ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 11, Number 3, Summer 1981

NONEXISTENCE OF PARTIAL TRACES FOR GROUP ACTIONS

R. M. GURALNICK, I. M. ISAACS and D. S. PASSMAN

ABSTRACT. Let G be a finite group acting on a ring R. If Λ is a nonempty subset of G, define the map $\mathrm{tr}_\Lambda\colon R \to R$ by $\mathrm{tr}_\Lambda(r) = \sum_{g\in\Lambda} r^g$. We say that G has a nontrivial partial trace on R if, for some Λ, $0 \neq \mathrm{tr}_\Lambda(R) \subseteqq R^G$, where R^G denotes the subring of elements fixed under G. We show here that if G is any nonsolvable group, then there exists a commutative reduced ring R of characteristic 2 acted upon by G such that G has no nontrivial partial trace on R. This is a converse to [**1**, Theorem 2.3].

Let G be a finite group acting as automorphisms on a ring R with 1 and let R^G denote the subring of fixed elements. The trace map $\mathrm{tr}_G\colon R \to R$ is defined by $\mathrm{tr}_G(r) = \sum_{g\in G} r^g$ and certainly $\mathrm{tr}_G(R) \subseteqq R^G$. However, it is quite possible that $\mathrm{tr}_G(R) = 0$. Because of this, one is led, as in [**1**], to define partial traces. Indeed if Λ is any nonempty subset of G, then the map $\mathrm{tr}_\Lambda\colon R \to R$ is given by $\mathrm{tr}_\Lambda(r) = \sum_{g\in\Lambda} r^g$. Of course, now it is not necessarily true that $\mathrm{tr}_\Lambda(R) \subseteqq R^G$. Nevertheless, we say that G has a nontrivial partial trace on R if there exists Λ with $0 \neq \mathrm{tr}_\Lambda(R) \subseteqq R^G$. The goal here is to discuss the existence, or more precisely the nonexistence, of these nontrivial partial traces.

A very informative example is given as [**1**, Examples 3.8]. Here $R = M_2(F)$ where F is a field of characteristic $p \geqq 5$ and $G \cong Z_p$ is a group of inner automorphisms. Yet even in this simple situation, G has no nontrivial partial trace on R. Thus it is clear that the allowable rings R for the study of these traces must be quite restrictive and it appears that the appropriate class to consider is the reduced rings, namely the rings with no nilpotent elements. Indeed, if R is such a ring and if G is solvable, then, by [**1**, Theorem 2.3], G has a nontrivial partial trace on R. On the other hand, an example of John Wilson, given as [**1**, Example 2.1], shows that the above result cannot be extended to all groups G since $G = \mathrm{Alt}_5$ acts on a particular reduced ring with no nontrivial partial trace.

In this paper, we extend Wilson's example in several different ways. We show first, in a fairly elementary manner that if G contains a non-

The latter two authors' research supported in part by NSF grant No. MCS77-01775 A02.

Received by the editors on December 26, 1979.

abelian simple group, then there is a reduced ring R acted upon by G such that G has no nontrivial partial trace on R. In this example, R is a ring of mixed characteristic. Examples with R of prime characteristic p are even more interesting and the case $p = 2$ turns out to be particularly easy. Since all nonabelian simple groups have even order, by the Feit-Thompson theorem [**2**], we can then rather quickly conclude the following theorem.

THEOREM. *If G is any nonsolvable group, then there exists a commutative reduced ring R of characteristic 2 acted upon by G such that G has no partial trace on R.*

Additional properties for p odd are considered, but no definitive results are obtained. Finally, we offer a brief example of a different nature which answers [**1**, Question 3.7] in the negative. We show that there exists a prime ring R of characteristic p and a group G of inner automorphisms having a normal Sylow p-subgroup p of order p, such that $\mathrm{tr}_P(R) \neq 0$ but $\mathrm{tr}_G(R) = 0$.

