon}^1 |\,[(A - B^+F)\psi](s)\,|^2\, ds\right)^{1/2}\right\} \\ &\leqq c_1\,|\psi| + c_2\,|\,(A - B^+G)\psi| \end{aligned}$$

for appropriate c_1, c_2.

Finally, we remark that a similar application of Theorem 1.2 (part (ii)) shows that T (with domain $\mathscr{D}(T)$) generates a C_0-group on L_2 provided C is nonsingular.

3. Proofs. The proof of Theorem 2.1 relies on the following technical lemma whose proof is left to the reader.

LEMMA 3.1. *For $\lambda \in \mathbf{C}$, define $M_\lambda: L_2 \to L_2$ by*

$$[M_\lambda\phi](t) = \int_0^t e^{\lambda(t-s)}\phi(s)ds.$$

Then

(i) $R(M_\lambda) = H_0^1 = \{\phi \in H^1 \,|\, \phi(0) = 0\}$,

(ii) $M_\lambda \in \mathscr{B}(L_2, H_0^1)$,

(iii) $[M_\lambda]^{-1}$ *exists and* $[M_\lambda]^{-1} \in \mathscr{B}(H_0^1, L_2)$.

PROOF OF THEOREM 2.1. Assume that T defined by (2.1)–(2.2) generates a C_0-semigroup on L_2. By the Hille-Yosida Theorem (see **[6]**), all large $\lambda > 0$ lie in the resolvent set of the operator T. That is, for all real $\lambda > 0$ sufficiently large

$$[T\phi](t) - \lambda\phi(t) = \psi(t), \qquad \phi(\cdot) \in \mathscr{D}(T) \tag{3.1}$$

has a unique solution that varies continuously in L_2 as ψ varies in L_2.

By the definition of T, (3.1) reads

$$\dot{\phi}(t) - H(t)L\phi - \lambda\phi(t) = \psi(t) \tag{3.2}$$

$$D\phi = 0, \qquad \phi \in H^1. \tag{3.3}$$

The equation (3.2) implies that ϕ has the general form

$$\begin{aligned} \phi(t) &= ae^{\lambda t} + \int_0^t e^{\lambda(t-s)}\,[H(s)b + \psi(s)]ds \\ &= ae^{\lambda t} + M_\lambda[H(\cdot)b + \psi(\cdot)](t) \end{aligned} \tag{3.4}$$

for some choice of $a, b \in \mathbf{R}^n$. In fact, a and b must satisfy

$$L\phi = b \tag{3.5}$$

$$D\phi = 0. \tag{3.6}$$

With a, b so chosen, clearly ϕ belongs to H^1.

Let λ be a point in the resolvent set of T. We claim that $a = a(\psi)$ and $b = b(\psi)$ belong to $\mathscr{B}(L_2, \mathbf{R}^n)$. Indeed, linearity in ψ follows from the uniqueness of solutions of (3.1). For continuity, note that the right side of

$$e^{\lambda\cdot}a(\psi) + M_\lambda[H(\cdot)](\cdot)b(\psi) = \phi(\cdot) - M_\lambda[\psi(\cdot)](\cdot)$$

is continuous in ψ since ϕ varies continuously with ψ (by assumption) and M_λ is a bounded linear operator on L_2. It follows that the map

$$\psi \to a(\psi) + \int_0^{\cdot} e^{-\lambda u}\, H(u)du\, b(\psi) \tag{3.7}$$

is a bounded map on L_2 with finite dimensional range. Accordingly, the map

$$\psi \to \frac{d}{ds}\left[a(\psi) + \int_0^s e^{-\lambda u}H(u)du\; b(\psi)\right] = e^{-\lambda s}H(s)\, b(\psi)$$

is continuous in ψ. By our assumption on $H(\cdot)$, it follows that $\psi \to b(\psi)$ is continuous; (3.7) implies $\psi \to a(\psi)$ is continuous as well.

