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FREDHOLM ALTERNATIVES FOR NONLINEAR DIFFERENTIAL EQUATIONS

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

1. Introduction. During the past decade there has been much work devoted to existence theory for nonlinear boundary value problems which are of nonresonance type and many different types of problems and approaches to such problems have been discussed in the literature. In this paper we present an approach which unifies much which has been written about such problems. Our approach is based on some fixed point theorems which have their origin in the work of Lasota on nonlinear mappings which are not necessarily differentiable but have multivalued derivatives. In each of the applications we consider, we show that the problem at hand may be formulated in such a way that one of the fixed point theorems proved in §2 may be applied to deduce the existence of solutions. While much of the work is of a survey nature it turns out that many of the original proofs may be very much simplified and many of the results are established in a somewhat more general framework.

To illustrate the types of results discussed we present the following example.

Consider the nonlinear oscillator

$$x'' + g(x) = p(t), \tag{1.1}$$

where $p(t)$ is a 2π-periodic forcing term and g is a nonlinear restoring force such that

$$n^2 < \nu \leqq (g(x) - g(y))/(x - y) \leqq \mu < (n + 1)^2, x \neq y \tag{1.2}$$

where n is an integer. One sees that the unforced problem has, because of assumption (1.2), at most one solution, and as is to be seen, (1.1) has a 2π-periodic solution for any L^2 forcing term p. Thus one has a nonlinear Fredholm type alternative for such problems. As will be seen, assumption (1.2), allows us to formulate a fixed point problem

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$$x = f(x), \tag{1.3}$$

where f is not differentiable but has a set valued derivative F which has certain compactness and homogeneity properties which are used to show the existence of solutions of (1.3) via continuation methods. The structure of (1.3) is furthermore such that in the case of parameter dependent problems

$$x = f(\lambda, x) \tag{1.4}$$

one may, by very similar arguments, obtain spectral and bifurcation type results.

2. Fixed point results. Let E be a real Banach space with norm $|\cdot|$ and let $n(E)$ denote the set of nonempty subsets of E. *A multi-valued* mapping with domain $U \subset E$ is a mapping F: $U \to n(E)$.

A multivalued mapping is called *upper semicontinuous on* U, *u.s.c*, for short (in the sense of Kakutani) if its graph $G(F) = \{(x, y): y \in F(x)\}$ is closed in $U \times E$, i.e., if for any two sequences $\{x_n\} \subseteq U$, $\{y_n\} \subseteq E$ such that $y_n \in F(x_n)$ and $\lim_{n\to\infty} x_n = x \in U$, $\lim_{n\to\infty} y_n = y \in E$, we have that $y \in F(x)$. It follows that if F is u.s.c., then for each $x \in U$, $F(x)$ is closed in E.

A multivalued mapping is called compact if for all bounded subsets $B \subset U$, the set $\bigcup_{x\in B} F(x)$ is precompact in E. If a multivalued mapping is both compact and u.s.c. it will be called *completely continuous*.

If f: $E \to E$ is a singlevalued mapping we use the term completely continuous in its usual sense, which of course, is simply a special case of the term just defined if one views a single valued mapping as a special multivalued mapping for which $U = E$.

A multivalued mapping F is called (*positive*) *homogeneous* if $tF(x) \subseteq F(tx)$ for all (positive) $t \in \mathbf{R}$ and $x \in U$. U is assumed to be homogeneous.

F is *starlike* with respect to 0 if $F(x)$ is starlike for each x.

Let f: $E \to E$ be a mapping. We call a multivalued mapping a *set valued derivative of f at* a if $f(x) - f(a) \in F(x - a)$ for all x near a.

F is called a *set valued derivative of f at* ∞ if $f(x) \in F(x)$ for all $|x|$ large.

THEOREM 2.1. *Let f, g: $E \to E$ be completely continuous mappings and let A be a linear compact operator on E. Assume that*

$$\lim_{|x|\to\infty} |g(x)|/|x| = 0, \tag{2.1}$$

and that there exists a completely continuous set valued derivative F of f at ∞ which is starlike with respect to 0 *and positive homogeneous. Furthermore assume that*

$$x \in Ax + F(x) \Rightarrow x = 0. \tag{2.2}$$

Then the equation

$$x = Ax + f(x) + g(x) \tag{2.3}$$

has a solution $x \in E$.

PROOF. We first show that the set $\{x \in E: x = Ax + \lambda(f(x) + g(x)), 0 \leqq \lambda \leqq 1\}$ is bounded in E. This we argue indirectly, i.e., we assume there exist sequences $\{\lambda_n\} \subseteqq [0, 1]$, $\{x_n\} \subseteqq E$ with $|x_n| \geqq n$ such that

$$x_n = Ax_n + \lambda_n(f(x_n) + g(x_n)).$$

Hence

$$x_n \in Ax_n + \lambda_n F(x_n) + \lambda_n g(x_n), \ n \text{ large}.$$

Since F is positive homogeneous we obtain, letting $y_n = x_n/|x_n|$, that

$$y_n \in Ay_n + \lambda_n F(y_n) + \lambda_n g(x_n)/|x_n|$$

or

$$y_n = Ay_n + \lambda_n \omega_n + \lambda_n g(x_n)/|x_n|,$$

where $\omega_n \in F(y_n)$. Since $g(x_n)/|x_n| \to 0$, since $U_{n \geqq 1}F(y_n)$ is precompact and since A is a compact operator it follows that $\{y_n\}$ is precompact. We thus may select convergent subsequences from $\{y_n\}$, $\{\omega_n\}$ and $\{\lambda_n\}$, which we relabel as the original sequences, having limits y, ω and λ, respectively, and satisfying $y = Ay + \lambda\omega$, also $\omega \in F(y)$, since F is u.s.c. Since F is starlike with respect to 0, it follows that $\lambda\omega \in F(y)$ and hence it follows that $y \in Ay + F(y)$, which by assumption implies that $y = 0$, contradicting $|y| = 1$.

Thus let $R > 0$ be a bound for the set $\{x \in E: x = Ax + \lambda(f(x) + g(x)), 0 \leqq \lambda \leqq 1\}$. It follows that the Leray-Schauder degree

$$d_{LS}(\text{id} - A - \lambda(f + g), B_{R'}(0), 0)$$

is defined and independent of λ, for any $R' > R$, where $B_R(a) = \{x \in E: |x - a| < R\}$. Hence this degree equals

$$d_{LS}(\text{id} - A, B_{R'}(0), 0) = \pm 1,$$

as follows from the Leray-Schauder formula, hence also

$$d_{LS}(\text{id-}A - f - g, B_{R'}(0), 0) = \pm 1$$

and the equation (2.3) has a solution.

REMARK 2.2. Using an indirect argument similar to the one used in the proof of theorem 2.1 one can show the following. Let A and F be as in theorem 2.1. Then there exists a constant $\gamma = \gamma(A, F)$ such that: If

$$x \in Ax + F(x) + y \tag{2.4}$$

then

$$|x| \leqq \gamma |y|. \tag{2.5}$$

We therefore may generalize theorem 2.1 somewhat, by replacing condition (2.1) by the more general assumption

$$\limsup_{|x|\to\infty} |g(x)|/|x| < 1/\gamma. \tag{2.6}$$

To see this, choose $R > 0$ so large that $|x| \geqq R$ implies that $|g(x)| < (1/\gamma)|x|$. If then $x = Ax + \lambda(f(x) + g(x))$ for some $\lambda \in (0, 1]$, then $|x| < R$, for otherwise $x \in Ax + F(x) + \lambda g(x)$ and hence

$$|x| \leqq \gamma|\lambda g(x)| < \gamma(1/\gamma)\lambda|x| < |x|,$$

a contradiction.

REMARK 2.3. If, in theorem 2.1,

$$f(x) - f(y) \in F(x - y), \quad x, y \in E, \tag{2.7}$$

and $g(x) = g =$ const., then equation (2.3) has a unique solution. To see this, let x and y be solutions of (2.3), then $x - y = A(x - y) + f(x) - f(y)$ and hence $x - y \in A(x - y) + F(x - y)$ which implies that $x - y = 0$ (see (2.2)).