1. **The mixed characteristic case.** Let G be a finite group and let p be a prime dividing its order. If Ω denotes the set of all right cosets of all subgroups of G of order p, then G acts on Ω by right multiplication. Furthermore, this action has the property that the stabilizer of any point is a subgroup of order p and that every subgroup of G of order p occurs in this manner. Now define the ring

$$R_p = R_p(G) = \oplus\textstyle\sum_{\omega\in\Omega} GF(p)_\omega$$

to be the direct sum of copies of $GF(p)$ indexed by the elements of Ω. Then certainly, R_p is a commutative reduced ring of characteristic p. Moreover, the action of G on Ω determines a natural action of G on R_p and in this way G acts as automorphisms on R_p. As we will see, these rings are the basic building blocks for constructing appropriate examples.

Now suppose Λ is a nonempty subset of G with $\mathrm{tr}_\Lambda(R_p) \subseteq R_p^G$. We then divide the subgroups L of G of order p into two classes according to the following scheme.

L is of nonzero type if for some $\omega \in \Omega$ with $L = G_\omega$, we have $\mathrm{tr}_\Lambda(GF(p)_\omega) \neq 0$.

L is of zero type if for all $\omega \in \Omega$ with $L = G_\omega$, we have $\mathrm{tr}_\Lambda(GF(p)_\omega = 0$. Here of course G_ω denotes the stabilizer of ω in G under the given permutation action. Furthermore, we set $H = \{x \in G | x\Lambda = \Lambda\}$ so that H is clearly a subgroup of G.

The following lemma lists some basic properties. Here all cosets and transversals are on the right. Furthermore, a k-fold transversal of a subgroup L of G is a subset of G meeting each coset of L in precisely k elements.

LEMMA 1.1. *Let Λ be a nonempty subset of G and suppose that* $\mathrm{tr}_\Lambda(R_p) \subseteqq R_p^G$. *Let L be a subgroup of G of order p.*

(i) *If L is of nonzero type, then Λ is a k-fold transversal of L. Here* $1 \leqq k \leqq p - 1$ *and* $k = p|\Lambda|/|G|$ *is independent of L.*

(ii) *L is of zero type if and only if Λ is a union of cosets of L.*

(iii) *Λ is a union of cosets of H and H contains all subgroups L of zero type.*

(iv) *If* $\Lambda = G$, *then* $\mathrm{tr}_\Lambda(R_p) = 0$.

PROOF. Choose $\omega \in \Omega$ with $G_\omega = L$ and write $\Lambda = \bigcup_i \Lambda_i x_i$ where $\{x_i\}$ is a transversal for L and $\Lambda_i \subseteqq L$. Then for $r \in \mathrm{GF}(p)_\omega$ we have clearly

$$\mathrm{tr}_\Lambda(r) = \textstyle\sum_i \mathrm{tr}_{\Lambda_i}(r)^{x_i} = \sum_i |\Lambda_i| r^{x_i}.$$

Observe that $r^{x_i} \in \mathrm{GF}(p)_{\omega x_i}$ and that the latter terms are all the distinct summands in R_p corresponding to the G-orbit of ω.

(i) If L is of nonzero type, we can assume ω and r so chosen that $\mathrm{tr}_\Lambda(r) \neq 0$. Then since $\mathrm{tr}_\Lambda(r) \in R_p^G$, it is clear that all terms r^{x_i} must occur in $\mathrm{tr}_\Lambda(r)$ and all with the same coefficient. Hence $|\Lambda_i|$ is a nonzero constant modulo p. But $|L| = p$ so $0 \leqq |\Lambda_i| \leqq p$ and this implies that, for all i. $|\Lambda_i| = k$ for some fixed k with $1 \leqq k \leqq p - 1$. Thus Λ is a k-fold transversal for L. Furthermore since

$$|\Lambda| = \textstyle\sum_i |\Lambda_i| = k[G\colon L] = k|G|/p,$$

we see that $k = p|\Lambda|/|G|$ is independent of L.

(ii) If L is of zero type, choose any ω with $L = G_\omega$ and any nonzero $r \in \mathrm{GF}(p)_\omega$. Then

$$0 = \mathrm{tr}_\Lambda(r) = \textstyle\sum_i |\Lambda_i| r^{x_i}$$

so we must have $|\Lambda_i| \equiv 0$ modulo p for all i. But $0 \leqq |\Lambda_i| \leqq p$ so either $\Lambda_i = \varnothing$ or $\Lambda_i = L$ Hence we see that Λ is a union of cosets of L. Conversely, if Λ is a union of cosets of L, then each Λ_i must satisfy $|\Lambda_i| = 0$ or p and hence $\mathrm{tr}_\Lambda(\mathrm{GF}(p)_\omega) = 0$.

