ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 12, Number 2, Spring 1982

ENDOMORPHISM RINGS AND SUBGROUPS OF FINITE RANK TORSION-FREE ABELIAN GROUPS

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Let A be a finite rank torsion-free abelian group and let $E(A)$ denote the endomorphism ring of A. Then $Q \otimes_Z E(A) = QE(A)$ and $E(A)/pE(A)$ are artinian algebras, where Z is the ring of integers, Q is the field of rationals, and p is a prime of Z.

Define A to be *Q-simple* if $QE(A)$ is a simple algebra, and *p-simple* for a prime p of Z if $pE(A) = E(A)$ or if $E(A)/pE(A)$ is a simple algebra. In contrast to finite rank torsion-free groups in general, groups that are p-simple for each p have some pleasant decomposition properties.

THEOREM I. *A reduced group A is p-simple for each prime p of Z if and only if $A = A_1 \oplus \cdots \oplus A_k$, where each A_i is fully invariant in A, each A_i is Q-simple and p-simple for each prime p of Z, and if p is a prime of Z then there is some j with $A/pA = A_j/pA_j$.*

THEOREM II. *A group A is Q-simple and p-simple for each prime p of Z if and only if $A = B_1 \oplus \cdots \oplus B_n$, where each B_i is strongly indecomposable, Q-simple and p-simple for each prime p of Z and B_i is nearly isomorphic to B_j (in the sense of Lady* [7]) *for each i and j.*

Suppose that A is Q-simple and p-simple for each prime p of Z. Then A is indecomposable if and only if A is strongly indecomposable. Furthermore, if S = Center $E(A)$, then S is a subring of an algebraic number field such that every element of S is a rational integral multiple of a unit of S, as described in **[1]**, and $E(A)$ is a maximal S-order in $QE(A)$.

Examples of groups that are Q-simple and p-simple for each prime p of Z include: indecomposable strongly homogeneous groups (characterized in **[1]**); indecomposable groups with p-rank $\leqq 1$ for each prime p of Z (Murley **[8]**); and indecomposable quasi-pure-projective and quasi-pure-injective groups (**[4]**).

Define A to be *irreducible* if QA is an irreducible $QE(A)$-module (Reid **[10]**) and *p-irreducible*, for a prime p of Z, if A/pA is an irreducible $E(A)/pE(A)$-module. If A is irreducible (p-irreducible), then A is Q-simple

*Research supported, in part, by N.S.F. Grant MCS 77–03458–A01.

Received by the editors on October 15, 1979 and in revised form on December 22, 1980.

(p-simple). Furthermore, each strongly homogeneous group is irreducible and p-irreducible for each prime of p of Z.

A group A is *finitely faithful* if $IA \neq A$ for each maximal right ideal I of finite index in $E(A)$. Define A, to be an *$\mathcal{S}$-group* if whenever B is a subgroup of finite index in A then $B = IA$ for some right ideal I of $E(A)$. The following theorem gives a class of groups irreducible and p-irreducible for each prime p of Z.

THEOREM III. *The following are equivalent*:

(a) *A is a finitely faithful $\mathcal{S}$-group*;

(b) *For each prime p of Z with $pA \neq A$, $E(A)/pE(A) \cong \mathrm{Mat}_m(Z/pZ)$, the ring of $m \times m$ matrices over Z/pZ, where $m = p\text{-rank}\, A$*;

(c) *$\mathrm{Ext}_Z(A, A)$ is torsion free*; *and*

(d) *A is finitely faithful and if B is a subgroup of finite index in A, then B is nearly isomorphic to A.*

B. Jónsson proved a uniqueness theorem for direct sum decompositions of finite rank torsion-free abelian groups up to quasi-isomorphism, where A and B are *quasi-isomorphic* if B is isomorphic to a subgroup of finite index in A (Fuchs [**6**]). Define A to be a *$\mathcal{J}$-group* if A is isomorphic to each subgroup of finite index in A (Warfield [**13**]). Each $\mathcal{J}$-group is an $\mathcal{S}$-group. Moreover, a reduced group A is a finitely faithful $\mathcal{J}$-group if and only if A is a $\mathcal{J}$-group and $QE(A)$ is a semi-simple algebra. If $A \simeq B^k$, where B is indecomposable with p-rank $B \leqq 1$ for each prime p of Z, then A is a $\mathcal{J}$-group, necessarily finitely faithful.

An example of an indecomposable finitely faithful $\mathcal{J}$-group with p-rank > 1 is constructed in §7. This is a counterexample to a conjecture of C.E. Murley: if A is an indecomposable $\mathcal{J}$-group, then p-rank $A \leqq 1$ for each prime p of Z.

In summary, the following implications are valid for Q-simple groups:

(a) Finite direct sum of copies of a indecomposable group with p-rank $\leqq 1$ for each $p \Rightarrow$ finitely faithful $\mathcal{J}$-group $\Rightarrow$ finitely faithful $\mathcal{S}$-group $\Rightarrow$ p-irreducible for each $p \Rightarrow$ p-simple for each p.

(b) Finite direct sum of copies of an indecomposable irreducible group with p-rank $\leqq 1$ for each $p \Leftrightarrow$ finitely faithful strongly homogeneous $\mathcal{J}$-group $\Leftrightarrow$ finitely faithful strongly homogeneous $\mathcal{S}$-group $\Rightarrow$ strongly homogeneous $\Rightarrow$ irreducible and p-irreducible for each $p \Rightarrow$ irreducible and p-simple for each p.

The abelian group terminology is as given in Fuchs [**6**]. The classical Wedderburn-Artin theory of semi-simple artinian algebras is assumed.

1. Q-simple and p-simple groups.

PROPOSITION 1.1. *The following are equivalent*:

(a) *A is Q-simple*;

(b) *If B is a fully invariant subgroup of A with* $\mathrm{Hom}(A, B) \neq 0$, *then* A/B *is finite; and*

(c) *If I is a non-zero ideal of $E(A)$, then $E(A)/I$ is finite.*

PROOF. (a) $\Rightarrow$ (b). Let $I = \mathrm{Hom}(A, B)$, a non-zero ideal of $E(A)$. Then $QI = QE(A)$, since $QE(A)$ is simple. Choose $0 \neq n \in Z$ with $n \cdot 1_A \in I$ so that $nE(A) \subseteqq I$. Then $nA \subseteqq IA \subseteqq B \subseteqq A$ and A/B is finite since A/nA is finite.

(b) $\Rightarrow$ (c). Let $B = IA$, a fully invariant subgroup of A. Then $0 \neq I \subseteqq \mathrm{Hom}(A, B)$ and A/B is finite.

Let $\mathscr{J}(QE(A))$ be the Jacobson radical of $QE(A)$ and $\mathscr{N}E(A) = \mathscr{J}(QE(A)) \cap E(A)$, the nil radical of $E(A)$. If $\mathscr{N}E(A) \neq 0$, then $A/\mathscr{N}E(A)A$ must be finite so that $Q\mathscr{N}E(A)A = \mathscr{J}(QE(A))QA = QA$, which is impossible by Nakayama's Lemma. Therefore, $QE(A)$ is semi-simple and artinian.

Now $QI = fQE(A)$ for some $f \in I$. Hence $QA = QB = fQA$ and f is an automorphism of QA. Thus, $fQE(A) = QI = QE(A)$ which implies that $E(A)/I$ is finite.

(c) $\Rightarrow$ (a). If $0 \neq I$ is an ideal of $QE(A)$, then $E(A)/(I \cap E(A))$ is finite and $I = Q(I \cap E(A)) = QE(A)$.

PROPOSITION 1.2. *A is p-simple for each prime p if and only if whenever I is an ideal of $E(A)$ with $E(A)/I$ finite, then $I = nE(A)$ for some $n \in Z$.*

PROOF. ($\Leftarrow$). If $I/pE(A)$ is an ideal of $E(A)/pE(A)$, then $I = pE(A)$ or $I = E(A)$.

