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A CONVERGENCE STRUCTURE ON A CLASS OF PROBABILITY MEASURES

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Let X denote a separable metric space and $\mathscr{B}$ the Borel σ-field generated by the set of all open subsets of X. Suppose that $\mathscr{P}$ denotes a collection of probability measures on $(X, \mathscr{B})$. A study is made in this note of some convergence structures on $\mathscr{P}$ which make the power function $\beta_\phi: \mathscr{P} \to [0, 1]$, defined by $\beta_\phi(P) = \int \phi dP$, continuous for each test ϕ. The reader is referred to Novak [3] for a detailed study of sequential convergence structures.

An equivalence relation in the set of all tests on $(X, \mathscr{B})$ is given by $\phi \sim \psi$ if and only if $\int \phi dP = \int \psi dP$ for each $P \in \mathscr{P}$; that is, two tests belong to the same equivalence class whenever their power functions are equal. The set of all equivalence classes is denoted by T. A natural convergence structure on T is defined as follows: $\phi_n \to \phi$ if and only if $\int \phi_n dP \to \int \phi dP$ for each $P \in \mathscr{P}$. The set T equipped with this convergence structure is denoted by $T_{\mathscr{P}}$.

Let μ be a σ-finite measure on $(X, \mathscr{B})$ and define $\phi \sim \psi$ if and only if $\phi = \psi$ a.e. $[\mu]$. Moreover, Lehmann [2, p. 349] defines $\phi_n \to \phi$ if and only if $\int \phi_n f d\mu \to \int \phi f d\mu$ for each μ-integrable function f. Let T_μ denote the set of equivalence classes equipped with this convergence structure. It is not difficult to show that $T_\mu = T_{\mathscr{P}}$ whenever $\mathscr{P}$ is the set of all probability measures on $(X, \mathscr{B})$ which are absolutely continuous with respect to μ; that is, the sets and convergence structures coincide. Hence the convergence space $T_{\mathscr{P}}$ seems to be a natural generalization of T_μ. It is known that the space T_μ is compact and metrizable (e.g., see [2, p. 354]. The following property concerning $T_{\mathscr{P}}$ is needed before investigating convergence structures on $\mathscr{P}$.

PROPOSITION 1. *Suppose that $\mathscr{P}$ is any subset of the set of all probability measures on $(X, \mathscr{B})$ which are absolutely continuous with respect to the σ-finite measure μ on $(X, \mathscr{B})$; then $T_{\mathscr{P}}$ is compact and metrizable.*

PROOF. Since each equivalence class re μ is contained in the corresponding equivalence class re $\mathscr{P}$, then let $j: T_\mu \to T_{\mathscr{P}}$ denote the natural mapping; j is continuous. Moreover, $j: \lambda T_\mu \to \lambda T_{\mathscr{P}}$ is also continuous, where $\lambda T_{\mathscr{P}}$

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denotes the set T equipped with the finest topology which is coarser than the convergence structure on $T_{\mathscr{P}}$. Then $\lambda T_{\mathscr{P}}$ is compact and since λT_μ is compact and metrizable, it follows easily that j: $\lambda T_\mu \to \lambda T_{\mathscr{P}}$ is a topological quotient mapping. It is known that a Hausdorff quotient of a compact metrizable space is metrizable (e.g., see [**1**, p. 159]). Hence $\lambda T_{\mathscr{P}}$ is metrizable. It can be shown that $\lambda T_{\mathscr{P}}$ and $T_{\mathscr{P}}$ agree on convergence of sequences and hence it follows that $T_{\mathscr{P}}$ is also metrizable. Therefore $T_{\mathscr{P}}$ is compact and metrizable.

Three convergence structures on any subset, $\mathscr{P}$, of the set of all probability measures on $(X, \mathscr{B})$ are defined below, each having the property that the power function β_ϕ: $\mathscr{P} \to [0, 1]$ is continuous for each test ϕ.