(iii) It follows immediately from the definition of H that Λ is a union of cosets of H. Furthermore, if L is of zero type, then Λ is a union of cosets of L. Thus $x\Lambda = \Lambda$ for all $x \in L$ and $L \subseteqq H$.

(iv) It $\Lambda = G$, then Λ is surely a union of cosets of L for each subgroup L of G of order p Thus, by (ii), each such L is of zero type and this implies that $\mathrm{tr}_G(R_p) = \mathrm{tr}_\Lambda(R_p) = 0$.

As a consequence we get the following generalization of Wilson's example.

LEMMA 1.2. *Let G be a finite group. Suppose there exist two distinct primes p and q such that G is generated by its elements of order p and by its elements of order q. Then G has no nontrivial partial trace on the ring $R = R_p \oplus R_q$. In particular this always occurs if G is nonabelian simple.*

PROOF. Suppose Λ is a nonempty subset of G with $\mathrm{tr}_\Lambda(R) \subseteqq R^G$. Then $\mathrm{tr}_\Lambda(R_p) \subseteqq R_p^G$ and Lemma 1.1 applies. If $\mathrm{tr}_\Lambda(R_p) \neq 0$. then some subgroup L of G of order p is of nonzero type and we conclude from Lemma 1.1(i) that Λ is a k-fold transversal for L. Thus $|G|/p$ divides $|\Lambda|$. On the other hand, if $\mathrm{tr}_\Lambda(R_p) = 0$, then all subgroups L are of zero type and H contains the subgroup of G generated by all elements of G of order p. Thus by assumption $H = G$ and Lemma 1.1(iii) implies that $\Lambda = G$. In particular, in either case we deduce that $|G|/p$ divides $|\Lambda|$.

Similarly since $\mathrm{tr}_\Lambda(R_q) \subseteqq R_q^G$ we deduce that $|G|/q$ divides $|\Lambda|$. But p and q are distinct primes so $|G|$ divides $|\Lambda|$ and hence $\Lambda = G$. Finally, by Lemma 1.1(iv),

$$\mathrm{tr}_\Lambda(R) = \mathrm{tr}_G(R_p) \oplus \mathrm{tr}_G(R_q) = 0$$

and we see that G has no nontrivial partial trace on R.

We can extend this result rather quickly by considering induced actions.

LEMMA 1.3. *Let U be a subgroup of G and suppose U acts on the ring S with no nontrivial partial trace. If $R = \oplus\sum_{i=1}^{n} S_i$ denotes the ring direct sum of $n = [G\colon U]$ copies of S, then G acts on R with no nontrivial partial trace.*

PROOF. The action of G on R is the usual induced representation given as follows. Let $\{x_1 = 1, x_2, \ldots, x_n\}$ be a transversal for U in G and for each i define S_i by

$$S_i = S \otimes x_i = \{s \otimes x_i | s \in S\} \cong S.$$

If $g \in G$ and i are given, then $x_i g \in U x_{i'}$ for some i', so $x_i g = u x_{i'}$ for some $u \in U$. We then define the action of G on R by $(s \otimes x_i)^g = s^u \otimes x_{i'}$. From this formula, it is easy to verify that G does indeed act on R and in fact as ring automorphisms.

Suppose by way of contradiction that there exists a nonempty subset Λ of G with $0 \neq \mathrm{tr}_\Lambda(R) \subseteqq R^G$. Then for some j we have $\mathrm{tr}_\Lambda(S_j) \neq 0$. Note that for any $x \in G$ the set $\Lambda' = x\Lambda$ also satisfies $0 \neq \mathrm{tr}_{\Lambda'}(R) \subseteqq R^G$ and hence, by choosing $x = x_j$ and replacing $\Lambda = \Lambda'$, we can clearly assume that $\mathrm{tr}_\Lambda(S_1) \neq 0$. Write $\Lambda = \bigcup_1^n \Lambda_i x_i$ with $\Lambda_i \subseteqq U$ and observe that, since $x_1 = 1$, we have

$$\mathrm{tr}_\Lambda(s \otimes 1) = \textstyle\sum_1^n \mathrm{tr}_{\Lambda_i}(s) \otimes x_i$$

for all $s \in S$. In particular, since $0 \neq \mathrm{tr}_\Lambda(S \otimes 1) \subseteqq R^G$, we see that

$0 \neq \mathrm{tr}_{\Lambda_1}(S) \subseteqq S$. Furthermore since $S \otimes 1$ is U-invariant, it follows that $\mathrm{tr}_{\Lambda_1}(S) \subseteqq S^U$. Thus Λ_1 gives rise to a nontrivial partial trace for U on S, a contradiction and the lemma is proved.