($\Rightarrow$). Let $0 < n$ be the least integer with $nE(A) \subseteqq I$ and let p be a prime divisor of n. Then $I + pE(A) = E(A)$ or $I + pE(A) = pE(A)$. In the latter case, $(n/p)E(A) \subseteqq (1/p)I \subseteqq E(A)$ so $(1/p)I = (n/p)E(A)$ by induction on n. In the former case, $(n/p)E(A) \subseteqq I$, contradicting the minimality of n.

THEOREM 1.3. *Assume that A is reduced. Then A is p-simple for each prime p if and only if $A = A_1 \oplus \cdots \oplus A_k$ where* (i) *each A_i is fully invariant in A*; (ii) *each A_i is Q-simple and p-simple for each prime p*; *and* (iii) *if p is a prime then there is some j with $A/pA = A_j/pA_j$.*

PROOF. ($\Leftarrow$). $E(A) = E(A_1) \times \cdots \times E(A_k)$ and if p is a prime, then $E(A)/pE(A) = E(A_j)/pE(A_j)$ for some j since $A/pA = A_j/pA_j$ implies that $pA_i = A_i$ and $pE(A_i) = E(A_i)$ for each $i \neq j$.

($\Rightarrow$). For each p, $(\mathscr{N}E(A) + pE(A))/pE(A) \subseteqq \mathscr{J}(E(A)/pE(A)) = 0$. Thus $\mathscr{N}E(A) \subseteqq pE(A) \cap \mathscr{N}E(A) = p\mathscr{N}E(A)$ for each p. Since A is reduced, $E(A)$ is reduced so that $\mathscr{N}E(A) = 0$ and $\mathscr{J}(QE(A)) = Q\mathscr{N}E(A) = 0$. Therefore, $QE(A)$ is a semi-simple artinian algebra.

Write $QE(A) = K_1 \times \cdots \times K_k$ as a product of simple algebras. Then $n\mathscr{R} \subseteqq E(A) \subseteqq \mathscr{R} = \mathscr{R}_1 \times \cdots \times \mathscr{R}_k$, where each $\mathscr{R}_i$ is a subring of K_i with $Q\mathscr{R}_i = K_i$ for some $0 \neq n \in Z$ (let $\mathscr{R}_i$ be the projection of $\mathscr{R}$ into K_i

and choose n with $n \cdot 1_{\mathscr{R}_i} \in E(A)$ for each i). Now $I = n\mathscr{R}$ is an ideal of $E(A)$ with $E(A)/I$ finite so $n\mathscr{R} = I = mE(A)$ for some $m \in Z$ (Proposition 1.2) whence $mE(A) = m\mathscr{R}E(A) = m\mathscr{R}$ and $E(A) = \mathscr{R}$.

Let $A_i = \mathscr{R}_i A$ so that $A = A_1 \oplus \cdots \oplus A_k$, $QE(A_i) = K_i$, and $E(A) = E(A_1) \times \cdots \times E(A_k)$. Clearly each A_i is Q-simple. If p is a prime, then $E(A)/pE(A) = E(A_j)/pE(A_j)$ for some j (since A is p-simple) so that $pA_i = A_i$ if $i \neq j$ and $A/pA = A_j/pA_j$. Since $E(A)/pE(A) = E(A_1)/pE(A_1) \times \cdots \times E(A_k)/pE(A_k)$, each A_i must be p-simple

COROLLARY 1.4. *Assume that A is Q-simple and p-simple for each prime p of Z and let $S =$* Center $E(A)$.

(a) *S is a principal ideal domain such that every element of S is a rational integral multiple of a unit of S.*

(b) *$E(A)$ is a maximal S-order in $QE(A)$.*

(c) *For some $0 < k \in Z$, $E(A) \cong S^k$ as S-modules.*

PROOF. (a) S is a domain since $QS =$ Center $(QE(A))$ is an algebraic number field. Let $0 \neq s \in S$. Then $sE(A)$ is an ideal of $E(A)$ with $E(A)/sE(A)$ finite (for some $s' \in S$, $0 \neq s's \in Z$). Write $sE(A) = nE(A)$ for some $n \in Z$ (Proposition 1.2). Now $s = nu$, $n = sv$ for some u, $v \in E(A) \cap$ Center $QE(A) = S$. Hence $s = nu = svu$ so that u is a unit of S. Clearly, S is a principal ideal domain.

(b) $E(A)$ is an *S-order* in $QE(A)$ ($E(A)$ is finitely generated as an S-module, Pierce [9]). If $E(A) \subseteq \mathscr{R} \subseteq QE(A)$, where $\mathscr{R}$ is an S-order, then $\mathscr{R}/E(A)$ is finite say $n\mathscr{R} \subseteq E(A) \subseteq \mathscr{R}$ for some $0 \neq n \in Z$. Thus $I = n\mathscr{R}$ is an ideal of $E(A)$ so $n\mathscr{R} = I = mE(A)$ for some $m \in Z$. But $mE(A) = \mathscr{R}(mE(A)) = m\mathscr{R}$ so $E(A) = \mathscr{R}$. and $E(A)$ is a maximal S-order in $QE(A)$.

(c) is a consequence of the fact that S is a principal ideal domain and $E(A)$ is a finitely generated torsion free S-module.

Two finite rank torsion free groups A and B are *quasi-isomorphic* if there is a monomorphism f: $A \to B$ with $B/f(A)$ finite and *nearly isomorphic* if for each $0 \neq n \in Z$ there is a monomorphism f_n: $A \to B$ such that $B/f_n(A)$ is finite with cardinality relatively prime to n (Lady [7]).

COROLLARY 1.5. (a) *If A and B are quasi-isomorphic and if A is Q-simple, then B is Q-simple.*

(b) *If A and B are nearly isomorphic and if A is p-simple, then B is p-simple.*

PROOF. Suppose that $nA \subseteq B \subseteq A$ for some $0 \neq n \in Z$. Then $QE(A)$ and $QE(B)$ are ring isomorphie. Furthermore, if p is a prime not dividing n then $E(A)/pE(A)$ and $E(B)/pE(B)$ are ring isomorphic.

The group A is *strongly indecomposable* if $0 \neq n \in Z$ and $nA \subseteq B \oplus$

$C \subseteqq A$ imply $B = 0$ or $C = 0$. Reid [10] proves that A is strongly indecomposable if and only if $QE(A)/\mathscr{J}(QE(A))$ is a division algebra.

THEOREM 1.6. *A is Q-simple and p-simple for each prime p of Z if and only if A is nearly isomorphic to B^k (the direct sum of k copies of B) where B is strongly indecomposable, Q-simple, and p-simple for each prime p.*

PROOF. ($\Leftarrow$). In view of Corollary 1.5, it is sufficient to assume that $A = B^k$. Thus $E(A) \cong \mathrm{Mat}_k(E(B))$, where $QE(B)$ is a division algebra and $E(B)/pE(B) \cong \mathrm{Mat}_{m_p}(F_p)$, F_p a finite field for each p. Hence, $QE(A) \cong \mathrm{Mat}_k(QE(B))$ and $E(A)/pE(A) \cong \mathrm{Mat}_{km_p}(F_p)$ so that A is Q-simple and p-simple for each p.

($\Rightarrow$). Since $QE(A)$ is a simple algebra, $QE(A) \cong \mathrm{Mat}_k(D)$ for some division algebra D. Write $QE(A) = I_1 \oplus \cdots \oplus I_k$ where each I_i is an irreducible right ideal of $QE(A)$, $I_i = e_iQE(A)$ for some $e_i^2 = e_i \in QE(A)$, and I_i is $QE(A)$-isomorphic to I_j for each i and j. Then A is quasi-isomorphic to $e_i(A) \oplus \cdots \oplus e_k(A)$, $e_i(A)$ and $e_j(A)$ are quasi-isomorphic for each i, j; and $QE(e_i(A)) \cong D$ for each i (Reid [10]).

