ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 12, Number 2, Spring 1982

PERTURBATIONS OF A BOUNDARY VALUE PROBLEM WITH POSITIVE, INCREASING AND CONVEX NONLINEARITY

B. ZWAHLEN

1. Introduction. Let ρ_t be a family of positive functions:

$$\rho_t(x) = \rho_0(x) + t\pi(x),\ x \in [-1, +1],\ t \in [-1, +1].$$

For a fixed t we consider the boundary value problem (BVP):

$$(\mathrm{BVP}t)\begin{cases} -u''(x) = \lambda\rho_t(x)f(u(x)),\ x \in (-1, +1) \\ u(-1) \quad = u(+1) = 0, \end{cases}$$

where λ is a non-negative parameter and f a positive, increasing and convex function. Under these conditions there is a critical value $\lambda_t^* > 0$ such that (BVPt) has at least one solution for $\lambda \in (0, \lambda_t^*)$ and no solution for $\lambda > \lambda_t^*$.

Thinking of (BVP$t = 0$) as the unperturbed problem, it is the purpose of this paper to study λ_t^* as a function of the perturbation parameter t. Our result is a condition which implies the inequality $\lambda_t^* < \lambda_0^*$ for small positive (or negative) t. This condition involves only the perturbation π and the solutions of (BVP0) at λ_0^* and of its linearization. The method which leads to this result is to develop (BVPt) around the unperturbed problem. Thus we find a bifurcation equation in t, which has to be discussed.

Our paper is organized as follows: §2 hypotheses; §3 here we reproduce some known results which we use in the next section; §4 statement and proof of our perturbation lemma.

2. Hypotheses. Let $I = \{x \in \mathbf{R}/|x| < 1\}$, $\bar{I}$ its closure, $\mathbf{R}_+ = \{\xi \in \mathbf{R}/\xi \geqq 0\}$, $\lambda \in R_+$. We make the following hypotheses:

H1) $\rho_0; \bar{I} \to \mathbf{R}$ continuous and positive.
π: $\bar{I} \to \mathbf{R}$ continuous and $|\pi(x)| < \rho_0(x)$, $x \in \bar{I}$.
$\rho_t(x) = \rho_0(x), + t\pi(x), x \in \bar{I}, t \in \bar{I}$.

H2) f: $\mathbf{R}_+ \to \mathbf{R}$ continuously differentiable and

$$f(0) > 0, \lim_{\xi\to+\infty} \frac{f(\xi)}{\xi} = \infty, f'(0) \geqq 0, f' \text{ strictly increasing.}$$

Thus f is positive, strictly increasing and strictly convex. We write

Received by the editors on February 15, 1980, and in revised form on March 31, 1981.

$$f(\xi) = f(\xi_0) + f'(\xi_0)(\xi - \xi_0) + r(\xi, \xi_0),\ \xi, \xi_0 \in \mathbf{R}_+$$
$$\text{with } r(\xi, \xi_0) > 0 \text{ for } \xi \neq \xi_0.$$

SOLUTIONS. For a fixed t and a given λ denote a solution by $u_t(\cdot, \lambda) \in C(\bar{I}) \cap C^2(I)$. The minimal solution (see §3) will be called $\hat{u}_t(\cdot, \lambda)$; for $\lambda = \lambda_t^*$ we write $\hat{u}_t(\cdot, \lambda_t^*) = u_t^*$.

INTEGRAL EQUATIONS. Equivalently to (BVPt) we may work with the integral equation

$$\text{(IE}t) \qquad u(x) = \lambda \int_{-1}^{+1} G(x, y)\rho_t(y)f(u(y))dy$$

where the Green's function is continuous in $\bar{I} \times \bar{I}$ and positive in $I \times I$.

3. Some known results. Problems such as (BVPt) have been studied extensively. From the known results about the solutions we only retain the following: For a fixed $t \in \bar{I}$ we have

1) $0 < \lambda_t^* < \infty$,

2) for all $\lambda \in [0, \lambda_t^*]$ there is a unique minimal solution $\hat{u}_t(\cdot, \lambda)$. This solution is found by monotone iterations:

$$\begin{aligned} -(u_t^{(n)}(x, \lambda))'' &= \lambda\rho_t(x)f(u_t^{(n-1)}(x, \lambda)),\ x \in I \\ u_t^{(n)}(\pm 1, \lambda) &= 0,\ \ n = 1, 2, 3, \ldots \\ u_t^{(0)}(x, \lambda) &= 0,\ \ x \in \bar{I}. \end{aligned}$$

The sequence $(u_t^{(n)}(\cdot, \lambda))$ is monotonically increasing and converges to $\hat{u}_t(\cdot, \lambda)$ in $C(\bar{I})$.

3) For $\lambda = \lambda_t^*$, u_t^* is the unique solution. u_t^* is weakly stable but not stable, that means the linearized BVP at u_t^*

$$-w''(x) = \mu\rho_t(x)f'(u_t^*(x))w(x),\ x \in I,\ w(\pm 1) = 0,$$

has λ_t^* as its lowest eigenvalue. The corresponding eigenfunction w_t can be chosen so as to be positive: $w_t(x) > 0$, $x \in I$.