The following result is now an immediate consequence of Lemmas 1.2 and 1.3.

THEOREM 1.4. *Let G be a finite group which contains a nonabelian simple subgroup. Then there exists a commutative reduced ring R acted upon by G such that G has no nontrivial partial trace on R.*

Of course we would like to extend the above to handle all nonsolvable groups, but the two obvious approaches do not seem to work in general. The first approach considers permutation representations of G where the point stabilizers are cyclic of order p^n. However here the ring R would then have to be built as a direct sum of copies of Z/p^nZ and R would not be reduced if $n > 1$. The second approach concerns using nonfaithful actions. In other words, if $G/N \cong \bar{G}$ and if $\bar{G}$ has a suitable action with no nontrivial partial trace, does the lifted action of G also have this property? The problem here is that if Λ is a subset of G, then Λ is really a subset of $\bar{G}$ with multiplicity and the multiplicities of the different elements could vary greatly. However, as we will see in the next section, this approach does work provided we deal with rings of characteristic 2.

2. **The prime characteristic case.** The rings constructed in Lemma 1.2 and Theorem 1.4 have mixed characteristic. Here we consider rings of characteristic p. Thus let G be a finite group whose order is divisible by p and let R_p be given as in §1. Furthermore, let Λ be a nonempty subset of G with $\mathrm{tr}_\Lambda(R_p) \subseteqq R_p^G$. We start with the case $p = 2$.

LEMMA 2.1 *Let $G \neq \langle 1 \rangle$ be generated by its elements of order 2. if G has no subgroup of index 2, then G has no nontrivial partial trace on R_2.*

PROOF. Let $x \in G$ be an element of order 2. If $\langle x \rangle$ is of zero type, then by Lemma 1.1(ii) Λ is a union of cosets of $\langle x \rangle$. Thus we see that $x\Lambda = \Lambda$ and hence also $x(G\backslash\Lambda) = G\backslash\Lambda$. On the other hand, if $\langle x \rangle$ is of nonzero type, then Λ is a 1-fold transversal for $\langle x \rangle$ in G, by Lemma 1.1(i). Since any coset $\langle x \rangle y$ of $\langle x \rangle$ contains just the two elements y and xy, it follows easily that $x\Lambda = G\backslash\Lambda$ and $x(G\backslash\Lambda) = \Lambda$. Thus each element x of G of order 2 permutes the set $\{\Lambda, G\backslash\Lambda\}$ by left multiplication and hence, since G is generated by these elements, G also permutes this set of size 2 by left multiplication. But G has no subgroup of index 2 so we conclude that $G\Lambda = \Lambda$. Thus $\Lambda = G$ and $\mathrm{tr}_\Lambda(R_2) = 0$ by Lemma 1.1(iv).

It is now a simple matter, using the Feit-Thompson Theorem **[2]**, to obtain our main result.

THEOREM 2.2. *If G is any nonsolvable group, then there exists a commutative reduced ring R of characteristic 2 acted upon by G such that G has no nontrivial partial trace on R.*

PROOF. Since G is nonsolvable, it clearly contains subgroups $N \lhd M$ with M/N nonabelian simple. Then, by [2], the group M/N has even order and certainly satisfies the hypotheses of Lemma 2.1. We conclude therefore that M/N has no nontrivial partial trace on $S = R_2(M/N)$.