(1) $P_n \to P$ in $\mathscr{P}$ if and only if $\int \phi dP_n \to \int \phi dP$ for each $\phi \in T$.
(2) $P_n \to P$ in $\mathscr{P}$ if and only if $\int \phi_n dP_n \to \int \phi dP$ whenever $\phi_n \to \phi$ in $T_{\mathscr{P}}$.
(3) $P_n \to P$ in $\mathscr{P}$ if and only if $\int \phi dP_n \to \int \phi dP$ uniformly in $\phi \in T$.

Convergence (2) has the desirable property that it is the coarsest convergence structure on $\mathscr{P}$ such that the mapping ω: $\mathscr{P} \times T_{\mathscr{P}} \to [0, 1]$, defined by $\omega(P, \phi) = \int \phi dP$, is jointly continuous. This is a desirable property to have when embedding $\mathscr{P}$ or $T_{\mathscr{P}}$ into function spaces. Convergences (1) and (3) are very familiar; the main interest is in convergence (2).

Consider the corresponding analogues of (1)–(3).

(1′) $P_n \to P$ in $\mathscr{P}$ if and only if $P_n(A) \to P(A)$ for each $A \in \mathscr{B}$.
(2′) $P_n \to P$ in $\mathscr{P}$ if and only if $P_n(A_n) \to P(A)$ whenever $A_n \to A$ in $\mathscr{B}$, i.e., lim sup A_n = lim inf $A_n = A$.
(3′) $P_n \to P$ in $\mathscr{P}$ if and only if $P_n(A) \to P(A)$ uniformly in $A \in \mathscr{B}$.

PROPOSITION 2. *The following implications are satisfied*: (3) $\Leftrightarrow$ (3′) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Leftrightarrow$ (1′) $\Leftrightarrow$ (2′).

The proof is supplied by using the results given in the appendix of Lehmann [**2**].

PROPOSITION 3. *Convergences* (2) *and* (3) *coincide whenever* $T_{\mathscr{P}}$ *is sequentially compact*; *in particular, they agree whenever* $\mathscr{P}$ *is dominated by a* σ-*finite measure* μ.

The proof of the above follows easily by using an indirect argument along with the definition of sequential compactness. The second part follows from Proposition 1.

EXAMPLE. Let X be the real line and consider the probability density functions re Lebesgue measure λ.

$$f_n(x) = \begin{cases} 1 + \sin 2\pi nx, & 0 \leqq x \leqq 1 \\ 0, & \text{otherwise} \end{cases}, \quad f(x) = \begin{cases} 1, & 0 \leqq x \leqq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Let $P_n(A) = \int_A f_n d\lambda$, $P(A) = \int_A f d\lambda$ for each $A \in \mathscr{B}$ and let $\mathscr{P} = \{P | P \ll \lambda\}$. Since (1) $\Leftrightarrow$ (1′) and (3) $\Leftrightarrow$ (3′), then from Robbins [**4**], $P_n \to P$ in (1) but $P_n \nrightarrow P$ in (3). However, by Proposition 3 above, (2) $\Leftrightarrow$ (3); hence $P_n \nrightarrow P$ in (2).

Furthermore, let $\mathscr{P}_1$ be the set of all probability measures on $(X, \mathscr{B})$. Then $\phi_n \to \phi$ in $T_{\mathscr{P}_1}$ if and only if $\phi_n \to \phi$ pointwise. The Dominated Convergence Theorem implies that $\int_0^1 (\phi_n - \phi) \sin 2\pi nx \, d\lambda \to 0$; the Riemann-Lebesgue Theorem implies that $\int_0^1 \phi \sin 2\pi nx \, d\lambda \to 0$. Hence it follows that $\int_0^1 \phi_n \sin 2\pi nx \, d\lambda \to 0$; consequently, $\int \phi_n dP_n \to \int \phi dP$ and so $P_n \to P$ in (2).

This argument shows that (3) is strictly stronger than (2), and (2) is strictly stronger than (1).

Note that only convergence (2) actually depends on the test space; convergences (1) and (3) are determined entirely by the σ-field on X. It might be of interest to investigate the convergence space properties of convergence (2).

References

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