It thus follows that in this case id $- A - f$ is a homeomorphism.

REMARK 2.4. A further extension of theorem 2.1 may be obtained by replacing condition (2.2) by the requirement

$$\inf\{\operatorname{dist}(x, Ax + F(x)) \colon |x| = 1\} = \alpha > 0, \tag{2.8}$$

in which case (2.6) may be replaced by

$$|g(x)| < \alpha|x|, |x| \geqq R. \tag{2.9}$$

If (2.8) holds, it is also not necessary to assume that F be completely continuous. If F, on the other hand, is completely continuous and if (2.2) holds, then one may easily verify that (2.8) holds.

Let us next consider the equation

$$x = f(x), \tag{2.10}$$

where $f\colon \operatorname{cl}(U) \to E$ is a completely continuous mapping and U is an open subset of E.

Let $x_0 \in U$ be a solution of (2.10) and assume there exists a completely continuous set valued mapping F such that

$$f(x) - f(x_0) \in F(x - x_0) + r(x - x_0), \tag{2.11}$$

where $r\colon \operatorname{cl}(U) - x_0 \to n(E)$ is such that

$$|r(x - x_0)| = o(|x - x_0|) \text{ as } x \to x_0.$$

We also assume here that F is convex-valued, homogeneous and satisfies

$$x \in F(x) \Rightarrow x = 0. \tag{2.12}$$

((2.12) may be replaced by the more general condition (2.8) with $A = 0$, and without assuming that F be completely continuous.)

THEOREM 2.5. *Let the above assumptions hold. Then*
(i) x_0 *is an isolated solution of* (2.10).
(ii) *If* V *is a bounded neighborhood of* x_0 *such that* x_0 *is the only solution of* (2.10) *in* $\mathrm{cl}(V)$, *then*

$$d_{LS}(\mathrm{id} - f, V, 0) = \text{odd}.$$

(iii) *If* $g\colon \mathrm{cl}(U) \times M \to E$ *is completely continuous,* (*where* M *is a normed vector space*) *with* $g(\cdot, 0) = 0$, *then for all* $v \in M$ *of sufficiently small norm, the equation*

$$x = f(x) + g(x, v) \tag{2.11}$$

has a solution x_v *which may be so chosen that* $|x_v - x_0| \to 0$ *as* $v \to 0$.

PROOF. (i) Assume the assertion is false. Then there exists a sequence $\{x_n\}$, $x_n \neq x_0$, $x_n \to x_0$, such that x_n solves (2.10). Hence

$$x_n - x_0 = f(x_n) - f(x_0) \in F(x_n - x_0) + r(x_n - x_0).$$

Letting $u_n = (x_n - x_0)/|x_n - x_0|$, we obtain, by the assumptions on F, that

$$u_n \in F(u_n) + r(x_n - x_0)/|x_n - x_0|,$$

since $|u_n| = 1$ and $|r(x_n - x_0)| = o(|x_n - x_0|)$ we obtain a contradiction using arguments similar to those already employed.

(ii) It follows from (i) that for $\varepsilon > 0$ sufficiently small $d_{LS}(I - f, B_\varepsilon(x_0), 0)$ is defined and independent of ε. Let $h(x) = f(x + x_0) - f(x_0)$, then

$$d_{LS}(\mathrm{id} - h, B_\varepsilon(0), 0) = d_{LS}(\mathrm{id} - f, B_\varepsilon(x_0), 0).$$

Consider now the family of vector fields,

$$\mathrm{id} - (1/(1 + \lambda))h(\cdot) + (\lambda/(1 + \lambda))h(-\,\cdot),\ 0 \leqq \lambda \leqq 1.$$

This family is zero free on $\partial B_\varepsilon(0)$ for all $\varepsilon > 0$, sufficiently small. For if not, we may find sequences $\{\varepsilon_n\}$, $\varepsilon_n \to 0$, $\{\lambda_n\} \subseteqq [0, 1]$ and $\{x_n\}$, $|x_n| = \varepsilon_n$, such that

$$x_n = (1/(1 + \lambda_n))h(x_n) - (\lambda_n/(1 + \lambda_n))h(-x_n).$$

It follows therefore that

$$x_n \in (1/(1 + \lambda_n))F(x_n) - (\lambda_n/(1 + \lambda_n))F(-x_n) + r_1(x_n),$$

where $|r_1(x_n)| = o(|x_n|)$, as $x_n \to 0$. Using the convexity and homogeneity of F we obtain $x_n \in F(x_n) + r_1(x_n)$, and letting $y_n = x_n/|x_n|$, we obtain $y_n \in F(y_n) + r_1(x_n)/|x_n|$, and by a convergence argument similar to the one used in the proof of theorem 2.1, one obtains a convergent subsequence of $\{y_n\}$ converging to an element $y \in E$ which must satisfy $y \in F(y)$, $|y| = 1$, a contradiction to what has been assumed about F.

We therefore obtain by the homotopy invariance principle of the Leray Schauder degree that

$$d_{LS}(\mathrm{id} - h, B_\varepsilon(0), 0) = d_{LS}(\mathrm{id} - (1/2)[h(\cdot) - h(- \cdot)], B_\varepsilon(0), 0]$$

for all small $\varepsilon > 0$; the latter, however, is an odd integer, as follows from Borsuk's theorem on odd vector fields [41].

(iii) This follows immediately from (ii), since for $|v|$ sufficiently small

$$d_{LS}(\mathrm{id} - f, B_\varepsilon(x_0), 0) = d_{LS}(\mathrm{id} - f - g(\cdot, v), B_\varepsilon(x_0), 0),$$

and hence (2.11) may be solved in $B_\varepsilon(x_0)$ for all small $|v|$. Letting $\varepsilon \to 0$, one obtains the remaining conclusion of (ii).

Assume now that E has the following property: There exists a sequence of closed subspaces $E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots \subset E$ such that $\bigcup_{i=1}^{\infty} E_i$ is dense in E, and continuous linear projections P_i: $E \to E_i$, $i = 1, 2, 3, \ldots$, which are uniformly bounded, i.e., there exists a constant K (independent of i) such that $|P_i| \leqq K$.

REMARK 2.6. In case E is a Banach space with a basis such a sequence of subspaces (finite dimensional) always exists and the "natural" projections defined by these subspaces have the required property (see, e.g., [43]).

Let f be as in theorem 2.5 and along with equation (2.10), we consider the sequence of equations

$$x = P_n f(x), x \in E_n, n = 1, 2, \ldots. \tag{2.12}$$

In case $E_1, E_2, \ldots$ are finite dimensional spaces, equation (2.12) is a finite system of equations. The following result shows that under the hypotheses of theorem 2.5, the solution x_0 may in fact be obtained as a limit of solutions of equation (2.11), in addition the theorem provides an error estimate.