Now let M act on S by means of the homomorphism $M \to \bar{M} = M/N$. Since S has characteristic 2, S can be viewed as a right module for the group rings $F[M]$ and $F[M/N]$ where $F = GF(2)$. Suppose Λ is any subset of M and let $\alpha = \sum_{x \in \Lambda} x \in F[M]$. Then

$$\operatorname{tr}_\Lambda(S) = S\alpha = S\bar{\alpha} = \operatorname{tr}_{\bar{\Lambda}}(S)$$

where $\bar{\alpha}$ is the image of α under the homomorphism $F[M] \to F[\bar{M}]$ and where $\bar{\Lambda} = \operatorname{Supp} \bar{\alpha}$. Thus since $S^M = S^{\bar{M}}$, we conclude immediately that M has no nontrivial partial trace on S. Finally, since S is a commutative reduced ring of characteristic 2, Lemma 1.3 yields the result.

Observe that characteristic 2 is used in a crucial manner in the above argument. In other characteristics, partial traces for M and for $\bar{M}$ do not correspond. We now consider nonexistence results which hold for all primes p. Again let G be a finite group whose order is divisible by p and let Λ be a nonempty subset of G with $\operatorname{tr}_\Lambda(R_p) \subseteq R_p^G$. Moreover, let H and the subgroups of zero and nonzero types be given as in §1. Observe that if $x \in G$, then Λx has the same properties as Λ with precisely the same H and subgroups of the two types. The following two lemmas show first that H is small in G and then that H is large. This allows us to show in various situations that H does not exist when $\operatorname{tr}_\Lambda(R_p) \neq 0$ and hence that Λ does not exist.

LEMMA 2.3. *Assume that* $0 \neq \operatorname{tr}_\Lambda(R_p) \subseteq R_p^G$.

(i) *$H \neq G$ and in fact $|H|_p < |G|_p$.*

(ii) *Let $|L_1| = |L_2| = p$. If L_1 is of zero type and L_2 is of nonzero type, then $\langle L_1, L_2 \rangle$ has order divisible by p^2.*

PROOF. (i) By Lemma 1.1(iii), Λ is a union of cosets of H so $|H|$ divides $|\Lambda|$. On the other hand, by Lemma 1.1(i), since $\operatorname{tr}_\Lambda(R_p) \neq 0$, Λ is a k-fold transversal for some subgroup L of G of order p and hence $|\Lambda| = k|G|/p$. Thus $|H|$ divides $k|G|/p$ and since $1 \leq k \leq p - 1$, we see that $|H|_p < |G|_p$.

(ii) Set $K = \langle L_1, L_2 \rangle$ and consider $\Lambda^* = \Lambda \cap K$. Since Λ is a k-fold transversal for L_2 in G, it is clear that Λ^* is a k-fold transversal for L_2 in K. Thus $|\Lambda^*| = k|K|/p$. On the other hand, Λ is a union of cosets of L_1, so Λ^* is a union of cosets of L_1. Thus $p = |L_1|$ divides $|\Lambda^*|$ and since $1 \leq k \leq p - 1$, we see that $p^2 \mid |K|$.

LEMMA 2.4. *Assume that* $0 \neq \mathrm{tr}_\Lambda(R_p) \subseteq R_p^G$.

(i) *If P is a subgroup of G of period p, then* $[P: P \cap H] \leqq p$.

(ii) *If* $K = QP$ *is a Frobenius subgroup of G with complement P of order p, then the Frobenius kernel Q is contained in H.*

PROOF. (i) Replacing Λ by Λx if necessary, we can assume that $1 \in \Lambda$. Write $A = P \cap H$ and set $n = [P: A]$. We may clearly assume that $n > 1$. We now compute the size of $\Lambda \cap P$ in two different ways. First since $P \neq A$ and P has period p, P contains a subgroup L of order p which is of nonzero type. Then, by Lemma 1.1(i), Λ is a k-fold transversal for L in G, so clearly $\Lambda \cap P$ is a k-fold transversal for L in P. Thus $|\Lambda \cap P| = k|P|/p$.

Next observe that P is the disjoint union of A and the nonidentity elements of all subgroups L of order p not contained in A. Now each such L must be of nonzero type, so Λ is a k-fold transversal of L and we have $|\Lambda \cap L| = k$. Thus $|\Lambda \cap L^\#| = k - 1$ since $1 \in \Lambda$. On the other hand, by Lemma 1.1(iii), Λ is a union of cosets of H and hence of A. In particular, since $1 \in \Lambda$, we see that $A \subseteq \Lambda$. It now follows that

$$\begin{aligned}|\Lambda \cap P| &= |\Lambda \cap A| + \textstyle\sum_L |\Lambda \cap L^\#| \\ &= |A| + (k-1)\,(|P| - |A|)/(p-1)\end{aligned}$$

since P contains precisely $(|P| - |A|)/(p-1)$ subgroups L of order p not contained in A.