Choose $0 \neq m \in Z$ with $mA \subseteqq C \subseteqq A$ where $C = B_1 \oplus \cdots \oplus B_k$ and $B_i \cong e_i(A)$ is strongly indecomposable. Let X be the pure subgroup of A generated by $\{B_i | i \neq 1\}$. Then $B = A/X$ is quasi-isomorphic to B_1, with $\mathrm{Hom}(A, B)A = B$.

It now suffices to assume that $mA \subseteqq C \subseteqq A$ for some $0 \neq m \in Z$ where $C = B_1 \oplus \cdots \oplus B_k$, each $B_i \cong B$ is strongly indecomposable and Q-simple, and $\mathrm{Hom}(A, C)A = C$ (replace each B_i by a subgroup of finite index isomorphic to B).

As a consequence of Corollary 1.4, $E(A)$ is a maximal S-order in $QE(A)$ and S is a principal ideal domain. Moreover, if p is a prime of Z with $pA \neq A$ and $E(A)_p = Z_p \otimes_Z E(A)$ (where Z_p is the localization of Z at p), then $E(A)_p$ is a maximal order over the discrete valuation ring S_p. Thus $\mathrm{Hom}(A, C)_p \cong E(A)_p$, since $\mathrm{Hom}(A, C)_p$ is a right ideal of $E(A)_p$ (Swan and Evans [12]). But $\mathrm{Hom}(A, C)$ and $E(A)$ are finitely generated S-modules so there are $E(A)$-maps θ: $\mathrm{Hom}(A, C) \to E(A)$ and ϕ: $E(A) \to \mathrm{Hom}(A, C)$ with $\phi\theta = m \in Z$ and m relatively prime to p. Since $\mathrm{Hom}(A, C)A = C$, θ and ϕ induce homomorphisms f: $C \to A$ and g: $A \to C$ with $gf = m$. It now follows that A is nearly isomorphic to $C \cong B^k$, where B is strongly indecomposable and Q-simple.

Finally, B^k is p-simple for each p (Corollary 1.5) from which it follows that B is p-simple for each p.

COROLLARY 1.7. *A is Q-simple and p-simple for each p if and only if $A \cong B^{k-1} \oplus B_0$ where B and B_0 are strongly indecomposable, Q-simple, p-simple for each p, and B is nearly isomorphic to B_0. Consequently, A is indecomposable if and only if A is strongly indecomposable.*

PROOF. A is nearly isomorphic to B^k if and only if $A \cong B^{k-1} \oplus B_0$ where B_0 is nearly isomorphic to B (Arnold [2]). Now apply Theorem 1.6.

2. Irreducible and p-irreducible groups. A is *irreducible* if B being a pure fully invariant subgroup of A implies $B = 0$ or $B = A$.

THEOREM 2.1. (REID [10]). *The following are equivalent*:

(a) *A is irreducible*;

(b) *QA is an irreducible left $QE(A)$-module*;

(c) *$QE(A) \cong \mathrm{Mat}_m(D)$, where D is a division algebra with rank $A = m \cdot \dim_Q D$; and*

(d) *A is quasi-isomorphic to B^m where B is a strongly indecomposable irreducible group.*

COROLLARY 2.2. (REID [10]). *Assume that A is strongly indecomposable. Then A is irreducible if and only if $QE(A)$ is a division algebra and* rank $E(A) =$ rank A.

Let p be a prime of Z. Then A is *p-irreducible* if B being a fully invariant subgroup of A with $pA \subseteqq B$ implies $B = pA$ or $B = A$. Define *p-rank* A to be the Z/pZ-dimension of A/pA.

THEOREM 2.3. *The following are equivalent*:

(a) *A is p-irreducible*;

(b) *A/pA is an irreducible left $E(A)/pE(A)$-module*; *and*

(c) *$E(A)/pE(A) \cong \mathrm{Mat}_m(F_p)$, F_p a finite field with* p-rank $A = m \cdot \dim F_p$.

PROOF. (a) $\Leftrightarrow$ (b) is routine.

(b) $\Rightarrow$ (c). If $f + pE(A) \in E(A)/pE(A)$ and $(f + pE(A)(A/pA) = 0$ for some $f \in E(A)$ then $f \in pE(A)$. Thus $E(A)/pE(A)$ is semi-simple since, if $I(pE(A) = \mathscr{J}(E(A)/pE(A))$, then $pA = IA$, in which case $I = pE(A)$; or else $IA = A$ which is impossible by Nakayama's Lemma. Therefore, $E(A)/pE(A)$ is a product of simple rings. In fact, A/pA irreducible implies that $E(A)/pE(A)$ is simple.

Write $E(A)/pE(A) \cong (A/pA)^m$, A/pA isomorphic to an irreducible left left ideal of $E(A)/pE(A)$. Then $E(A)/pE(A) \cong \mathrm{Mat}_m(F_p)$ where $F_p = \mathrm{End}_{E(A)/pE(A)}(A/pA)$. Now p-rank $E(A) = m(p\text{-rank } A) = m^2 \dim F_p$ so p-rank $A = m\dim F_p$.

(c) $\Rightarrow$ (b). Write $E(A)/pE(A) \cong I^m$, I an irreducible left ideal of $E(A)/pE(A)$. Since $E(A)/pE(A)$ is simple, $A/pA \cong I^k$ for some k. But dim $I = m\dim F_p = p$-rank A so $k \cong 1$ and A/pA is $E(A)/pE(A)$-irreducible.

COROLLARY 2.4. (a) *If A is irreducible (p-irreducible), then A is Q-simple (p-simple).*

(b) *If A is quasi-isomorphic to B and A is irreducible, then B is irreducible.*

(c) *If A is nearly isomorphic to B and if A is p-irreducible, then B is p-irreducible.*

PROOF. (a) Follows from Theorems 2.1 and 2.3.

(b) If A is quasi-isomorphic to B, then $QA \cong QB$ and $QE(A) \cong QE(B)$.

(c) If $nA \subseteqq B \cong A$ and if p is a prime not dividing n, then $B/pB \cong A/pA$ and $E(A)/pE(A) = E(B)/pE(B)$.

COROLLARY 2.5. *Assume that A is reduced. Then A is p-irreducible for each p if and only if $A = A_1 \oplus \cdots \oplus A_k$ where* (i) $\mathrm{Hom}(A_i, A_j) = 0$ *if $i \neq J$*; (ii) *each A_i is Q-simple and p-irreducible for each p*; (iii) *if p is a prime then there is some j with $A/pA = A_j/pA_j$.*

PROOF. Apply Theorem 1.3 and Theorem 2.3.

COROLLARY 2.6. *A is Q-simple (irreducible) and p-irreducible for each p if and only if $A \cong B^{k-1} \oplus B_0$ where B and B_0 are strongly indecomposable, Q-simple (irreducible), and p-irreducible for each p.*

PROOF. Apply Corollary 1.7 and the preceding results.

PROPOSITION 2.7. *Assume that A is Q-simple and p-irreducible for each prime p. Let B be a pure fully invariant subgroup of A and assume that $C = A/B \neq 0$ and $B \neq 0$.*

(a) *B and C are p-irreducible for each p and there are ring monomorphisms $E(A) \to E(B)$ and $E(A) \to E(C)$.*

(b) *If p is a prime, then either $pB = B$, $A/pA \cong C/pC$, and there is a ring monomorphism $E(A)/pE(A) \to E(C)/pE(C)$; or else $pC = C$, $A/pA \cong B/pB$ and there is a ring monomorphism $E(A)/pE(A) \to E(B)/pE(B)$.*

PROOF. Let p be a prime with $pB \neq B$. Since A is p-irreducible $B + pA = A$ (the case $B + pA = pA$ is impossible). Therefore, the natural map $B/pB \to A/pA$ is an isomorphism of $E(A)/pE(A)$-modules. Hence $pC = C$ and B is p-irreducible since any $E(B)/pE(B)$-submodule of B/pB is an $E(A)/pE(A)$-submodule of $B/pB \cong A/pA$.