Substituting (3.4) into (3.5) and (3.6) we see that a, b must solve simultaneously

$$L(e^{\lambda \cdot})a + L(M_\lambda[H(\cdot)])b - b = L(M_\lambda[\psi(\cdot)])$$

and

$$D(e^{\lambda \cdot})a + D(M_\lambda[H(\cdot)])b = D(M_\lambda[\psi(\cdot)]).$$

By the previous claim, the left sides of these equations vary continuously as ψ varies in L_2. Therefore $LM_\lambda \in \mathscr{B}(L_2, \mathbf{R}^n)$ and $L = LM_\lambda M_\lambda^{-1} \in \mathscr{B}(H_0^1, \mathbf{R}^n)$. Because $H^1 \cong \mathbf{R}^n \times H_0^1$, we conclude that $L \in \mathscr{B}(H^1, \mathbf{R}^n)$. The argument is the same for D.

The final assertion follows easily from the fact that (again by the Hille-Yosida Theorem) $\mathscr{D}(T)$ must be dense in L_2 if T generates a C_0-semigroup on L_2.

Recall that if $L \in \mathscr{B}(H^1, \mathbf{R}^n)$, then the representation (2.3) is not unique. We take advantage of this fact to establish the following lemma that is needed in the proof of Lemma 2.1.

LEMMA 3.2. *If $L \in \mathscr{B}(H^1, \mathbf{R}^n)$, then there exist $n \times n$ matrix-valued functions $A(\cdot)$ and $B(\cdot)$ whose columns belong to L_2 such that for $\phi \in H^1$, $L\phi = \int_0^1 \{A(s)\phi(s) + B(s)\dot{\phi}(s)\}ds$ and the matrix $I - \int_0^1 B(s)H(s)ds$ is nonsingular.*

PROOF. The Riesz Representation Theorem applied to H^1 implies that there exists an $n \times n$ matrix-valued function $Z(\cdot)$ whose columns belong to H^1 such that L has the representation

$$L\phi = \int_0^1 \{Z(s)\phi(s) + \dot{Z}(s)\dot{\phi}(s)\}ds, \qquad \phi \in H^1. \tag{3.8}$$

Let $Y(\cdot)$ be any $n \times n$ matrix-valued function whose columns belong to H^1 satisfying

$$Y(0) = Y(1) = 0 \tag{3.9}$$

and

$$\int_0^1 |\dot{Z}(s) - Y(s)|\,|H(s)|ds < 1/2. \tag{3.10}$$

Define the matrices $A(\cdot)$ and $B(\cdot)$ by $A(s) = Z(s) - \dot{Y}(s)$ and $B(s) = \dot{Z}(s) - Y(s)$, respectively. Observe that $(\dot{Z}(\cdot) - B(\cdot)) = Y(\cdot)$ is absolutely continuous, $(\dot{Z}(t) - B(t)) = \int_0^t (Z(s) - A(s))ds$ and $(\dot{Z} - B)(0) = (\dot{Z} - B)(1) = 0$. Consequently, it follows from Remark 2.9 that L also has the representation

$$L\phi = \int_0^1 \{A(s)\phi(s) + B(s)\dot{\phi}(s)\}ds.$$

Moreover, since

$$\left|\int_0^1 B(s)H(s)ds\right| \leqq \int_0^1 |\dot{Z}(s) - Y(s)||H(s)|ds < 1/2,$$

the matrix $I - \int_0^1 B(s)H(s)ds$ is nonsingular.

PROOF OF LEMMA 2.2. The lemma follows almost directly from Theorem 1.3. Let $X = \mathbf{R}^n$, $Y = L_2$, $\mathscr{D}(A) = \mathscr{D}(B) = \mathscr{D}(F) = H^1$, $[A\phi](t) = \dot{\phi}(t)$, $F\phi = KL\phi$ and $B\phi = D\phi$. The operator B has a bounded right inverse $[B^+\eta](t) = H(t)K^{-1}\eta$ with range in H^1. Clearly AB^+ and FB^+ are bounded.