Combining these two formulas for $|\Lambda \cap P|$, dividing by $|A|$ and using $[P: A] = n$, we obtain

$$kn/p = 1 + (k-1)(n-1)/(p-1)$$

which simplifies to $(n-p)(p-k) = 0$. But $p \neq k$ so we conclude that $n = p$.

(ii) Since $p^2 \nmid |K|$, it follows from Lemma 2.3(ii) that either all subgroups of K of order p are of zero type or all are of nonzero type. If all are of zero type, then they are all contained in H, by Lemma 1.1(iii), and hence $Q \subseteq K \subseteq H$. We may therefore assume that all are of nonzero type.

Suppose first that $1 \in \Lambda$. We compute the size of $\Lambda \cap K$ in two different ways. Let L be any subgroup of K of order p. Then L is of nonzero type so Λ is a k-fold transversal for L in G, and thus $\Lambda \cap K$ is a k-fold transversal for L in K. Hence $|\Lambda \cap K| = k|K|/p = k|Q|$. Furthermore, since $1 \in \Lambda$, we have $|\Lambda \cap L^\#| = k - 1$. Now K is a Frobenius group, so K is the disjoint union $K = Q \cup \bigcup_L L^\#$ where the subgroups L are the Frobenius complements, all of order p. Thus, since there are precisely $|Q|$ such complements, we have

$$\begin{aligned}k|Q| = |\Lambda \cap K| &= |\Lambda \cap Q| + \textstyle\sum_L |\Lambda \cap L^\#| \\ &= |\Lambda \cap Q| + |Q|(k-1).\end{aligned}$$

This yields $|\Lambda \cap Q| = |Q|$, so $Q \subseteq \Lambda$.

Finally write $\Lambda = \bigcup_i \Lambda_i x_i$ where $\Lambda_i \subseteq Q$, $1 \in \Lambda_i$ and the x_i are in distinct cosets of Q in G. Then $1 \in \Lambda x_j^{-1}$ so, by the above, $Q \subseteq \Lambda x_j^{-1}$ and $Qx_j \subseteq \Lambda$. Thus $\Lambda_i = Q$ for all i and Λ is a union of cosets of Q. Hence clearly $Q \subseteq H$.

As a consequence we have the following theorem.

THEOREM 2.5. *Let G be a simple group whose order is divisible by p. Suppose that G contains either a nonabelian p-group of period p or a Frobenius group QP with complement P of order p. Then G has no nontrivial partial trace on R_p.*

PROOF. Suppose by way of contradiction that Λ is a nonempty subset of G with $0 \neq \mathrm{tr}_\Lambda(R_p) \subseteq R_p^G$. The two cases are quite similar.

Assume first that G contains P, a nonabelian p-group of period p. Then, by Lemma 2.4(i), $[P\colon P \cap H] \leqq p$ and hence $H \supseteq P \cap H \supseteq P'$, the commutator subgroup of P. Furthermore, for all $x \in G$, $G \supseteq P^x$ so $H \supseteq (P^x)' = (P')^x$. Thus H contains the normal closure of $P' \neq \langle 1 \rangle$, so $H = G$ and this contradicts Lemma 2.3(i).

Now assume that G contains the Frobenius group $K = QP$. Then by Lemma 2.4(ii), $H \supseteq Q$. Furthermore, for all $x \in G$, $G \supseteq K^x = Q^x P^x$ so $H \supseteq Q^x$. Thus H contains the normal closure of $Q \neq \langle 1 \rangle$, so $H = G$ and this contradicts Lemma 2.3(i).

We remark that the first part of the above is vacuous for $p = 2$. A slight modification of the argument shows that for $p = 2$, if G contains the dihedral group of order 8, then G has no nontrivial partial trace on R_2. Of course this is all unnecessary in view of Lemma 2.1. Some examples where the above applies are as follows:

COROLLARY 2.6. *Let $p > 2$ and let $G \cong \mathrm{PSL}_n(p^f)$ with $n \geqq 3$ or $G \cong \mathrm{Alt}_n$ with $n \geqq 2p$. Then G has no nontrivial partial trace on R_p.*

PROOF. The first case follows from Theorem 2.5 since the Sylow p-subgroup of $\mathrm{PSL}_3(p)$ is the nonabelian p-group of order p_3 and period p. The second case follows since Alt_{2p} contains a Frobenius group of order $2^{p-1}.\,p$ with complement of order p.