THEOREM 2.7. *Let the hypotheses of theorem* 2.5 *hold and let E have the property described above. Let V be an open neighborhood of* x_0 *such that* x_0 *is the only solution of* (2.10) *in* cl(V). *Then there exists* n_0 *such that for* $n \geqq n_0$ (2.12) *has a solution* $x_n \in V \cap E_n$ *and there exists a constant Q and a sequence* $\{\varepsilon_n\}$, $\varepsilon_n \to 0$ *such that*

$$|x_n - x_0| \leqq Q(1 + \varepsilon_n)|P_n x_0 - x_0|. \tag{2.13}$$

PROOF. Since $\lim_{n\to\infty}|P_nf(x) - f(x)| = 0$ uniformly for $x \in \mathrm{cl}(V)$ we obtain that for all n sufficiently large $d_{LS}(\mathrm{id} - P_nf, V \cap E_n, 0) = d_{LS}(\mathrm{id} - f, V, 0)$. Hence equation (2.12) has a solution $x_n \in V \cap E_n$. Now

$$x_n - x_0 = P_n(f(x_n) - f(x_0)) + P_nx_0 - x_0$$

and hence

$$x_n - x_0 \in P_nF(x_n - x_0) + P_nx_0 - x_0 + P_nr(x_n - x_0).$$

The set valued mapping P_nF has the same properties as F; we may thus apply remark 2.3 and conclude that

$$|x_n - x_0| \leqq \gamma(P_nF)(|P_nx_0 - x_0| + |P_n||r(x_n - x_0)|). \tag{2.14}$$

One next shows that there exists a constant M such that $\gamma(P_nF) \leqq M$, $n = 1, 2, \ldots$ and hence concludes that $x_n \to x_0$. Let δ_n be a sequence of positive real numbers such that $\delta_n \to 0$ and $|r(x_n - x_0)| \leqq \delta_n|x_n - x_0|$. One then obtains from (2.14) that $|x_n - x_0|(1 - KM\delta_n) \leqq M|P_nx_0 - x_0|$, i.e.,

$$|x_n - x_0| \leqq (M/(1 - KM\delta_n))|P_nx_0 - x_0|,$$

for n large, this yields (2.14) with $Q = M$ and $\varepsilon_n = KM\delta_n/(1 - KM\delta_n)$.

REMARK 2.8. We observe that in theorem 2.7 not all the properties of F were needed, namely F need only be positive homogeneous and $d_{LS}(\mathrm{id} - f, V, 0) \neq 0$.

We conclude this section by establishing some well-known existence theorems as consequences of the above results.

EXAMPLE 2.9. Let $E = L^2(a, b)$, $-\infty < a < b < \infty$ and let $K: E \to E$ be a compact Hermitian operator with characteristic values

$$\cdots \leqq \lambda_{-2} \leqq \lambda_{-1} < 0 < \lambda_0 \leqq \lambda_1 \leqq \cdots$$

and let $h: [a, b] \times \mathbf{R} \to \mathbf{R}$ be a continuous function such that

$$\lambda_n < \mu_n \leqq (h(t, y_2) - h(t, y_1))/(y_2 - y_1) \leqq \mu_{n+1} < \lambda_{n+1}, \tag{2.15}$$

$t \in [a, b]$, $y_1, y_2 \in \mathbf{R}$, where μ_n, μ_{n+1} are fixed constants. Let $\tilde{h}: E \to E$ be the mapping

$$\tilde{h}(y)(t) = h(t, y(t)).$$

This mapping is easily seen to be continuous. Consider now the equations

$$x = K\tilde{h}(x) \tag{2.16}$$

and

$$x = K\tilde{h}(x) + w(x), \tag{2.17}$$

where $w: E \to E$ is completely continuous and satisfies $w(x)/|x| \to 0$ as $|x| \to \infty$.

We observe first that (2.16) and (2.17) are respectively equivalent to

$$x = \lambda Kx + K(\tilde{h}(x) - \tilde{h}(0) - \lambda x) + K\tilde{h}(0) \tag{2.18}$$

and

$$x = \lambda Kx + K(\tilde{h}(x) - \tilde{h}(0) - \lambda x) + K\tilde{h}(0) + w(x), \tag{2.19}$$

for any λ. We choose

$$\begin{aligned} \lambda &= (\mu_{n+1} + \mu_n)/2; \\ Ax &= \lambda Kx; \\ f(x) &= K(\tilde{h}(x) - \tilde{h}(0) - \lambda x); \\ g(x) &= \begin{cases} K\tilde{h}(0), \text{ in (2.18)}, \\ K\tilde{h}(0) + w(x), \text{ in (2.19)}. \end{cases} \end{aligned} \tag{2.20}$$

Define $F: E \to n(E)$ by

$$\begin{aligned} F(x) = \{v \in E: v = Kmx, &\text{ where } m \in L^\infty(a, b), \\ &\text{with } |m|_{L^\infty} \leqq (\mu_{n+1} - \mu_n)/2\}. \end{aligned} \tag{2.21}$$

It follows from the assumptions on K and some elementary properties of L^2 functions that F is completely continuous, it also easily follows that F is positive homogeneous and convex valued, and that

$$f(x) - f(y) \in F(x - y). \tag{2.21}$$

Thus the existence of solutions of (2.16) and (2.17) will follow from theorem 2.1, once we check the condition (2.2). Thus assume there exists $x \in E$ such that $x \in Ax + F(x)$. Then there exists $m \in L^\infty(a, b)$, $|m|_{L^\infty} \leqq (\mu_{n+1} - \mu_n)/2$ such that $x = \lambda Kx + Kmx$ or $x = (\mathrm{id} - \lambda K)^{-1}Kmx$. Hence

$$|x| \leqq |(\mathrm{id} - \lambda K)^{-1}K||m|_{L^\infty}|x|.$$

Since

$$|(\mathrm{id} - \lambda K)^{-1}K| = (1/\mathrm{dist}(\lambda, \{\lambda_i\})) < 2/(\mu_{n+1} - \mu_n) \tag{2.22}$$

(see e.g. [**14**]), it follows that $|x| = 0$. Thus theorem 2.1 and remark 2.3 may be applied to conclude the existence of a unique solution of (2.18), hence of (2.16). For (2.19), hence for (2.17), we can only assert existence of a solution. We note that if one is only interested in existence of solutions one can weaken the requirement (2.15) to

$$\lambda_n < \mu_n \leqq (h(t, y) - h(t, 0))/y \leqq \mu_{n+1} < \lambda_{n+1}; \tag{2.23}$$

of course (2.21) no longer will hold.

The requirement, $w(x)/|x| \to 0$ as $|x| \to \infty$, may be weakened as in remark 2.2. Let us, in this case, compute a permissible value for $\gamma(F)$. Thus suppose $x \in Ax + F(x) + y$. Then as above $x = \lambda Kx + Kmx + y$ or some $m \in L^\infty(a, b)$, $|m|_{L^\infty} \leqq (\mu_{n+1} - \mu_n)/2$, or

$$x = (\mathrm{id} - \lambda K)^{-1}Kmx + (\mathrm{id} - \lambda K)^{-1}y.$$

Hence

$$\begin{aligned}|x| &\leqq |(\mathrm{id} - \lambda K)^{-1}K||m|_{L^\infty}|x| + |(\mathrm{id} - \lambda K)^{-1}||y| \\ &\leqq (1/\mathrm{dist}(\lambda, \{\lambda_i\}))((\mu_{n+1} - \mu_n|)/2)|x| + \sup_i | \lambda_i/(\lambda_i - \lambda) | \, ||y|,\end{aligned}$$

where we have used the spectral theorem for self-adjoint operators to estimate $|(\mathrm{id} - \lambda K)^{-1}|$ (see again **[14]**). Hence $|x| \leqq \gamma|y|$, where

$$\begin{aligned}\gamma = &(1/\min\{1 - \lambda/\lambda_{n+1}, 1 - \lambda/\lambda_n\}) \\ &\cdot[1 - (\mu_{n+1} - \mu_n)/2 \min\{\lambda - \lambda_n, \lambda_{n+1} - \lambda\}]^{-1}.\end{aligned}$$

We observe that (2.18) is equivalent to

$$x = (\mathrm{id} - \lambda K)^{-1}K(\tilde{h}(x) - \tilde{h}(0) - \lambda x) + (\mathrm{id} - \lambda K)^{-1}K\tilde{h}(0), \tag{2.24}$$

where the right hand side may be shown to be a contraction mapping. Thus the existence and uniqueness of a solution of (2.18) will also follow from the contraction mapping principle.