See for instance **[1]**, **[2]**, [3], and **[4]**.

4. Perturbations. In order to study the effect of a perturbation on the critical value we first establish an estimate for the norm of a solution, and then we transform the (BVPt) at $\lambda = \lambda_0^*$. Instead of (BVPt) we consider the equivalent integral equation

$$\text{(IE}t) \qquad u(x) = \lambda \int_{-1}^{+1} G(x, y)(\rho_0(y) + t\pi(y))f(u(y))dy.$$

By hypothesis H1) there exists a $\beta > 0$ such that

$$\rho_0(x) + t\pi(x) \geqq \beta, \text{ for all } x \in \bar{I}, \text{ for all } t \in \bar{I}.$$

For $\lambda > 0$ every solution u of (IEt) is positive, concave and $u(\pm 1) = 0$. Therefore

$$u(x) \geqq \frac{1}{2} \|u\|(1 - |x|),\ x \in \bar{I} \tag{1}$$

and

$$u(x) \geqq \frac{1}{4} \|u\|,\ x \in \left[-\frac{1}{2}, +\frac{1}{2}\right]. \tag{2}$$

Introducing (1) and (2) into (IEt) gives

$$u(x) \geqq \lambda\beta \int_{-1}^{+1} G(x, y) f\left(\frac{1}{2} \|u\|(1 - |y|\right) dy \geqq \lambda\beta f\left(\frac{1}{4} \|u\|\right) \int_{-1/2}^{+1/2} G(x, y) dy$$

and by taking the maximum norm on both sides

$$\frac{\frac{1}{4} \|u\|}{f\left(\frac{1}{4} \|u\|\right)} \geqq \frac{\lambda\beta}{4} \left\| \int_{-1/2}^{+1/2} G(x, u) dy \right\|. \tag{3}$$

There is a $\gamma > 0$ such that $\gamma \leqq \lambda_t^*$, $t \in \bar{I}$.

Thus the inequality (3) together with the hypothesis $\lim_{\xi \to \infty}(f(\xi)/\xi) = \infty$ tell us that all solutions (λ, u) of (IEt) with $t \in \bar{I}$ and $\lambda \geqq \gamma$ are bounded by a constant $C > 0$:

$$\|u\| \leqq C. \tag{4}$$

We consider now (BVPt) at $\lambda = \lambda_0^*$.

$$\text{(BVP}t\text{)} \quad \begin{aligned} -u''(x) &= \lambda_0^*(\rho_0(x) + t\pi(x)) f(u(x)),\ -1 < x < +1 \\ u(\pm 1) &= 0 \end{aligned}$$

with $u = u_0^* + v$ and $f(u) = f(u_0^* + v) = f(u_0^*) + f'(u_0^*)v + r(u_0^*, v)$ (BVPt) becomes

$$\begin{aligned} &-v''(x) - \lambda_0^* \rho_0(x) f'(u_0^*(x)) v(x) \\ &= \lambda_0^*(\rho_0(x) + t\pi(x)) r(u_0^*(x), v(x)) + t\lambda_0^* \pi(x)[f(u_0^*(x)) + f'(u_0^*) v(x)] \\ &\qquad v(\pm 1) = 0. \end{aligned} \tag{5}$$

By 3.3. we have a) for $t = 0$ the only solution of (5) is $v(x) \equiv 0$, and b) $Lv = -v''(x) - \lambda_0^* \rho_0(x) f'(u_0^*(x)) v(x)$ is a selfadjoint operator $L: D \subset L^2(I) \to L^2(I)$ with $N(L) = \{v \in D / Lv = 0\}$ the one dimensional linear subspace spanned by w_0.

By multiplying (5) with w_0 and integrating we get

$$0 = \int_{-1}^{+1} (\rho_0(x) + t\pi(x)) r(u_0^*(x), v(x)) w_0(x) dx \tag{6}$$
$$+ t \int_{-1}^{+1} \pi(x)(f(u_0^*(x)) + f'(u_0^*(x))v(x)) w_0(x) dx.$$

Let

$$\Pi = \int_{-1}^{+1} \pi(x) f(v_0^*(x)) w_0(x) dx.$$

We are now able to state the following result.