Lemma 2.4(i) can be used in additional ways. For example suppose we know that the Sylow p-subgroups of G form a TI set and contain a subgroup of type (p, p). Then it follows easily from Lemma 2.4(i) that the map $P \mapsto P \cap H$ yields a one-to-one correspondence between the Sylow p-subgroups of G and of H. Hence we deduce that $G = H \cdot \mathrm{N}_G(P)$, a rather rare occurrence. In particular, this handles $G \cong \mathrm{PSL}_2(p^f)$ with $p > 2$, $f \geqq 2$.

On the other hand, nontrivial partial traces certainly can exist for $p > 2$. Indeed, suppose G has a Sylow p-subgroup of order p and that G has a p-complement H. Then $\Lambda = H$ is a transversal for all subgroups of order p in G and hence clearly $0 \neq \mathrm{tr}_\Lambda(R_p) \subseteqq R_p^G$. In particular, this applies to $G \cong \mathrm{Alt}_p$ with H corresponding to Alt_{p-1}.

3. **A final example.** We close this paper with a brief example which answers [**1**, Question 3.7] in the negative.

LEMMA 3.1. *Let R be a ring of prime characteristic p and let $\lambda \in R$ with $\lambda \neq 0$, $\lambda^p = 0$. Set $x = 1 + \lambda$.*

(i) *x is a unit of order p.*

(ii) *For any $\alpha \in R$, $\mathrm{tr}_{\langle x\rangle}\alpha = x\sum_{i=0}^{p-1}\lambda^{p-1-i}\alpha\lambda^i$.*

(iii) *If $p > 2$ and $\lambda^{(p+1)/2} = 0$, then $\mathrm{tr}_{\langle x\rangle}\alpha = \lambda^{(p-1)/2}\alpha\lambda^{(p-1)/2}$.*

PROOF. (i) This is clear since $x^p = (1 + \lambda)^p = 1 + \lambda^p = 1$.

(ii) For any $y \in R$ let $\mathscr{L}_y$ and $\mathscr{R}_y$ denote left and right multiplication by y respectively and observe that, as operators on R, $\mathscr{L}_y$ and $\mathscr{R}_y$ commute. Since $pR = 0$, we therefore have

$$\begin{aligned}\mathrm{tr}_{\langle x\rangle}\alpha &= \textstyle\sum_{i=0}^{p-1}x^{p-i}\alpha x^i = x\sum_{i=0}^{p-1}x^{p-1-i}\alpha x^i \\ &= x\cdot(\textstyle\sum_{i=0}^{p-1}\mathscr{L}_x^{p-1-i}\mathscr{R}_x^i)\alpha = x\cdot(\mathscr{L}_x - \mathscr{R}_x)^{p-1}\alpha \\ &= x\cdot(\mathscr{L}_\lambda - \mathscr{R}_\lambda)^{p-1}\alpha = x\cdot\textstyle\sum_{i=0}^{p-1}\lambda^{p-1-i}\alpha\lambda^i.\end{aligned}$$

(iii) Now suppose $\lambda^{(p+1)/2} = 0$. Then the i-th term in the above sum is zero if either $i \geqq (p + 1)/2$ or $p - 1 - i \geqq (p + 1)/2$. This leaves only $i = (p - 1)/2$ and we have

$$\mathrm{tr}_{\langle x\rangle}\alpha = (1 + \lambda)\lambda^{(p-1)/2}\alpha\lambda^{(p-1)/2} = \lambda^{(p-1)/2}\alpha\lambda^{(p-1)/2}.$$

PROPOSITION 3.2. *Let F be a finite field of characteristic $p > 2$ and set $R = M_n(F)$ for $n = (p + 1)/2$. Then there exists a group G of inner automorphisms of R with a normal Sylow p-subgroup P of order p, such that $\mathrm{tr}_P(R) \neq 0$ but $\mathrm{tr}_G(R) = 0$.*