The natural maps $E(A)/pE(A) \to E(B)/pE(B)$ and $E(A) \to E(B)$ are non-zero, hence monic, since $E(A)/pE(A)$ and $QE(A)$ are simple algebras.

Similarly, if $pC \neq C$ then C is p-irreducible, $A/pA \cong C/pC$, $E(A) \to E(C)$ is monic and $E(A)/pE(A) \to E(C)/pE(C)$ is monic.

COROLLARY 2.8. *Suppose that A is Q-simple and p-irreducible for each prime p. Then there are subgroups $B_1, \ldots, B_k$ of A such that*

(i) *$A/(B_1 \oplus \cdots \oplus B_k)$ is torsion divisible,*

(ii) *For each i, B_i is a minimal non-zero pure fully invariant subgroup of A, B_i is irreducible and p-irreducible for each i,*

(iii) *For each i, A/B_i is p-irreducible for each p,*

(iv) *For each p and each i, either* $pB_i = B_i$ *or else* $p(A/B_i) = A/B_i$,

(v) *For each i and j,* QB_i *and* QB_j *are isomorphic as* $QE(A)$-modules, *hence* rank B_i = rank B_j, *and*

(vi) *If A is strongly indecomposable, then* rank $A = k$ rank $E(A)$.

PROOF. Since $QE(A)$ is a simple algebra, $QA = M_1 \oplus \cdots \oplus M_k$ as left $QE(A)$-modules where each $M_i \cong I$, an irreducible left ideal of $QE(A)$. Let $B_i = M_i \cap A$, a minimal non-zero pure fully invariant subgroup of A. Then B_i is irreducible since any $E(B_i)$-module of B_i is an $E(A)$-submodule of B_i. Moreover, $QB_i = M_i \cong M_j = QB_j$ as $QE(A)$-modules. In view of Proposition 2.7, each B_i and each A/B_i is p-irreducible for each p and either $pB_i = B_i$ or else $p(A/B_i) = A/B_i$.

Since $QA = QB_1 \oplus \cdots \oplus QB_k = M_1 \oplus \cdots \oplus M_k$, $A/(B_1 \oplus \cdots \oplus B_k)$ is torsion. If p is a prime with $pA \neq A$, then $pB_i \neq B_i$ for some i. Thus $p(A/B_i) = A/B_i$ so that $A/(B_1 \oplus \cdots \oplus B_k)$ is p-divisible. If A is strongly indecomposable, then $QE(A)$ is a division algebra so rank A = rank $B_1 + \cdots +$ rank $B_k = k$ rank $E(A)$ (since $M_i = QB_i \cong QE(A)$).

3. Strongly homogeneous groups. The group A is *strongly homogeneous* if whenever X and Y are two pure rank 1 subgroups of A then there is an automorphism f of A with $f(X) = Y$.

THEOREM 3.1. (ARNOLD [1]): *The group A is strongly homogeneous if and only if A is isomorphic to the direct sum of finitely many copice of* $\mathscr{R} \otimes_Z X$ *where* $\mathscr{R}$ *is a subring of an algebraic number field such that every element of* $\mathscr{R}$ *is a rational integral multiple of a unit of* $\mathscr{R}$ *and X is a rank* 1 *group. Moreover,* $\mathscr{R}$ *may be chosen so that* $E(A) \cong \mathrm{Mat}_m(\mathscr{R})$ *and* $\mathscr{R} \otimes_Z X$ *is strongly indecomposable.*

COROLLARY 3.2. *If A is strongly homogeneous, then A is irreducible and p-irreducible for each p.*

PROOF. As a consequence of Theorem 3.1, $QE(A) \cong \mathrm{Mat}_m(Q\mathscr{R})$ where $Q\mathscr{R}$ is a field and rank $A = m$ rank $\mathscr{R} = m \dim_Q Q\mathscr{R}$. Furthermore, $E(A)/pE(A) \cong \mathrm{Mat}_m(\mathscr{R}/p\mathscr{R})$ where $\mathscr{R}/p\mathscr{R}$ is a field, if $\mathscr{R} \neq p\mathscr{R}$, and p-rank $A = m$ p-rank $\mathscr{R}$.

COROLLARY 3.3. *Suppose that A is a finitely generated E(A)-module. Then the following are equivalent*:

(a) *A is strongly homogeneous*;

(b) *A is irreducible and p-irreducible for each p*; *and*

(c) *A is Q-simple and p-simple for each p.*

PROOF. (a) ⇒ (b) Corollary 3.2.

(b) ⇒ (c) Corollary 2.4.

(c) ⇒ (a). Let S = Center $E(A)$. Then $E(A)$ is a free S-module and S

is a subring of an algebraic number field such that every element of S is an integral multiple of a unit of S (Corollary 1.4). Since A is a finitely generated $E(A)$-module, A is a finitely generated torsion free S-module. Therefore, $A \cong S^m$ so that Theorem 3.1 applies.

The ring $E(A)$ is *sub-commutative* if whenever $f, g \in E(A)$, then there is $h \in E(A)$ with $fg = hf$. Examples of sub-commutative rings are given by Reid [**11**].

LEMMA 3.4. *Suppose that $E(A)$ is sub-commutative. Then*
(a) *$QE(A)/\mathscr{J}(QE(A))$ is a product of division algebras, and*
(b) *$(E(A)/pE(A))/\mathscr{J}(E(A)/pE(A))$ is a product of fields.*

PROOF. $\mathscr{R} = QE(A)/\mathscr{J}(QE(A))$ is a semi-simple artinian sub-commutative ring hence a product of division algebras (Reid [**11**]). The proof of (b) is similar.

PROPOSITION 3.5. *A is strongly homogeneous and strongly indecomposable if and only if A is irreducible and p-irreducible for each p, and $E(A)$ is sub-commutative. In this case, $E(A)$ is commutative.*

PROOF. ($\Rightarrow$). Theorem 3.1 implies that $E(A)$ is commutative, hence sub-commutative. Now apply Corollary 3.2.

($\Leftarrow$). As a consequence of Corollary 2.4.a and Lemma 3.4, $QE(A)$ is a division algebra and $E(A)/pE(A)$ is a field for each p. Thus A is strongly indecomposable.

Let X be a pure rank 1 subgroup of A and ϕ: $E(A) \otimes X \to A$ defined by $\phi(f \otimes x) = f(x)$. Then ϕ is monic and $A/E(A)X$ is torsion since rank $E(A)$ = rank A (Corollary 2.2). Let p be a prime and $pa \in E(A)X$ for $a \in A$. Since X has rank 1, $pa = f(x)$ for some $f \in E(A)$. Now $fE(A) \subseteqq E(A)f$, since $E(A)$ is subcommutative, so $E(A)fE(A) \subseteqq E(A)f \subseteqq E(A)fE(A)$ and $E(A)fE(A) = nE(A)$ for some $n \in Z$ (Proposition 1.2). Since $QE(A)$ is a division algebra, $f = nu$ for some unit u of $E(S)$. Now $pa = f(x) = nu(x)$ so $p|n$, or else $u(x)/p \in A$ implies that $x \in pA \cap X = pX$, since u is a unit of $E(A)$. In either case, $a \in E(A)X$ and $A \cong E(A) \otimes_Z X$.

Since A is strongly indecomposable, $E(A)$ must be strongly indecomposable. In view of Theorem 3.1 and Corollary 1.4 it suffices to prove that $E(A)$ is commutative. For $0 \neq f \in E(A)$ define α_f: $E(A) \to E(A)$ by $\alpha_f(g) = fg - gf$. Then α_f induces an endomorphism of $A \cong E(A) \otimes_Z X$ via $g \otimes x \to \alpha_f(g) \otimes x$. Since $\alpha_f(1) = 0$ and $QE(A)$ is a division algebra, $\alpha_f = 0$. Thus $fg = gf$ for all $g \in E(A)$.