EXAMPLE 2.10. Let E be a Hilbert space and let L: dom $L \subset E \to E$ be a linear self-adjoint operator and let $N: E \to E$ be a continuous nonlinear operator. Let $\lambda, \mu \in \sigma(L) =$ spectrum of L be such that $(\lambda, \mu) \subset \rho(L) =$ resolvent of L and assume there exists $\xi \in (\lambda, \mu)$ such that $(L - \xi\mathrm{id})^{-1}$ is compact. We assume that

$$|(N - \xi\mathrm{id})(x)| \leqq k|x|, \tag{2.25}$$

where

$$k < \mathrm{dist}(\xi, \sigma(L)). \tag{2.26}$$

Under these conditions $L - N$ maps dom L onto E. To see this we consider for a given $y \in E$ the equation $Lx = Nx - y$ which is equivalent to $(L - \xi\mathrm{id})x = (N - \xi\mathrm{id})(x) + y$ or $x = (L - \xi\mathrm{id})^{-1}(N - \xi\mathrm{id})(x) + (L - \xi\mathrm{id})^{-1}y$. We now apply theorem 2.1 with

$$\begin{aligned}f(x) &= (L - \xi\mathrm{id})^{-1}(N - \xi\mathrm{id})(x), \\ g(x) &= (L - \xi\mathrm{id})^{-1}y,\end{aligned}$$

and

$$F(x) = \{v: v = (L - \xi\mathrm{id})^{-1}w, |w| \leqq k|x|\},$$

where we use the fact that

$$|(L - \xi \mathrm{id})^{-1}| = 1/\mathrm{dist}(\xi, \sigma(L))$$

(see [14]). We note that (2.25) may be realized if for example N is Gâteaux differentiable with a symmetric Gâteaux derivative $N'(x)$ satisfying $q \cdot \mathrm{id} \leqq N'(x) \leqq p\ \mathrm{id}$, where $\lambda < q \leqq p < \mu$ and where for two operators C and D, $C \leqq D$ is defined in the usual way, i.e. $(Cx, x) \leqq (Dx, x)$ for all $x \in E$.

Since

$$\begin{aligned} |(N - \xi \mathrm{id})(x)| &\leqq |(N'(x\tau) - \xi \mathrm{id})||x|, \quad 0 < \tau < 1, \\ &\leqq \sup_{z \in E} |N'(z) - \xi \mathrm{id}||x|, \end{aligned}$$

and

$$(q - \xi)\mathrm{id} \leqq N'(z) - \xi \mathrm{id} \leqq (p - \xi)\mathrm{id},$$

if follows that

$$(q - \xi)|x|^2 \leqq ((N'(z) - \xi \mathrm{id})x, x) \leqq (p - \xi)|x|^2,$$

and thus since

$$|N'(z) - \xi \mathrm{id}| \leqq \sup_{|x|=1} |((N'(z) - \xi \mathrm{id})x, x)|$$

we obtain

$$\begin{aligned} |N'(z) - \xi \mathrm{id}| &\leqq \max\{|q - \xi|, |p - \xi|\} = k \\ &< \mathrm{dist}(\xi, \sigma(L)). \end{aligned}$$

If one replaces (2.26) by the more restrictive Lipschitz condition

$$|N(x) - N(y) - \xi(x - y)| \leqq k|x - y|,$$

$k < \mathrm{dist}(\xi, \sigma(L))$, then the result just proven remains valid without the compactness assumption on the resolvent, since in this case the mapping f will be a contraction.

REMARK 2.11. The idea of set valued derivative in the study of fixed point problems was first introduced by Lasota [17, 18]. Less general versions of theorem 2.1 are given in [17] and [18]. The idea of remark 2.4 is due to Szostak [45], where also an implicit function theorem of the type of theorem 2.5 is established. Theorem 2.5 was first established by somewhat different means by Chow and Lasota in [7]. Theorem 2.7 is taken from Schmitt [38]; it represents a generalization of a result of Krasnosel'skii (see [38]) and provides a convergence result and error estimates for a Galerkin method for the solution of such nonlinear equations. Example

2.9 extends results which are originally due to Dolph [8] and example 2.10 is based on a result of Mawhin [27]. This later result has nicely been extended by Bates in [3, 4].

3. Nonlinear differential equations subject to linear endpoint constraints. Let

$$L = \sum_{i=0}^{n} a_i(t)D^i, \ D = \frac{d}{dt} \tag{3.1}$$

be an nth order formal differential operator with continuous coefficients $a_i(\cdot)$, $0 \leqq i \leqq n$, and $a_n(t) \neq 0$, $0 \leqq t \leqq 2\pi$.

Let

$$B_i u = \sum_{j=0}^{n-1} \alpha_{ij} u^{(j)}(0) + \sum_{j=0}^{n-1} \beta_{ij} u^{(j)}(2\pi), \ 1 \leqq i \leqq n \tag{2.3}$$

be a set of linearly independent boundary conditions, where $\alpha_{ij}, \beta_{ij} \in \mathbf{R}$. We assume that L together with the boundary conditions

$$B_i u = 0, \ 1 \leqq i \leqq n \tag{3.3}$$

gives rise to a self-adjoint operator on $L^2(0, 2\pi)$ whose spectrum we denote by $\sigma(L)$.

We consider now the problem

$$\begin{aligned} &Lu + h(t, u, \ldots, u^{(n-1)})u = k(t, u, \ldots, u^{(n-1)}) \\ &B_i u = 0, \ 1 \leqq i \leqq n \end{aligned} \tag{3.4}$$

where $h, k\colon [0, 2\pi] \times \mathbf{R}^n \to \mathbf{R}$ are such that $u_1 h(t, u_1, \ldots, u_n)$ and $k(t, u_1, \ldots, u_n)$ are continuous.

We establish now the following nonresonance result.

THEOREM 3.1. *Assume that there exist $p, q \in \mathbf{R}$ such that*

$$p \leqq h(\cdot) \leqq q, \tag{3.5}$$

$$[p, q] \cap \sigma(L) = \emptyset, \tag{3.6}$$

$$\lim_{|u_1| + \cdots + |u_n| \to \infty} k(t, u_1, \ldots, u_n)/(|u_1| + \cdots + |u_n|) = 0 \tag{3.7}$$

uniformly with respect to $t \in [0, 2\pi]$.

Then (3.4) *has a solution $u \in C^n[0, 2\pi]$.*

PROOF. Let $\lambda = (p + q)/2$ and rewrite (3.4) as

$$(L - \lambda)u = (h - \lambda)(u) + k(u), \ B_i u = 0. \tag{3.8}$$

Let $E = \{u \in C^{n-1}[0, 2\pi]\colon B_i u = 0, \ 1 \leqq i \leqq n\}$. Then E becomes a real Banach space when equipped with the usual norm

$$|u| = \sum_{i=0}^{n-1} \max_{[0, 2\pi]} |u^{(i)}(t)].$$

Since $\lambda \notin \sigma(L)$, (3.8) is equivalent to

$$\begin{aligned} u &= (L - \lambda)^{-1}(h - \lambda)(u) + (L - \lambda)^{-1}k(u) \\ &= f(u) + g(u). \end{aligned}$$

The operator f: $E \to E$ is completely continuous and has the following set-valued derivative

$$\begin{aligned} F(u) &= \{v \in E\colon v = (L - \lambda)^{-1}au, \text{ where} \\ &\quad a \in L^\infty(0, 2\pi) \text{ with } |a|_{L^\infty} \leqq (q - p)/2. \end{aligned} \tag{3.9}$$

The multivalued mapping F is clearly completely continuous, convex valued and positive homogeneous. If now $u \in F(u)$, then

$$\begin{aligned} |u|_{L^2} &\leqq |(L - \lambda)^{-1}|_{L^2}|au|_{L^2} \\ &\leqq (1/\text{dist}(\lambda, \sigma(L)))((q - p)/2)|u|_{L^2}, \end{aligned} \tag{3.10}$$

where we use the fact that

$$\begin{aligned} |(L - \lambda)^{-1}|_{L^2} &= \text{ spectral radius of } (L - \lambda)^{-1} \\ &= \sup\{1/|\lambda - \mu|\colon \mu \in \sigma(L)\}. \end{aligned}$$

But $\inf\{|\lambda - \mu|\colon \mu \in \sigma(L)\} > (q - p/2$, hence (3.10) implies $u = 0$. Because of (3.7) it follows that $g(u) = o(|u|)$ as $|u| \to \infty$. Hence we may apply theorem 2.1 to deduce the existence of a solution of (3.8). Since $(L - \lambda)^{-1}\colon L^2(0, 2\pi) \to C^{n-1}[0, 2\pi] \cap \{u\colon u^{(n)}$ is absolutely continuous$\}$, it follows that in fact $u \in C^n[0, 2\pi]$ and satisfies (3.4).