PERTURBATION LEMMA. *Suppose $\Pi > 0$. Then there exists a $\delta > 0$ such that for all t with $0 < t < \delta$ we have $\lambda_t^* < \lambda_0^*$. Similarly for $\Pi < 0$ there exists a $\delta > 0$ such that $\lambda_t^* < \lambda_0^*$ for $-\delta < t < 0$.*

PROOF. A) Suppose to the contrary that there is a sequence $(t_k)_{k \geqq 1}$, $1 \geqq t_k > t_{k+1} > 0$, $\lim_{k \to \infty} t_k = 0$ with $\lambda_{t_k}^* \geqq \lambda_0^*$. (BVP$t_k$) has at least one solution at $\lambda = \lambda_0^*$ and to simplify the notation call the minimal solution $\hat{u}_{t_k}(\cdot, \lambda_0^*) = u_k = u_0^* + v_k$. Equation (6) now reads

$$0 = \int_{-1}^{+1} (\rho_0(x) + t_k\pi(x)) r(u_0^*(x), v_k(x)) w_0(x) dx \tag{7}$$
$$+ t_k \Pi + t_k \int_{-1}^{+1} \pi(x) f'(u_0^*(x)) v_k(x) w_0(x) dx.$$

As $t_k \Pi \neq 0$ we have $v_k(x) \not\equiv 0$, for all k. Therefore

$$\int_{-1}^{+1} (\rho_0(x) + t_k\pi(x)) r(u_0^*(x), v_k(x)) w_0(x) dx > 0,$$

for all k. With this we get from (7)

$$\Pi + \int_{-1}^{+1} \pi(x) f'(u_0^*(x)) v_k(x) w_0(x) dx < 0, \text{ for all } k. \tag{8}$$

From (8) we get immediately

$$\Pi \leqq C \|v_k\|, \text{ for all } k. \tag{9}$$

C is independent of k. The inequalities (4) and (9) together tell us $\hat{C} \leqq \|v_k\| \leqq \bar{C}$, $\hat{C} > 0$, $\bar{C} > 0$.

B) Using $\rho_0(y) + t_k\pi(y) \leqq 2\rho_0(y)$, the continuity of $G(x, y)$ and inequality (4) it follows from (IEt_k) that

$$u_k(x) \leqq 2f(\tilde{C})\, \lambda_0^* \int_{-1}^{+1} G(x, y)\rho_0(y)dy, \; \tilde{C} > 0,$$

hence the sequences $(u_k)_{k\geqq 1}$ and $(v_k)_{k\geqq 1}$ are bounded and equicontinuous in $C(\bar{I})$.

Let $(v_{k\ell})_{\ell\geqq 1}$ be a convergent subsequence, $\lim_{\ell\to\infty} v_{k\ell} = \bar{v}(x)$. Evidently $\|\bar{v}\| \geqq \hat{C}$. Taking the limit in (7) as $\ell \to \infty$ we get

$$0 = \int_{-1}^{+1} \rho_0(x) r(u_0^*(x), \bar{v}(x)) w_0(x) dx$$

or

$$\int_{-1}^{+1} \rho_0(x) r(u_0^*(x), \bar{v}(x)) w_0(x) dx = 0$$

is in contradiction with the facts that $\rho_0(x) > 0$, $x \in I$, $w_0(x) > 0$, $x \in I$ and $r(u_0^*(x), \bar{v}(x)) > 0$ for $\|\bar{v}(x)\| \geqq \hat{C}$. Thus the lemma is proved by contradiction.

REMARK. Now let f be an asymptotically linear function, which satisfies all the other hypotheses, that is f: $\mathbf{R}_+ \to \mathbf{R}$ continuously differentiably $f(0) > 0, f'(0) \geqq 0, f'$ strictly increasing and $\lim_{\xi\to\infty}(f(\xi)/\xi) = \ell$, $0 < \ell < \infty$. Let μ^∞ be the principal characteristic value of the linearized equation "at infinity" (for $t = 0$):

$$w(x) = \mu\ell \int_{-1}^{+1} G(x, y)\rho_0(y) w(y) dy.$$

If $\mu^\infty < \lambda_0^*$ then the perturbation lemma remains true. In fact, writing $f(\xi) = \ell\xi + g(\xi)$, $\lim_{\xi\to\infty}(g(\xi)/\xi) = 0$, it can easily be shown, that there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that all solutions of (IEt) with $|t| < \delta_1$ and $\lambda \geqq \mu^\infty + \delta_2$ stay bounded (inequality (4)). Therefore the proof of the perturbation lemma remains unchanged.

EXAMPLES. $\rho_0(x) = \rho_0 = \text{cst} > 0$. $u^*(x)$ and $w_0(x)$ are concave and symmetric. Therefore Π is positive in the examples: $0 < \mathscr{H} < \rho_0$.
a) $\pi(x) = \mathscr{H} \cos(\pi x)$, b) $\pi(x) = \mathscr{H}(1 - 2|x|)$, c) $\pi(x) = \mathscr{H}(1 - 2x^2)$.

REFERENCES

1. H. Amann, *Supersolutions, monotone iterations, and stability*, J. Differential Equations **21** (1976,), 363–377.

2. ———, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review, **18** (1976), 620–709.

3. ———, and T. Laetsch, *Positive solution of convex nonlinear eigenvalue problems*, Indiana Univ. Math. J. **25** (1976), 259–270.

4. J.P. Keener and H.B. Keller, *Positive solutions of convex nonlinear eigenvalue problems*, J. Differential Equations **16** (1974), 103–125.

DÉPARTMENT DE MATHÉMATIQUES, ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE, LAUSANNE, SWITZERLAND.