4. Finitely faithful groups.

LEMMA 4.1. *Suppose that $QE(A)$ is semi-simple and that A is finitely faithful. Every exact sequence of groups*

$$O \to B \to G \overset{\Pi}{\to} A \to 0$$

such that Hom(A, G)$A + B = G$ *is split exact.*

PROOF. Let $I = \{\Pi h: h \in \mathrm{Hom}(A, G)\}$, a right ideal of $E(A)$ with $IA = A$. Then $E(A)/I$ is finite since $QE(A)$ is semi-simple (Arnold and Lady [3], Corollary 2.3). But A is finitely faithful so $I = E(A)$, i.e., there is $h: A \to G$ with $\Pi h = 1_A$.

PROPOSITION 4.2. *The following are equivalent*:

(a) *A is finitely faithful*;

(b) $I = \mathrm{Hom}(A, IA)$ *for each maximal right ideal I of finite index in* $E(A)$; *and*

(c) $J_p = \mathrm{Hom}(A, J_pA)$ *for each prime p where* $J_p/pE(A) = \mathscr{J}(E(A)/pE(A))$.

PROOF. (a) $\Rightarrow$ (b). Note that $I \subseteqq \mathrm{Hom}(A, IA)$. a right ideal of $E(A)$. Since A is finitely faithful, $\mathrm{Hom}(A, IA) = E(A)$ is impossible. By the maximality of I, $I = \mathrm{Hom}(A, IA)$.

(b) $\Rightarrow$ (c). Clearly, $J_p = \bigcap\{I \mid I$ is a maximal right ideal of $E(A)$ containing $pE(A)\}$. If $f \in \mathrm{Hom}(A, J_pA)$ then $f(A) \subseteqq IA$ so that $f \in I$ for each maximal right ideal $I \subseteqq pE(A)$. Thus $J_p = \mathrm{Hom}(A, J_pA)$.

(c) $\Rightarrow$ (a). Let I be a maximal right ideal of $E(A)$ with $E(A)/I$ finite and $IA = A$. Then $pE(A) \subseteqq J_p \subseteqq I$ for some prime p of Z. Since $E(A)/J_p$ is semi-simple, $I = eE(A) + J_p$ for some $e \in E(A)$, $e^2 - e \in J_p$. But $A = IA = eA + J_pA$ so $(1 - e)(A) \subseteqq J_pA$. By (c), $1 - e \in J_p$ so $E(A) \subseteqq eE(A) + J_p = I$.

COROLLARY 4.3. *Assume that A is not divisible. If $E(A)/pE(A)$ is semi-simple for each p, then A is finitely faithful. Moreover, $\mathscr{N}E(A)A$ is the maximal divisible subgroup of A.*

PROOF. In this case, $J_p = pE(A)$ and $pE(A) = \mathrm{Hom}(A, pA)$. Also $(\mathscr{N}E(A) + pE(A))/pE(A) \subseteqq J_p/pE(A) = 0$ so $\mathscr{N}E(A) = p\mathscr{N}E(A)$ for each p. Thus $\mathscr{N}E(A)$ is divisible and $\mathscr{N}E(A)A \subseteqq D$, the maximal divisible subgroup of A. Write $A = B \oplus D$, B reduced. Then $D = \mathrm{Hom}(B, D)B \subseteqq \mathscr{N}E(A)A$ since $\mathrm{Hom}(B, D)$, regarded as a left ideal of $E(A)$, is nilpotent.

5. $\mathscr{S}$-groups.

THEOREM 5.1. *The following are equivalent*:

(a) *For each prime p*, p-rank$(E(A)) = (p$-rank$(A))^2$;

(b) *For each prime p with $pA \neq A$*, $E(A)/pE(A) = \mathrm{Mat}_{m_p}(Z/pZ)$ *where* $m_p = p$-rank A; *and*

(c) *A is a finitely faithful $\mathscr{S}$-group.*

PROOF. (a) $\Rightarrow$ (b). There is a monic ring homomorphism $E(A)/pE(A) \to$

$E(A/pA)$. But $E(A/pA) \cong \mathrm{Mat}_{m_p}(Z/pZ)$, where $m_p = p\text{-rank}(A)$, has dim $= m_p^2$ so $E(A)/pE(A) \cong E(A/pA)$.

(b) $\Rightarrow$ (c). Since $E(A)/pE(A)$ is simple for each p, A is finitely faithful (Corollary 4.3).

Suppose that $nA \subseteqq B \subseteqq A$ for some $0 \neq n \in Z$. $B/nA = B_1/nA \oplus \cdots \oplus B_k/nA$ where each B_i/nA is cylic of prime power order.

It suffices to assume that $p^jA \subseteqq B \subseteqq A$ for some prime p and that $B/p^jA \cong Z/p^iZ$ for some i; since if $B_i/nA \cong Z/p^jZ$, then $p^jA \subseteqq (p^j/n) B_i \subseteqq A$ with $(p^j/n)B_i \cong B_i$, and if $\mathrm{Hom}(A, B_i)A = B_i$ for $1 \leqq i \leqq k$, then $\mathrm{Hom}(A, B)A = B$.

As a consequence of (b) there is an isomorphism ϕ: $E(A)/p^jE(A) \to E(A/p^jA)$. Write $B = Zb + p^jA$ and choose $f' \in E(A/p^jA)$ with $f'(A/p^jA) = B/p^jA$. Then $f' = \phi(f + p^jE(A))$ for some $f \in E(A)$ so $b \in f(A) + p^jA$. Thus $B = Zb + p^jA \subseteqq (fE(A) + p^jE(A))A \subseteqq \mathrm{Hom}(A, B)A \subseteqq B$ so that $B = \mathrm{Hom}(A, B)A$ and A is an $\mathcal{S}$-group.

(c) $\Rightarrow$ (a). Write $A/pA = B_1/pA \oplus \cdots \oplus B_m/pA$ where $m = p\text{-rank } A$ and $B_i/pA \cong Z/pZ$. For each i, choose a right ideal I_i of $E(A)$ minimal with respect to $pE(A) \subseteqq I_i$ and $I_iA = B_i$. Then $A = B_1 + \cdots + B_m = (I_1 + \cdots + I_m)A$ so $E(A) = I_1 + \cdots + I_m$ since A is finitely faithful. Also $E(A)/pE(A) = I_1/pE(A) \oplus \cdots \oplus I_m/pE(A)$ and each $I_i/pE(A)$ is a minimal right ideal of $E(A)/pE(A)$ by the choice of I_i and the fact that $I_iA = B$ and $B_i/pA \cong Z/pZ$. Therefore $E(A)/pE(A)$, being the direct sum of minimal right ideals, must be semi-simple.

In fact, $E(A)/pE(A)$ is simple. Otherwise, $E(A)/pE(A) = I/pE(A) \oplus J/pE(A)$ is the direct sum of non-zero ideals. Since A is finitely faithful, $IA \neq A$, $JA \neq A$. Choose $a_1 \in IA \backslash JA$, $a_2 \in JA \backslash IA$, noting that $A = IA + JA$. Let $a = a_1 + a_2 \in A \backslash (IA \cup JA)$ and $B = Za + pA$. Then $LA = B$ for some right ideal L of $E(A)$. Since $E(A) = I + J$, $L = L \cap I + L \cap J$ so $L/pE(A) = (L \cap I)/pE(A) \oplus (L \cap J)/pE(A)$. But $B/pA \cong Z/pZ \cong LA/pA \cong (L \cap I)(A)/pA \oplus (L \cap J)(A)/pA$. Thus, for example, $(L \cap I) A = B$ and $a \in B \subseteqq (L \cap I)A \subseteqq IA$, a contradiction.