REMARK 3.2. Theorem 3.1 remains valid without the assumption of self-adjointness provided we assume the following: There exists $\lambda \in (p, q)$ such that $|(L - \lambda)^{-1}|_{L^2}|h - \lambda|_{L^\infty} < 1$. In this case we define F in exactly the same way and note that a formula like (3.10) may be used to conclude that $u \in F(u)$ implies $u = 0$.

As an example for the above existence theorem we consider a result of Loud **[26]** which has been the source of many papers on periodically perturbed conservative systems.

The problem is to establish the existence of periodic solutions (of period 2π) of the equation

$$u'' + g(u) = p(t), \tag{3.11}$$

where p is a given 2π periodic continuous function. We assume that g satisfies the Lipschitz condition

$$(n + \delta)^2 \leqq (g(u) - g(0))/u \leqq (n + 1 - \delta)^2, \tag{3.12}$$

where δ is a small positive constant. We rewrite (3.11) as

$$u'' + h(u)u = p(t) + g(0), \tag{3.13}$$

where $h(u) = (g(u) - g(0))/u$ and thus h satisfies (3.12). We may therefore apply theorem 3.1 and obtain the result.

COROLLARY 3.3. *Let g satisfy* (3.12), *then for any 2π-periodic function p there exists a 2π-periodic solution of* (3.11).

If (3.12) *is replaced by the stronger requirement*

$$n^2 < p \leqq (g(u) - g(v))/(u - v) \leqq q < (n + 1)^2, \tag{3.14}$$

then in fact, this solution is unique.

We also note that theorem 2.1 shows that Theorem 3.1 and hence Corollary 3.3 is true if it is required that (3.5) and (3.12) hold for large $|u|$ only.

Another way of stating (3.12) is the following: There exists $\xi \in (n^2, (n + 1)^2)$ such that

$$|g(u) - g(0) - \xi u| < k|u|, \ |u| \text{ large},$$

where

$$k < \min\{\xi - n^2, (n + 1)^2 - \xi\},$$

in which case Corollary 3.3 immediately fits into the context of example 2.10.

Let us next consider the case where equation (3.11) is a coupled system of equations and p may depend upon x (and also on x').

$$x'' + \operatorname{grad} G(x) = p(t, x). \tag{3.15}$$

THEOREM 3.4. *Let there exist constant symmetric matrices A and B with eigenvalues $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$, $\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_n$, respectively, which satisfy*

$$N_n^2 < \lambda_k \leqq \mu_k < (N_k + 1)^2, \tag{3.16}$$

where $N_1, \ldots, N_n$ are integers. Furthermore assume that

$$A \leqq \left(\frac{\partial^2 G}{\partial u_i \partial u_j}\right) \leqq B, \ u \in \mathbf{R}^n. \tag{3.17}$$

Let p be 2π-periodic with respect to t and be such that $|p(t, x)|/|x| \to 0$ as $|x| \to \infty$ uniformly with respect to $t \in [0, 2\pi]$. Then (3.15) *has a 2π-periodic solution.*

PROOF. We rewrite (3.15) as

$$x'' + Bx = (Bx - \operatorname{grad} G(x) + \operatorname{grad} G(0)) - \operatorname{grad} G(0) + p(t, x). \tag{3.18}$$

By the assumption concerning the eigenvalues of B it follows that the left hand side of (3.18) defines an invertible self-adjoint operator in $L^2(0, 2\pi)$ whose inverse, denoted by K, can be viewed as a compact operator on $C_{2\pi}[0, 2\pi] = \{x \in C[0, 2\pi]: x(0) = x(2\pi)\}$. Hence (3.18) may be written as $x = f(x) + g(x)$, where $f(x) = K(Bx - \text{grad } G(x) + \text{grad } G(0))$ and $g(x) = K(p(\cdot, x) - \text{grad } G(0))$. Now

$$\begin{aligned} Bx - \text{grad } G(x) + \text{grad } G(0) &= Bx - \int_0^1 \frac{\partial^2 G}{\partial x^2}(sx)\, x ds \\ &= \int_0^1 \left(B - \frac{\partial^2 G}{\partial x^2}(sx)\right) x ds = P(x)x, \end{aligned}$$

where $\partial^2 G/\partial x^2$ is the Hessian matrix $(\partial^2 G/\partial x_i \partial x_j)$, and $0 \leqq P \leqq B - A$. Hence $f(x) \in F(x) = \{v \in C_{2\pi}[0, 2\pi]: v = KQx$, where $Q(t)$ is a symmetric matrix with measurable entries, such that $0 \leqq Q(t) \leqq B - A\}$. The theorem will therefore follow from theorem 3.1 provided we show that (2.2) holds with $A = 0$. Thus let $x \in F(x)$, i.e., $x = KQx$ or $x'' + (B - Q(t))x = 0$, where Q is a measurable symmetric matrix, with $0 \leqq Q(t) \leqq B - A$. Since $A \leqq B - Q(t) \leqq B$, the conclusion will follow from the next lemma.

LEMMA 3.5. *Let $P(t)$, $-\infty < t < \infty$, be a 2π-periodic measurable symmetric $n \times n$ matrix such that $A \leqq P(t) \leqq B$ where A and B are as above. Then the only 2π-periodic solution of $u'' + P(t)u = 0$ is the trivial solution.*

PROOF. It suffices to show that the operator

$$L - P: \text{dom } L = \{u \in AC^1(-\infty, \infty): u(t + 2\pi) = u(t),\ u'' \in L^2(0, 2\pi)\}$$

does not have zero as an eigenvalue, where $Lu = u''$, $(Pu)(t) = P(t)u(t)$. Thus let μ be an eigenvalue. Then because of the symmetry of P we get that μ is real and $-u'' - P(t)u = \mu u$ has a nontrivial 2π-periodic solution. Hence

$$-(u'', u)_{L^2} - (P(t)u, u)_{L^2} = \mu(u, u)_{L^2}$$

and since $(Au, u) \leqq (P(t)u, u) \leqq (Bu, u)$ it follows that

$$-(u'', u)_{L^2} - (Bu, u)_{L^2} \leqq \mu(u, u)_{L^2} \leqq -(u'', u)_{L^2} - (Au, u)_{L^2},$$

which by the variational characterization of the eigenvalues implies that for some k and some integer n, $n^2 - \mu_k \leqq \mu \leqq n^2 - \lambda_k$, which by (3.16) implies that $\mu \neq 0$. Hence the lemma follows.

REMARK 3.6. One may also treat perturbation terms of the form $p(t, x, x')$ as long as $p(t, x, x')/(|x| + |x'|) \to 0$ as $|x| + |x'| \to \infty$. In this case one employs for E the space $C^1_{2\pi}[0, 2\pi]$ and arguments similar to the above.

Let us next consider nonlinear Sturm-Liouville problems of the form

$$
\begin{aligned}
&Lu = \lambda ru + f(t, u, u', \lambda),\\
(3.19)\qquad &au(0) + bu'(0) = 0,\\
&cu(1) + du'(1) = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&Lu = \lambda ru + f(t, u, u', \lambda) + g(t)\\
(3.20)\qquad &au(0) + du'(0) = 0\\
&cu(1) + du'(1) = 0,
\end{aligned}
$$

where L is the regular differential operator

$$Lu = -(pu')' + qu,$$

where p and r are positive and continuous on $[0, 1]$, q is continuous and $(a^2 + b^2)(c^2 + d^2) > 0$. We assume there exists $M > 0$ such that

$$(3.21)\qquad |f(t, u, v, \lambda)| \leqq M|u|,\ (t, u, v, \lambda) \in [0, 1] \times \mathbf{R}^3,$$

and that f is continuous.