To see that F is $(A_0 - B^+F)$-bounded we select a representation (2.3) for L for which $I - \int_0^1 B(s)H(s)ds$ is nonsingular and compute

$$\begin{aligned} KL\phi &= K\int_0^1 A(s)\phi(s)ds + K\int_0^1 B(s)\dot{\phi}(s)ds \\ &= K\int_0^1 A(s)\phi(s)ds + K\int_0^1 B(s)[\dot{\phi}(s) - H(s)K^{-1}KL\phi]ds \qquad (3.11)\\ &\quad + K\int_0^1 B(s)H(s)K^{-1}KL\phi ds. \end{aligned}$$

It follows that

$$\left[I - K\int_0^1 B(s)H(s)K^{-1}ds\right]KL\phi = K\int_0^1 A(s)\phi(s)ds + K\int_0^1 B(s)[A_0\phi - B^+F\phi](s)ds$$

for all $\phi \in \mathscr{D}(A) \supseteqq \mathscr{D}(A_0)$. Since $I - K\int_0^1 B(s)H(s)dsK^{-1} = K[I - \int_0^1 B(s)H(s)ds]K^{-1}$, this matrix is nonsingular and we obtain

$$\begin{aligned} F\phi = KL\phi &= \left[I - K\int_0^1 B(s)H(s)K^{-1}ds\right]^{-1} K\int_0^1 A(s)\phi(s)ds \\ &+ \left[I - K\int_0^1 B(s)H(s)K^{-1}ds\right]^{-1} K\int_0^1 B(s)[A_0\phi - B^+F\phi](s)ds. \end{aligned}$$

The Cauchy-Schwarz inequality yields the existence of constants c_1, c_2 such that

$$|F\phi| \leqq c_1|\phi(\cdot)| + c_2|(A_0 - B^+F)\phi|.$$

This establishes the $(A_0 - B^+F)$-boundedness of F. The result now follows by setting $\hat{A} = A_k$ and applying Theorem 1.3.

The assertions concerning $-T$ and $-A_k$ are proved similarly; i.e., let $[A\phi](t) = -\dot{\phi}(t)$, $F\phi = -KL\phi$, $B\phi = D\phi$, and $[B^+\eta](t) = H(t)K^{-1}\eta$. From (3.11) the equation

$$\left[I-K\int_0^1 B(s)H(s)dsK^{-1}\right](-KL\phi) = -K\int_0^1 A(s)\phi(s)ds$$
$$+ K\int_0^1 B(s)\{-\dot{\phi}(s)-[H(s)K^{-1}](-KL\phi)\}ds$$

leads to the $(A_0 - B^+F)$-boundedness of F and Theorem 1.3 can again be applied with $\hat{A} = -A_k$.

PROOF OF THEOREM 2.3. By Lemma 2.2 we have that T generates a C_0-semigroup on L_2 if and only if A_k generates a C_0-semigroup on $\mathbf{R}^n \times L_2$. We define $L_k \in \mathscr{B}(H^1, \mathbf{R})^n$ by $L_k\phi = KL\phi$. Then Theorem 1.2 (part (i)) implies that A_k generates a C_0-semigroup on $\mathbf{R}^n \times L_2$ if D is atomic at 1.

If in addition D is atomic at 0, then Theorem 1.2 (part (ii)) implies that $-A = -A_k$ generates a C_0-semigroup on $\mathbf{R}^n \times L_2$. Consequently, both T and $-T$ are generators which implies that T generates a C_0-group on L_2.

PROOF OF THEOREM 2.8. Let (ξ, ψ) belong to $\mathscr{D}(A_k^*)$. By definition there exists $(\tilde{\xi}, \tilde{\psi}) \in \mathbf{R}^n \times L_2$ such that

$$0 = \langle A_k(\eta, \phi), (\xi, \psi)\rangle - \langle(\eta, \phi), (\tilde{\xi}, \tilde{\psi})\rangle \tag{3.12}$$

for all $(\eta, \phi) \in \mathscr{D}(A_k)$ and

$$A_k^*(\xi, \psi) = (\tilde{\xi}, \tilde{\psi}). \tag{3.13}$$

We assume that L and D have the representations $D\phi = \int_0^1 d\mu(s)\phi(s)$ and $L\phi = \int_0^1\{A(s)\phi(s) + B(s)\dot{\phi}(s)\}ds$, respectively. Therefore (3.12) can be written as

$$0 = \int_0^1 \langle\phi(s), A^*(s)K^*\xi - \tilde{\psi}(s)\rangle\, ds + \int_0^1 \langle\dot{\phi}(s), B^*(s)K^*\xi + (s)\rangle ds$$
$$- \int_0^1 \langle d\mu(s)\phi(s), \xi\rangle. \tag{3.14}$$