Write $E(A)/pE(A) = \mathrm{Mat}_m(F)$, $F = \mathrm{End}_{E(A)/pE(A)} (I_i/pE(A))$, recalling that $m = p\text{-rank } (A)$. Thus $p\text{-rank } (E(A)) = m^2 \dim(F) \leqq m^2$ so $\dim(F) = 1$ and $p\text{-rank}(E(A)) = (p\text{-rank}(A))^2$.

COROLLARY 5.2. *A is a finitely faithful* $\mathcal{S}$*-group if and only if* $\mathrm{Ext}(A, A)$ *is torsion free.*

PROOF. Warfield [13] proves that $\mathrm{Ext}(A, A)$ is torsion free if p-rank $(E(A)) = (p\text{-rank}(A))^2$ for each prime p.

COROLLARY 5.3. *Assume A is a finitely faithful* $\mathcal{S}$*-group.*

(a) *A is p-irreducible for each p.*

(b) *If B is quasi-isomorphic to A, then B is a finitely faithful* $\mathcal{S}$*-group.*

PROOF. (a) follows from Theorem 5.1 and Theorem 2.3.

(b) Note that p-rank $(B) = p$-rank (A) and p-rank $(E(B)) = p$-rank$(E(A))$ for each prime p and apply Theorem 5.1.

COROLLARY 5.4. (a) *A is a finitely faithful $\mathcal{S}$-group if and only if $A = A_1 \oplus \cdots \oplus A_k$ where* $\text{Hom}(A_i, A_j) = 0$ *if $i \neq j$; each A_i is a Q-simple finitely faithful $\mathcal{S}$-group; and if p is a prime, then is some j with $A/pA = A_j/pA_j$.*

(b) *A is a Q-simple finitely faithful $\mathcal{S}$-group if and only if $A \cong B^{k-1} \oplus B_0$ where B and B_0 are Q-simple strongly indecomposable finitely faithful $\mathcal{S}$-groups and B is nearly isomorphic to B_0. In this case, if $S =$* Center$E(A)$, *then $S/pS \cong Z/pZ$ for each prime p with $pA \neq A$.*

(c) *Assume that A is a Q-simple finitely faithful $\mathcal{S}$-group. Then there are pure fully invariant subgroup $B_1, \ldots, B_k$ of A such that $A/(B_1 \oplus \cdots \oplus B_k)$ is torsion divisible; for each i, B_i is an irreducible finitely faithful $\mathcal{S}$-group; and if p is a prime then either $pB_i = B_i$ or else $p(A/B_i) = A/B_i$.*

PROOF. Apply Theorem 5.1 and the results of §2.

COROLLARY 5.5. *Assume that A is finitely faithful. Then A is an $\mathcal{S}$-group if and only if whenever B is a subgroup of finite index in A then B is nearly isomorphic to A.*

PROOF. ($\Leftarrow$). Let B be a subgroup of finite index in A. Since B is nearly isomorphic to A, $B \oplus B_0 \cong A \oplus A$ for some B_0 (Lady [7]). Then $\text{Hom}(A, B)A = B$ so that A is an $\mathcal{S}$-group.

($\Rightarrow$). As a consequence of Corollary 5.4, it suffices to assume that A is Q-simple. Let B be a subgroup of finite index in A. Then $\text{Hom}(A, B)$ is a right ideal of $E(A)$ and $E(A)$ is a maximal S-order in $QE(A)$ (Corollary 1.4). Thus $\text{Hom}(A, B)$ is a projective right ideal of $E(A)$ (Swan and Evans [12]). Since $\text{Hom}(A, B)A = B$, B is nearly isomorphic to A (as in the proof of Theorem 1.6).

6. $\mathcal{J}$-groups.

THEOREM 6.1. *If A is reduced, then the following are equivalent:*

(a) *A is a finitely faithful $\mathcal{J}$-group;*

(b) *A is a finitely faithful $\mathcal{S}$-group and every right ideal of finite index in $E(A)$ is principal; and*

(c) *A is a $\mathcal{J}$-group and $QE(A)$ is semi-simple.*

PROOF. (a) $\Rightarrow$ (b). Every finitely faithful $\mathcal{J}$-group is a finitely faithful $\mathcal{S}$-group. Let I be a right ideal of $E(A)$ with $E(A)/I$ finite. Then A/IA is finite so choose $f \in E(A)$, $f(A) = IA$. Then $A = f^{-1}IA$ so $f^{-1}I = E(A)$, since A is finitely faithful, and $I = fE(A)$.

(b) $\Rightarrow$ (c). Clearly, A is a $\mathcal{J}$-group and $QE(A)$ is semi-simple (Theorem 5.1 and Corollary 4.3).

(c) $\Rightarrow$ (a). Since $QE(A)$ is semi-simple, there is $0 \neq n \in Z$, $n\mathscr{R} \subseteqq E(A) \subseteqq \mathscr{R} = \mathscr{R}_1 \times \cdots \times \mathscr{R}_k$, $Q\mathscr{R}_i$ a simple algebra, $\mathscr{R}_i$ an S_i-order where $S_i =$ Center $\mathscr{R}_i$ (as in Theorem 1.3). There is a maximal S_i-order $\bar{\mathscr{R}}_i$ in $Q\mathscr{R}_i$ with $\bar{\mathscr{R}}_i/\mathscr{R}_i$ finite (Swan and Evans [12]). Hence there is $0 \neq m \in Z$, $m\bar{\mathscr{R}} \subseteqq E(A) \subseteqq \bar{\mathscr{R}} = \bar{\mathscr{R}}_1 \times \cdots \times \bar{\mathscr{R}}_k$. Let $B_i = \bar{\mathscr{R}}_i A$ and $B = B_1 \oplus \cdots \oplus B_k$. Them $mB \subseteqq A \subseteqq B = B_1 \oplus \cdots \oplus B_k$ and $E(B_i) = \bar{\mathscr{R}}_i$. Since A is a $\mathscr{J}$-group, $A \cong B$ and $E(A) \cong \bar{\mathscr{R}}_1 \times \cdots \times \bar{\mathscr{R}}_k$. Consequently, every right ideal of $E(A)$ is projective, since every right ideal of the maximal order $\bar{\mathscr{R}}_i$ is projective (Swan and Evans [12]). Let I be a maximal right ideal of finite index in $E(A)$. Since I is $E(A)$-projective, $I = \text{Hom}(A, IA)$ (Arnold and Lady [3]). Thus A is finitely faithful by Proposition 4.2.

The group A is a *Murley group* if p-rank $(A) \leqq 1$ for each prime p of Z. If A is a Murley group, then A is a $\mathscr{J}$-group. If, in addition, A is indecomposable, then $E(A)$ is an integral domain with $E(A)/pE(A) \cong Z/pZ$ or 0. Furthermore, an indecomposable Murley group is irreducible if and only if it is strongly homogeneous (Murley [8]).

COROLLARY 6.2. *Suppose that $A \cong B^k$ where B is an indecomposable Murley group. Then A is a finitely faithful $\mathscr{J}$-group. Moreover A is irreducible if and only if A is strongly homogeneous.*

COROLLARY 6.3. *A is a strongly homogenecus finitely faithful S-group if and only if $A \cong B^k$ where B is a strongly homogeneous indecomposable Murley group.*

PROOF. ($\Leftarrow$). Corollary 6.2.

($\Rightarrow$). Write $A \cong (R \otimes_Z X)^m$ so that $E(A) \cong \text{Mat}_m(\mathscr{R})$ and $\mathscr{R} \otimes_Z X$ is strongly indecomposable. Then $E(A)/pE(A) \cong \text{Mat}_m(\mathscr{R}/p\mathscr{R}) \cong \text{Mat}_n(Z/pZ)$ where p-rank $(A) = m \cdot p$-rank $(\mathscr{R}) = n$. The uniqueness of Wedderburn-Artin theory implies that $m = n$ and $\mathscr{R}/p\mathscr{R} \cong Z/pZ$. Thus, $\mathscr{R} \otimes_Z X$ is a strongly indecomposable Murley group since $\dim(\mathscr{R} \otimes_Z X)/p(\mathscr{R} \otimes_Z X) \leqq 1$ for each p. Also $\mathscr{R} \otimes_Z X$ is a strongly homogeneous group.