We shall be concerned with the question of finding all real values of λ for which (3.29) has nontrivial solutions for some f satisfying (3.21) and finding those λ for which (3.20) has solutions for all $g \in L^2(0, 1)$ and all f satisfying (3.21).

Assuming that $\lambda = 0$ is not an eigenvalue of the associated linear problem ($f \equiv 0$), the operator L subject to the boundary conditions has an inverse on $L^2(0, 1)$ which we view as a completely continuous operator L^{-1} on $E = C^1[0, 1]$.

Then (3.19) and (3.20) may be rewritten as

$$(3.22)\qquad u = \lambda Au + h(\lambda, u)$$

$$(3.23)\qquad u = \lambda Au + h(\lambda, u) + e,$$

where

$$
\begin{aligned}
Au &= L^{-1}ru\\
h(\lambda, u)(t) &= L^{-1}f(t, u, u', \lambda)\\
e &= L^{-1}g.
\end{aligned}
$$

Clearly h is completely continuous and

$$
\begin{aligned}
h(\lambda, u) \in F(u) = \{v \in E\colon v = Ah, &\text{ where } h \in L^\infty(0, 1)\\
&\text{is such that } r_0|h(s)| \leqq M|u(s)| \text{ a.e.,}\\
&r_0 = \min r(s)\}.
\end{aligned}
$$

It follows immediately that F is completely continuous, positive homogeneous and F is starlike with respect to 0.

THEOREM 3.7. *Let $\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$ be the eigenvalues of the linear problem subject to the boundary conditions. If f satisfies* (3.21) *and* (λ, u) *satisfies* (3.19) *with $u \neq 0$, then*

$$\lambda \in \bigcup_{k=1}^{\infty} [\lambda_k - M/r_0, \lambda_k + M/r_0].$$

If $\lambda \notin \bigcup_{k=1}^{\infty}[\lambda_k - M/r_0, \lambda_k + M/r_0]$, then (3.20) *has a solution for every $g \in L^2(0, 1)$.*

PROOF. If (λ, u) satisfies (3.19), then $u = \lambda Au + h(\lambda, u)$. Thus either $\lambda = \lambda_k$ for some k or $u = (\mathrm{id} - \lambda A)^{-1}h(\lambda, u)$. Taking L^2 norms and viewing $(\mathrm{id} - \lambda A)^{-1}$ as an L^2 operator we get

$$|u|_{L^2} \leqq |(\mathrm{id} - \lambda A)^{-1}A|_{L^2}(M/r_0)|u|_{L^2}.$$

Since $u \neq 0$, it follows that

$$\mathrm{dist}(\lambda, \{\lambda_i\}) = 1/|(\mathrm{id} - \lambda A)^{-1}A_{L^2}| \leqq M/r_0,$$

proving the first assertion. In fact, we have shown that if $u \in \lambda Au + F(u)$, $u \neq 0$, then $\lambda \in \bigcup_{i=0}^{\infty}[\lambda_i - M/r_0, \lambda_i + M/r_0]$. Thus the second assertion follows immediately from theorem 2.1.

REMARK 3.8. Assumption (3.21) may be relaxed to

$$|f(t, u, u', \lambda)| \leqq M|u| + K|u'| \tag{3.24}$$

provided K is sufficiently small. In this case $\bigcup_{k=0}^{\infty}[\lambda_k - M/r_0, \lambda_k + M/r_0]$ must be replaced by $\bigcup_{k=0}^{\infty}[\lambda_k - a_k, \lambda_k + b_k]$ where it is known that these intervals are eventually pairwise disjoint.

REMARK 3.9. If we consider the linear problem ($f \equiv 0$), then to each λ_k there corresponds a one dimensional space of solutions where each nontrivial solution has precisely k nodes interior to $(0, 1)$. If f is differentiable at 0, then it is known that in fact each λ_k is a bifurcation point for the nonlinear problem, and the global bifurcation theorem of Krasnosel'skii and Rabinowitz may be applied. If, on the other hand, f satisfies the Lipschitz condition (3.21) or (3.24), one may show that each interval $I_k = [\lambda_k - M_0/r_0, \lambda_k + M_0/r_0]$ (or $I_k = [\lambda_k - a_k, \lambda_k + b_k]$) which is disjoint from the other such intervals will be a bifurcation interval and an unbounded continuum of nontrivial solutions having k nodes in $(0, 1)$ bifurcate from this interval, i.e., there exists a set of solutions $S_k = \{(\lambda, u)\colon \lambda \in I_k$ and u has exactly k nodes in $(0, 1)\}$ which is such that $\mathrm{cl}(S_k) \cap (I_k \times \{0\}) \neq \emptyset$ and $\mathrm{cl}(S_k)$ is an unbounded continuum. If (3.21) only holds near the origin one may still conclude all the above properties about $\mathrm{cl}(S_k)$, except that λ no longer need be such that $\lambda \in I_k$ for all $(\lambda, u) \in \mathrm{cl}(S_k)$.

REMARK 3.10. The problems discussed in this section have been written about extensively. Theorem 3.1 presents a generalization of a result of Kannan and Locker [13]. The results on periodically perturbed conservative systems have been established in various forms and by various means by Loud [26], Leach [25], Ahmad [1], Mawhin [27], Kannan [12], Ward [48], Brown and Liu [5]. Lemma 3.5 which is crucial for our method of attack is due to Lazer [23]. The case of periodically perturbed nonconservative systems which has been extensively studied by Reissig [32–35] may also be treated by the results of section 2 by employing arguments based on Remark 3.2. The results covered by Theorem 3.7, and Remarks 3.8, 3.9 are due to Schmitt and Smith [40] (see also Schmitt [39]). Results analogous to that established by Castro [6] for weak solutions of nonlinear Dirichlet problems may also be established by means of the methods of §2 and §3 by using the appropriate function space setup.

4. Weak solutions of nonlinear elliptic equations. In this chapter we consider some boundary value problems for nonlinear elliptic partial differential equations which are motivated by the nonresonance results of Landesman and Lazer [16] and which have been extended in various directions by several authors. The setup again is such that these results may be put into the framework of the theory developed in chapter 2. We shall restrict ourselves here to equations of order 2, though elliptic equations of higher order may be treated in much the same way.

Let G be a bounded domain in $\mathbf{R}^n$ and let $\mathbf{H}^i(G)$ be the Sobolev space of all real valued $L^2(G)$ functions having weak derivatives up to order i which belong to $L^2(G)$, i.e.,

$$H^i(G) = \{u \in L^2(G) \colon \partial^\alpha u \in L^2(G);\ |\alpha| \leqq 1\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex, $|\alpha| = \sum_{i=1}^n \alpha_i$ and $\partial^\alpha u$ is the weak derivative of u of order α, and of course, $\partial^0 u = u$. The inner product in $H^i(G)$ is given by

$$(u, v)_{H_i} = \sum_{|\alpha| \leqq i} \partial^\alpha u \, \partial^\alpha v.$$

Furthermore, $H_0^i(G)$ shall denote the closure of $C_0^\infty(G)$ in $H^i(G)$.

Let V be a sub-Hilbert space of $H^1(G)$ which contains $H_0^1(G)$ and H a sub-Hilbert space of $L^2(\partial G)$ and let $\gamma\colon V \to H$ be a linear surjection with kernel $H_0^1(G)$ and assume that H is isomorphic to $V/H_0^1(G)$.