Integrating the last term by parts we obtain

$$0 = \int_0^1 \langle\phi(s), A^*(s)K^*\xi - \tilde{\psi}(s)\rangle ds$$
$$+ \int_0^1 \langle\dot{\phi}(s), B^*(s)K^*\xi + \psi(s) + \mu^*(s)\xi\rangle ds \tag{3.15}$$
$$+ \langle\mu(0^-)\phi(0) - \mu(1)\phi(1), \xi\rangle.$$

Note that each element of the form $(D\phi, \phi)$ with $\phi \in H^1$, $\phi(0) = \phi(1) = 0$, belongs to $\mathscr{D}(A_k)$ and for such an element the last term in (3.15) becomes zero. Consequently, the Fundamental Lemma of the Calculus of Variations yields the absolute continuity of $[B^*K^*\xi + \psi + \mu^*\xi](\cdot)$ and

$$(3.16) \qquad [B^*K^*\xi + \psi + \mu^*\tilde{\xi}]'(t) = [A^*K^*\xi - \tilde{\psi}](t)$$

holds almost everywhere in [0, 1]. Moreover, the left side of (3.16) belongs to L_2 and since $\mu^*(\cdot)\tilde{\xi}$ is of bounded variation, we have that

$$(3.17) \qquad \Psi(t) = [B^*K^*\xi + \psi](t)$$

is of bounded variation and

$$(3.18) \qquad \tilde{\psi}(t) = A^*(t)K^*\xi - [\Psi(t) + \mu^*(t)\tilde{\xi}]'.$$

Equation (3.15) can now be written as

$$0 = \int_0^1 \langle \phi(s), (\Psi(s) + \mu^*(s)\tilde{\xi})\rangle' ds + \langle (\mu(0^-)\phi(0) - \mu(1)\phi(1)), \tilde{\xi}\rangle,$$

which implies that for each $\phi \in H^1$

$$(3.19) \qquad \begin{aligned} 0 = {} & \langle \phi(0), -(\Psi(0^+) + \mu^*(0^+)\tilde{\xi} + \mu^*(0)\tilde{\xi})\rangle \\ & + \langle \phi(1), (\Psi(1^-) + \mu^*(1^-)\,\tilde{\xi} - \mu^*(1)\tilde{\xi})\rangle. \end{aligned}$$

Consequently, it follows that

$$(3.20) \qquad \Psi(0^+) = [\mu^*(0^+) - \mu^*(0)]\tilde{\xi} = J^*(0)\tilde{\xi}$$

and

$$(3.21) \qquad \Psi(1^-) = [\mu^*(1) - \mu^*(1^-)]\tilde{\xi} = J^*(1)\tilde{\xi}.$$

Since $J^*(1)$ is nonsingular, we can solve (3.21) for $\tilde{\xi}$ and substitute this solution into (3.20) to obtain

$$(3.22) \qquad \tilde{\xi} = [J^*(1)]^{-1}\,\Psi(1^-)$$

and

$$(3.23) \qquad \Psi(0^+) = J^*(0)[J^*(1)]^{-1}\,\Psi(1^-).$$

Substituting (3.22) into (3.17) yields

$$(3.24) \qquad \tilde{\psi}(t) = A^*(t)K^*\xi - [\Psi(t) + \mu^*(t)[J^*(1)]^{-1}\,\Psi(1^-)]'.$$

Thus we have shown that if $(\xi, \psi) \in \mathscr{D}(A_k^*)$, then $\Psi(\cdot)$ is of bounded variation, $\Psi(\cdot) + \mu^*(\cdot)[J^*(1)]^{-1}\Psi(1^-)$ belongs to H^1, $\Psi(\cdot)$ satisfies the boundary condition (3.23) and $A_k^*(\xi, \psi) = (\tilde{\xi}, \tilde{\psi})$ where $\tilde{\xi}$ and $\tilde{\psi}$ are defined by (3.22) and (3.24), respectively. To complete the proof there remains only an elementary calculation to show that all (ξ, ψ) that satisfy these conditions belong to $\mathscr{D}(A_k^*)$.

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