COROLLARY 6.4. *Assume that A is semi-local ($pA = A$ for all but a finite number of primes p). Then A is a finitely faithful $\mathscr{S}$-group if and only if A is a finitely faithful $\mathscr{J}$-group.*

PROOF. Apply Theorem 6.1, Corollary 5.4, and Theorem 1.4, noting that if A is Q-simple and semi-local, then $A =$ Center $E(A)$ has only finitely many maximal ideals and $E(A)$ is a maximal S-order. Thus every right ideal of $E(A)$ is principal (Swan and Evans [12]).

COROLLARY 6.5. *Assume that $E(A)$ is subcommutative. Then A is a finitely faithful S-group if and only if $A = A_1 \oplus \cdots \oplus A_k$ where each A_i is fully invariant in A and each A_i is an indecomposable Murley group.*

PROOF. ($\Leftarrow$). Each A_i is a finitely faithful $\mathscr{S}$-group (Corollary 6.2) so that $A = A_1 \oplus \cdots \oplus A_k$ is a finitely faithful $\mathscr{S}$-group (Corollary 5.4).

($\Rightarrow$). As a consequence of Theorem 5.1 and Lemma 3.4, $E(A)/pE(A)$ is a field isomorphic to Z/pZ for each prime p with $pA \neq A$ and $\dim(A/pA) = 1$. Thus $A = A_1 \oplus \cdots \oplus A_k$ where each A_i is a fully invariant indecomposable Murley group.

7. Examples. Let S be a commutative ring and $H(S) = S \oplus Si \oplus Sj \oplus Sk$ the ring of Hamiltonian quaternions over S. Then $H(S) \subseteqq QH(S) = H(QS)$. Moreover, $H(Z)/pH(Z) \cong H(Z/pZ) \cong \mathrm{Mat}_2(Z/pZ)$ for each prime $p \neq 2$ of Z and $H(Q)$ is a division algebra. Let $Z[1/2]$ be the subring of Q generated by Z and $1/2$. Then $H(Z[1/2])/pH(Z[1/2]) \cong H(Z/pZ)$ if $p \neq 2$ while $2H(Z[1/2]) = H(Z[1/2])$.

EXAMPLE 7.1. There is a torsion free group A of rank 8 with $E(A) \cong H(Z[1/2])$, p-rank $A = \dim E(A)/pE(A) = 4$ for $p \neq 2$, and 2-rank$(A) = 0$ (Corner [5]). Then A is strongly indecomposable, Q-simple, p-simple for each p but is not irreducible or p-irreducible for $p \neq 2$.

EXAMPLE 7.2. There is a torsion free group A of rank 4 with $E(A) \cong H(Z)$ and p-rank $A = p$-rank $H(Z) = 4$ for each p (Zassenhaus [**14**]). Then A is strongly indecomposable, irreducible, p-simple for $p \neq 2$, but not p-irreducible for any p (note that $H(Z)/2H(Z)$ is not semi-simple).

EXAMPLE 7.3. There is a torsion free group A of rank 2 with $E(A) \cong Z$ and p-rank $A = 1$ for each p (Corner [5]). Then A is a strongly indecomposable Q-simple Murley group that is not irreducible. Moreover, A satisfies the conclusions of Corollary 2.8 with $k > 1$.

Assume that $\mathscr{R}$ is a subring of $Q\mathscr{R}$ and that $Q\mathscr{R}$ is a finite dimensional Q-algebra. For a prime p of Z, define $\mathscr{R}_p^*$ to be the p-adic completion of $\mathscr{R}$. Then $\mathscr{R}_p^*$ is complete in the p-adic topology. Moreover, if $S = $ Center $\mathscr{R}$ then $S_p^* = $ Center $\mathscr{R}_p^*$.

LEMMA 7.4. *Suppose that* $\mathscr{R}/p\mathscr{R} \cong \mathrm{Mat}_m(F)$, *where F is a finite field. Then* $\mathscr{R}_p^* = \mathscr{R}_p^* e_1 \oplus \cdots \oplus \mathscr{R}_p^* e_m$, *where* $e_i^2 = e_i$ *and* $\dim(\mathscr{R}_p^* e_i/p\mathscr{R}_p^* e_i) = m \dim(\mathscr{R}/p\mathscr{R})$.

PROOF. Write $\mathscr{R}/p\mathscr{R} = I_1 \oplus \cdots \oplus I_m$, where $I_i = (\mathscr{R}/p\mathscr{R})f_i$ is an irreducible left ideal, $f_i^2 = f_i$, and $\dim(I_i) = m \dim(F)$ for each i. Then $\mathscr{R}_p^*/p\mathscr{R}_p^*$ is isomorphic to $\mathscr{R}/p\mathscr{R}$. Since $\mathscr{R}_p^*$ is complete in the p-adic topology, $\mathscr{R}_p^* = \mathscr{R}_p^* e_1 \oplus \cdots \oplus \mathscr{R}_p^* e_m$, where $e_i^2 = e_i$ and $e_i + p\mathscr{R}_p^* = f_i$ for each i. Thus $\dim(\mathscr{R}_p^* e_i/p\mathscr{R}_p^* e_i) = \dim(I_i) = m \dim(F)$ for each i.

LEMMA 7.5. *Suppose that* $\mathscr{R} \subseteqq Q\mathscr{R}$, $Q\mathscr{R}$ *is a division algebra,* $S =$ Center $\mathscr{R}$, $\mathscr{R}$ *is p-local* ($q\mathscr{R} = \mathscr{R}$ *for each prime* $q \neq p$) *and that* $\mathscr{R}/p\mathscr{R} \cong \mathrm{Mat}_m(F)$, *$F$ a finite field. Then there is a finite rank torsion free p-local* *group A such that* $E_S(A) \cong \mathscr{R}$ *and* p-rank $(A) = m \dim(F)$.

PROOF. The construction of A is a mild variation of the construction given by Corner [5]. As in Lemma 7.4, write $\mathscr{R}_p^* = \mathscr{R}_p^* e_1 \oplus \cdots \oplus \mathscr{R}_p^* e_m$ where $e_i^2 = e_i$ and $\dim(\mathscr{R}_p^* e_i/p\mathscr{R}_p^* e_i) = m \dim(F)$ for each i. Then $\mathscr{R}$ is a free S-module, a maximal S-order in $Q\mathscr{R}$, and S is a discrete valuation ring with unique maximal ideal pS (Corollary 1.4).

Now $\mathscr{R}e_1 \subseteqq \mathscr{R}_p^* e_1$ and $\mathscr{R} \cong \mathscr{R}e_1$ since $Q\mathscr{R}$ is a division algebra. Let $\{\xi_1, \ldots, \xi_\ell\}$, where $\ell \dim(S_p^*/pS_p^*) = \dim(\mathscr{R}_p^* e_1) = m \dim(F)$, be an S_p^*-basis for $\mathscr{R}_p^* e_1$ and choose $\{\xi_{\ell+1}, \ldots \xi_k\} \subseteqq \mathscr{R}_p^* e_2 \oplus \cdots \mathscr{R}_p^* e_m$ such that $\{\xi_1, \ldots, \xi_k\}$ is an S_p^*-basis for $\mathscr{R}_p^*$ where $k \dim(S_p^*/pS_p^*) = m^2 \dim(F)$.

Define T to be the pure subring of S_p^* generated by $\{\Pi_i | r = \Pi_1\xi_1 + \cdots + \Pi_k\xi_k$ for some $r \in \mathscr{R}\}$. Then T is countable and S_p^* is uncountable so there are $n = S\text{-rank}(\mathscr{R})$ elements $\rho_1, \ldots \rho_n$ of S_p^* algebraically independent over T.