Let $a_1\colon V \times V \to \mathbf{R}$ and $a_2\colon H \times H \to \mathbf{R}$ be continuous sesquilinear forms and put

$$a(u, v) = a_1(u, v) + a_2(\gamma u, \gamma v),\ u, v \in V. \tag{4.1}$$

Furthermore let $f \in L^2(G)$ be given and consider the problem:

(4.2) Find $u \in V$ such that $a(u, v) = (f, v)_{L^2(G)}$, $v \in V$.

Define the linear operators L_1 and L_2 by domain $L_1 = \mathrm{dom}(L_1) = \{u \in V$: $v \to a_1(u, v)$ is a continuous linear functional of $L^2(G)\}$ and

$$a_1(u, v) = (L_1 u, v)_{L^2(G)},\ v \in V,$$

and $L_2\phi(\psi) = a_2(\phi, \psi)$, $\phi, \psi \in H$. Further let the operator ρ be defined by

$$a_1(u, v) - (L_1 u, v)_{L^2(G)} = \rho u(\gamma v),\ u \in \mathrm{dom}(L_1),\ v \in V. \tag{4.3}$$

Then it is well-know (see, *e.g.*, Showalter **[42]**) that problem (4.2) is equivalent to the problem:

(4.4) Find $u \in V$ such that $L_1 u = f$, $\rho u + L_2(\gamma u) = 0$.

Concerning G we assume that ∂G is a C^2 manifold of dimension $n - 1$ such that G lies on one side of ∂G, and we define the operator L similar to L_1 by

$$a(u, v) = (Lu, v)_{L^2(G)},\ u \in \mathrm{dom}\ L,\ v \in V, \tag{4.5}$$

and we shall assume that a is symmetric, i.e. $a(u, v) = a(v, u)$, $u, v \in V$. Then it is true that A is self-adjoint, also because of the regularity assumptions on the boundary it follows that the injection of $H^1(G)$ into $L^2(G)$ is compact, and hence V is compactly injected into $L^2(G)$.

REMARK 4.1. If $V = H_0^1(G)$, then the injection of V into $L^2(G)$ is compact; we therefore can treat the case of Dirichlet boundary conditions in a somewhat more general setting.

Let us further assume that a is V-elliptic, i.e., there exists a constant $c > 0$ such that

$$a(u, v) \geqq c|u|^2_{H^2(G)},\ u \in V.$$

In this case it is known that there exists an infinite sequence of eigenvalues

$$0 < \lambda_0 \leqq \lambda_1 \leqq \cdots \leqq \lambda_N \leqq \cdots \lim_{N\to\infty} \lambda_N \to \infty$$

with associated eigenfunctions $\{v_j\}_{j\to 0}^{\infty}$, such that $(v_i, v_j)_{L^2(G)} = \delta_{ij}$ (Kronecker delta) of a, i.e for $i = 0, 1, 2, \ldots$,

$$a(v_i, v) = \lambda_i(v_i, v),\ v \in V,$$

and $\{v_j\}_{j=0}^{\infty}$ is a basis for $L^2(G)$.

Let $f\colon G \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ and $f\colon G \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ be functions which satisfy Carathéodory conditions and are such that

$$\lambda_N \leqq \alpha(x) \leqq f(x, u, v) \leqq \beta(x) \leqq \lambda_{N+1}, \tag{4.6}$$

where $\alpha, \beta \in L^\infty(G)$ are such that they differ from λ_N, respectively λ_{N+1}, on a set of positive measure, and there exists $c \in L^\infty(G)$ and for all $\varepsilon > 0$, $d_\varepsilon \in L^2(G)$ such that

$$|g(x, u, v)| \leqq \varepsilon\, c(x)|(u, v)| + d_\varepsilon(x). \tag{4.7}$$

We now consider the problem:
Find $u \in V$ such that

$$a(u, v) = (f(\cdot, u, \partial u)u, v)_{L^2(G)} + (g(\cdot, u, \partial u), v)_{L^2(G)},\ v \in V,$$

where $\partial u = (\partial_1 u, \ldots, \partial_n u)$ is the distributional gradient vector of u. As was observed earlier, this problem is equivalent to the boundary value problem

$$\begin{aligned} L_1 u &= (f(\cdot, u, \partial u)u + g(\cdot, u, \partial u) \\ \rho u + L_2(\gamma u) &= 0,\ u \in V. \end{aligned} \tag{4.8}$$

Before proving an existence theorem for this problem we verify some auxiliary results.

LEMMA 4.2. *Let $p \in L^\infty(G)$ be such that*

$$\lambda_N \leqq p(x) \leqq \lambda_{N+1}, \tag{4.9}$$

and suppose that p differs from λ_N and λ_{N+1} on a set of positive measure. Then the only solution $u \in V$ of

$$a(u, v) = (p(\cdot)u, v)_{L^2(G)},\ v \in V \tag{4.10}$$

is $u = 0$.

PROOF. If $u \neq 0$, then because of the hypotheses on p we conclude that

$$\lambda_N |u|^2_{L^2(G)} < (p(\cdot)u, u)_{L^2(G)} < \lambda_{N+1}|u|^2_{L^2(G)} \tag{4.11}$$

(u cannot vanish on sets of positive measure unless $u \equiv 0$), hence

$$a(u, v) = \sum_{i=0}^{\infty} \lambda_i \alpha_i^2 \tag{4.12}$$

must satisfy

$$\lambda_N |u|^2_{L^2(G)} < \sum_{i=0}^{\infty} \lambda_i \alpha_i^2 < \lambda_{N+1}|u|^2_{L^2(G)}, \tag{4.13}$$

but since $\{v_i\}_{i=0}^\infty$ is an orthonormal basis for $L^2(G)$ we get also that $|u|^2_{L^2(G)} = \sum_{i=0}^\infty \alpha_i^2$.

If now $u \in \operatorname{span}\{v_0, \ldots, v_N\} = H_1$ then $\alpha_i = 0$ for $i \geqq N + 1$ and the left side of (4.11) implies that $\alpha_i = 0$ for $i = 0, \ldots, N$. Similarly if $u \in \operatorname{cl}(\operatorname{span}\{v_{N+1}, v_{N+2}, \ldots\}) = H_2$, then the right side of (4.11) implies that $\alpha_i = 0$, $i = 0, 1, \ldots$.

Hence $u = u_1 + u_2$, $u_1 \in H_1$, $u_2 \in H_2$, where neither u_1 nor u_2 equal 0.

On the other hand it follows from (4.10), that

$$\begin{aligned} 0 &= a(u_1 + u_2, u_1 - u_2) - (p(\cdot)(u_1 + u_2), u_1 - u_2)_{L^2(G)} \\ &= a(u_1, u_1) - a(u_2, u_2) - (p(\cdot)u_1, u_1) - (p(\cdot)u_2, u_2) \\ &= a(u_1, u_1) - (p(\cdot)u_1, u_1) - (a(u_2, u_2) - (p(\cdot)u_2, u_2)). \end{aligned} \tag{4.14}$$

For any nontrivial u (4.11) holds with $<$ replaced by $\leqq$, this coupled with (4.12) yields that

$$a(u_1, u_1) - (p(\cdot)u_1, u_1) \leqq 0 \leqq a(u_2, u_2) - (p(\cdot)u_2, u_2), \tag{4.15}$$

hence because of (4.14) we must have that equality holds in (4.15). This on the other hand implies that $\alpha_0 = \alpha_1 = \cdots = \alpha_N = \cdots = 0$, contradicting that $u \neq 0$.

Let us now consider the problem (4.7) or equivalently (4.8). Let $\lambda = (\lambda_N + \lambda_{N+1})/2$, Then the resolvent $(L - \lambda)^{-1}$ is a compact mapping from $L^2(G)$ to V and (4.8) is equivalent to

$$u = (L - \lambda)^{-1}(\tilde{f}(u) - \lambda)u + (L - \lambda)^{-1}\tilde{g}(u), \tag{4.16}$$

where in fact because of the assumptions on the boundary the right hand side belongs to $H^2(G) \cap V$, where we have let $\tilde{f}(u)u$ and $\tilde{g}(u)$ to be the Nemitskii operators associated with $f(\cdot, u, \partial u)u$ and $g(\cdot, u, \partial u)$. Since, because of the Carathéodory assumptions upon f and g, $\tilde{f}(u)u$ and $\tilde{g}(u)$ are continuous from V to $L^2(G)$, it follows that the right hand side of (4.16) is a completely continuous mapping from V to V (here we have used the fact that the injection of $H^2(G)$ into $H^1(G)$ is compact (see, e.g., **[42]**)).

We now wish to apply theorem 2.1, more specifically the more general result pointed out in remark 2.4.

We define F by

$$F(u) = \{v \in V : v = (L - \lambda)^{-1}(p(\cdot) - \lambda)u, \tag{4.17}$$
where $p \in L^\infty(G)$ is such that $\lambda_N \leqq p(x) \leqq \lambda_{N+1}$, and p differs from each of λ_N and λ_{N+1} on sets of positive measure$\}$.

Then F is completely continuous and homogeneous. Further it follows that if $u \in F(u)$, then $Lu = p(\cdot)u$, for some such p, and hence by lemma 4.2 it follows that $u = 0$.

One now proves (see, e.g., lemma 2.2 of **[30]**) that if u is a solution of

$$u = (L - \lambda)^{-1}(p(\cdot) - \lambda)u + (L - \lambda)^{-1}y, \; y \in L^2(G),$$

then there exists a constant $\delta > 0$ such that

$$|y|_{L^2} \geqq \delta|u|_{H^1},$$

for any p satisfying the above requirement. Hence for $y = \tilde{g}(u)$ it follows that

$$\delta|u|_{H^1} \leqq \varepsilon|c|_{L^\infty}|u|_{H^1} + |d_\varepsilon|_{L^2}$$

for any $\varepsilon > 0$. Hence if $\varepsilon|c|_{L^\infty} < \delta$ it follows that

$$|u|_{H^1} \leqq |d_\varepsilon|_{L^2}/(\delta - \varepsilon|c|_{L^\infty}),$$

and we may apply a remark analogous to remark 2.4 to conclude that (4.8) has a solution. These remarks may be summarized in the following theorem.

THEOREM 4.3. *Let f and g satisfy* (4.6), *respectively* (4.7), *then there exists a solution $u \in V$ of* (4.8).

Various results for elliptic partial differential equations of second order may now be obtained by choosing V and the sesquilinear quadratic forms accordingly (see, e.g., [**42**]).

In particular, the results of Castro [**6**], Landesman and Lazer [**16**] and Mawhin and Ward [**30**] may be put into the above framework.

5. Some results on boundary value problems for nonlinear systems. We consider the boundary value problem of Nicoletti type

$$(5.1) \qquad \begin{aligned} x' &= f(t, x), & a \leqq t \leqq b \\ x_i(t_i) &= r_i, & t_i \in [a, b],\; 1 \leqq i \leqq n \end{aligned}$$

where f: $[a, b] \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous and $r_i \in \mathbf{R}$ are given real numbers.

We assume that

$$(5.2) \qquad |f_i(t, x) - f_i(t, 0)| \leqq \sum_{j=1}^{n} P_{ij}(t)|x_j|, \; 1 \leqq i \leqq n.$$

where P_{ij}: $[a, b] \to [0, \infty)$ $1 \leqq i, j \leqq n$ are continuous functions. Furthermore we require that if $u(t)$ is an absolutely continuous function satisfying almost everywhere

$$(5.3) \qquad |u_i'(t)| \leqq \sum_{j=1}^{n} P_{ij}(t)|u_j(t)|$$

and

$$(5.4) \qquad u_i(t_i) = 0, \quad 1 \leqq i \leqq n,$$

then $u \equiv 0$.

THEOREM 5.1. *Let the above conditions hold. Then for any choice of* $r_i \in \mathbf{R}$ *the problem* (5.1) *has a solution.*

PROOF. (5.1) is equivalent to the operator equation

$$x = h(x) + g, \tag{5.5}$$

where $x \in C([a, b], \mathbf{R}^n)$ and h and g are given by

$$h(x)(t) = \begin{bmatrix} \int_{t_1}^{t} (f_1(s, x(x)) - f_1(s, 0))ds \\ \vdots \\ \int_{t_n}^{t} (f_n(s, x(s)) - f_n(s, 0))ds \end{bmatrix}$$

$$g(t) = \begin{bmatrix} r_1 + \int_{t_1}^{t} f_1(s, 0)ds \\ \vdots \\ r_n + \int_{t_n}^{t} f_n(s, 0)ds \end{bmatrix}$$

It is clear that h is a completely continuous operator and has a set valued derivative given by

$$H(x) = \{v \in C[a, b]\colon v_i(t) = \int_{t_i}^{t} u_i(s)ds, \text{ where } u_i(t) \text{ is measurable on } [a, b] \text{ and satisfies } |u_i(t)| \leqq \sum_{j=1}^{n} P_{ij}(t)|x_j(t)|\}.$$

If $x \in H(x)$, then it follows from the definition of H that

$$|x_i'(t)| \leqq \sum_{j=1}^{n} P_{ij}(t)|x_j(t)|, \quad 1 \leqq i \leqq n,$$

and $x_i(t_i) = 0$, hence $x \equiv 0$ by hypothesis. The other requirements are equally easy to verify.

REMARK 5.2. If one requires that any function u which satisfies (5.3) must vanish identically, then one may also conclude that the set $\{t_i\}$ may be specified arbitrarily. It also should be clear how one may obtain results similar to the one above for the problem

$$\begin{aligned} x' &= f(t, x), \\ Lx &= r, \end{aligned} \tag{5.6}$$

where L: $C([a, b], \mathbf{R}^n) \to \mathbf{R}^n$ is a continous linear mapping. Further if (5.2) is replaced by a Lipschitz condition

$$|f_i(t, x) - f_i(t, y)| \leqq \sum_{j=1}^{n} P_{ij}(t)|x_j - y_j|,$$

then under the above conditions one may again deduce uniqueness.

Let f: $(a, b) \times \mathbf{R}^n \to \mathbf{R}$ be continuous and satisfy

$$|f(t, y_1, \ldots, y_n) - f(t, 0, \ldots, 0)| \leqq \sum_{j=1}^{n} k_j(t)|y_j|. \tag{5.7}$$

The boundary value problem

$$\begin{aligned} x^{(n)} &= f(t, x, \ldots, x^{(n-1)}) \\ x^{(i-1)}(t_j) &= c_{ij}\ 1 \leqq i \leqq n_j,\ \sum_j n_j = n \end{aligned} \tag{5.8}$$

may then be viewed as a special case of a problem of the form (5.6) and one obtains the result that if $x \in C^{n-1}(a, b)$ has an absolutely continuous $(n - 1)$st derivative and satisfies

$$\begin{aligned} |x^{(n)}(t)| &\leqq \sum_{j=1}^{n} k_j(t)|x^{(j-1)}| \text{ a.e.,} \\ x^{(i-1)}(t_j) &= 0, 1 \leqq i \leqq n_j,\ \sum_j n_j = n \end{aligned} \tag{5.9}$$

implies that x must vanish identically, then (5.8) will have a solution.

REMARK 5.3. Results for boundary value problems of Nicoletti and more general types such as those discussed here may be found in Lasota and Opial [21, 22] Lasota [19], Krakowiak [15]. Two point and many point problems of the type of (5.8) have been discussed by Jackson [9, 10] where also some interesting applications of the Pontryagin maximum principle are given to conclude that (5.9) have a unique solution. The more general problem area, where uniqueness of solutions does imply existence has not been touched upon here. The interested reader is referred to papers by Lasota [20], Jackson [9], Jackson and Schrader [11], where such problems are discussed.

Many of the results on differential inequalities which are primarily motivated by the work of Bailey, Shampine and Waltman [2], may indeed by derived by the methods discussed here; we refer the interested reader to Schmitt [36] and Szafraniec [44], where this has been done.

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