Let $\{\alpha_1 = 1, \alpha_2, \ldots \alpha_n\}$ be an S-basis of $\mathscr{R}$ and let $e = \rho_1\alpha_1 + \cdots + \rho_n\alpha_n \in \mathscr{R}_p^*$. Define A to be the pure S_p-submodule of $\mathscr{R}_p^* e_1$ generated by $\{\mathscr{R}e_1, \mathscr{R}ee_1\}$. Then $A_p^* = \mathscr{R}_p^* e_1$ since $\mathscr{R}_p^*/\mathscr{R}$, hence $(\mathscr{R}_p^* e_1)/(\mathscr{R}e_1)$, is divisible and A is pure in $\mathscr{R}_p^* e_1$. Thus $\dim(A/pA) = \dim(A_p^*/pA_p^*) = m \dim(F)$.

There is a ring monomorphism $\phi\colon \mathscr{R} \to E_S(A)$, given by $\phi(r) =$ left multiplication by r since $\mathscr{R}A \subseteqq A$. To show that ϕ is into, let $f \in E_S(A)$. Then f lifts to an S_p^*-homomorphism $f^*\colon \mathscr{R}_p^* e_1 \to \mathscr{R}_p^* e_1$. Hence $f(ee_1) = \rho_1 f(\alpha_1 e_1) + \cdots + \rho_n f(\alpha_n e_1)$ and for some $0 < j \in Z$, $p^j f(ee_1) = \beta_0 + \gamma_0 ee_1$ and $p^j f(\alpha_i e_1) = \beta_i + \gamma_i ee_1$ for $\beta_i \in \mathscr{R}e_1$, $\gamma_i \in \mathscr{R}$, $0 \leqq i \leqq n$. Therefore, $\beta_0 + \gamma_0(\rho_1\alpha_1 e_1 + \cdots + \rho_n\alpha_n e_1) = \rho_1(\beta_1 + \gamma_1(\rho_1\alpha_1 e_1 + \cdots + \rho_n\alpha_n e_1)) + \cdots + \rho_n(\beta_n + \gamma_n(\rho_1\alpha_1 e_1 + \cdots + \rho_n\alpha_n e_1))$. Note that $\rho_1, \ldots \rho_n$ are algebraically independent over the pure subring of S_p^* generated by $\{\Pi_i | re_1 = \Pi_1\xi_1 + \cdots + \Pi_\ell\xi_\ell$ for some $re_1 \in \mathscr{R}e_1\}$ since $r = re_1 + \cdots + re_m = (\Pi_1\xi_1 + \cdots + \Pi_\ell\xi_\ell) + (\Pi_{\ell+1}\xi_{\ell+1} + \cdots + \Pi_k\xi_k)$ by the choice of the ξ_i's. It follows that $p^j f(ee_1) = \gamma_0 ee_1$ and $p^j f(\alpha_i e_1) = \beta_i = \gamma_0(\alpha_i e_1)$ for $1 \leqq i \leqq n$.

Since $\{\alpha_1, \ldots \alpha_n\}$ is an S-basis of $\mathscr{R} \cong \mathscr{R}e_1$, $S = \text{Center } \mathscr{R}$, and $f \in E_S(A)$, $p^j f(re_1) = \gamma_0(re_1)$ for all $re_1 \in \mathscr{R}e_1$. Now $p^j f^* - \phi(\gamma_0)\colon \mathscr{R}_p^* e_1 \to \mathscr{R}_p^* e_1$ with $(p^j f^* - \phi(\gamma_0))(\mathscr{R}e_1) = 0$. Since $\mathscr{R}_p^* e_1/\mathscr{R}e_1$ is divisible and $\mathscr{R}_p^* e_1$ is reduced, $p^j f^* = \phi(\gamma_0)$ and $p^j f = \phi(\gamma_0)$. Thus $E_S(A)/\phi(\mathscr{R})$ is torsion so it suffices to assume $\mathscr{R} \subseteqq E_S(A) \subseteqq Q\mathscr{R} = QE_S(A)$. But $S = \text{Center } \mathscr{R}$, $S \subseteqq \text{Center } E_S(A) \subset \text{Center } (Q\mathscr{R}) = QS$ and S is a discrete valuation ring. Thus $S = \text{Center } E_S(A)$ so that $\mathscr{R}$ and $E_S(A)$ are S-orders in $Q\mathscr{R}$ (Pierce [9]). But $\mathscr{R}$ is a maximal S-order in $Q\mathscr{R}$ so that $\mathscr{R} = E_S(A)$.

EXAMPLE 7.6. For each $p \neq 2$ there is a p-local strongly indecomposable $\mathscr{J}$-group A such that $QE(A)$ is a division algebra but A is not a Murley group.

PROOF. Let $\mathscr{R} = H(Z_p)$ so that $\mathscr{R}/p\mathscr{R} \cong \text{Mat}_2(Z/pZ)$ and $S =$ Center $\mathscr{R} \cong Z_p$. By Lemma 7.5 there is a p-local group A with $E(A) \cong \mathscr{R}$ and $\dim(A/pA) = 2$. Since $\dim(E(A)/pE(A)) = 4 = (\dim(A/pA)^2$, A is a finitely faithful $\mathscr{S}$-group, hence a finitely faithful $\mathscr{J}$-group (Theorem 5.1 and Corollary 6.4). Also A is strongly indecomposable since $QE(A) \cong Q\mathscr{R} \cong H(Q)$ is a division algebra.

The following question, arising from Theorem II, remains open: If A is strongly indecomposable, Q-simple, p-simple for each p and if B is nearly isomorphic to A, then is $B \cong A$? Equivalently, if R is a maximal Center (R)-order in a finite dimensional division algebra QR and if every ideal in R is generated by an element of Z, then is every right ideal of R principal?

REFERENCES

1. D. Arnold, *Strongly homogeneous torsion free abelian groups of finite rank*, Proc. A.M.S. **56** (1975), 67–72.

2. ———, *Genera and direct sum decompositions of torsion free modules*, Lecture Notes in Mathematics #616, Springer, New York, (1977), 197–218.

3. ——— and E. Lady, *Endomorphism rings and direct sums of torsion free abelian groups*, Trans. A.M.S., **211** (1975), 225–237.

4. ———, B. O'Brien and J. Reid, *Quasi-pure injective and projective torsion free abelian groups of finite rank*, Proc. London Math. Soc. **38** (1979), 532–544.

5. A.L.S. Corner, *Every countable reduced torsion free ring is an endomorphism ring*, Proc. London Math. Soc. **13** (1963), 687–710.

6. L. Fuchs, *Infinite Abelian Groups*, Vol. I and II, Academic Press, New York, 1970, 1973.

7. E.L. Lady, *Nearly isomorphic torsion-free abelian groups*, J. Alg. **35** (1975), 235–238.

8. C.E. Murley, *The classification of certain classes of torsion free abelian groups of finite rank*, Pac. J. Math. **40** (1972), 647–665.

9. R.S. Pierce, *Subrings of simple algebras*, Mich. Math. J. **7** (1960), 241–243.

10. J. Reid, *On the ring of quasi-endomorphisms of torsion free groups*, *Topics in Abelian Groups*, Scott-Foresman, Chicago, 1963, 51–68.

11. ———, *On subcommutative rings*, Acta Math. Acad. Sci. Hungar **16** (1965), 23–26.

12. R.G. Swan and E.G. Evans, *K-Theory of finite groups and orders*, Lecture Notes in Mathematics 149, Springer, New York, 1970.

13. R.B. Warfield, Jr., *Extensions of torsion free abelian groups of finite rank*, Arch. Math. **23** (1972), 145–150.

14. H.J. Zassenhaus, *Orders as endomorphism rings of modules of the same rank*, J. London Math. Soc. **42** (1967), 180–182.

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