PROOF. Set

$$\lambda = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & 0 & 1 & \\ & & \ddots & \ddots \;\; 1 \\ 0 & & & 0 \end{pmatrix} \in R$$

and $x = 1 + \lambda$. Then

$$\lambda^{(p-1)/2} = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix} \neq 0$$

and $\lambda^{(p+1)/2} = 0$. Moreover, if $\alpha = (\alpha_{ij}) \in R$, then by Lemma 3.1(iii)

$$\mathrm{tr}_{\langle x\rangle}\alpha = \lambda^{(p-1)/2}\alpha\lambda^{(p-1)/2} = \alpha_{n1}\lambda^{(p-1)/2}.$$

Thus we see that $0 \neq \mathrm{tr}_{\langle x\rangle}(R) \subseteq F\lambda^{(p-1)/2}$.

Let t be an integer which generates the cyclic group $\mathrm{GF}(p)\backslash\{0\}$ of order $p - 1$. Then $t^{(p-1)/2}$ has order 2 so $t^{(p-1)/2} \equiv -1 \bmod p$. Observe that

$$x^t = (1 + \lambda)^t = 1 + t\lambda + \lambda^2 T = 1 + \lambda(t + \lambda T)$$

where T denotes a sum of other terms. In particular, if we write $x^t = 1 + \mu$, then $\mu = \lambda(t + \lambda T)$ so

$$\mu^{(p-1)/2} = \lambda^{(p-1)/2}(t^{(p-1)/2} + \lambda T^*) = -\lambda^{(p-1)/2}.$$

One consequence of the above is that x^t and x have the same Jordan block structure and thus there exists $y \in \mathrm{GL}_n(F) \subseteq R$ with $y^{-1}xy = x^t$. In particular, y normalizes $\langle x\rangle$ and hence y^{p-1} centralizes $\langle x\rangle$. Thus by replacing y by its p'-part if necessary, we can assume that y is a p'-element. Let $G = \langle x, y\rangle \subseteq R$ so that G is the semidirect product of $\langle x\rangle$ by $\langle y\rangle$ and G acts on R by conjugation. Certainly $P = \langle x\rangle$ is the normal Sylow p-subgroup of G and, as we have seen above, $\mathrm{tr}_p(R) \neq 0$.

Now, since $(1 + \lambda)^y = x^y = x^t = 1 + \mu$, we have $\lambda^y = \mu$ and hence

$$(\lambda^{(p-1)/2})^y = \mu^{(p-1)/2} = -\lambda^{(p-1)/2}.$$

Thus we conclude that y has even order and then that $\mathrm{tr}_{\langle y\rangle}\lambda^{(p-1)/2} = 0$. But $\mathrm{tr}_{\langle x\rangle}(R) \subseteq F\lambda^{(p-1)/2}$ and $\mathrm{tr}_G = \mathrm{tr}_{\langle y\rangle} \cdot \mathrm{tr}_{\langle x\rangle}$ since G is the semidirect product of $\langle x\rangle$ by $\langle y\rangle$. Therefore

$$\begin{aligned}\mathrm{tr}_G(R) &= \mathrm{tr}_{\langle y\rangle}(\mathrm{tr}_{\langle x\rangle}(R)) \\ &\subseteq \mathrm{tr}_{\langle y\rangle}(F\lambda^{(p-1)/2}) = F\,\mathrm{tr}_{\langle y\rangle}(\lambda^{(p-1)/2}) = 0\end{aligned}$$

and G is an appropriate example.

For $p = 2$, a simple example is as follows. Take $R = M_3(F)$ where $F = \mathrm{GF}(2)$ and let

$$x = \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, y = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}, t = \begin{pmatrix}0 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Then $P = \langle x, y\rangle$ is a four's group normalized by $T = \langle t\rangle$ a cyclic group of order 3. Hence $G = \langle x, y, t\rangle = PT \cong \mathrm{Alt}_4$ and P is the normal Sylow 2-subgroup of G. Now

$$\mathrm{tr}_T(R) \subseteq R^T \subseteq \left\{\begin{pmatrix}A & 0\\ 0 & b\end{pmatrix}\middle|\, A \in M_2(F),\ b \in F\right\}$$

and the elements of P all look like $\begin{pmatrix}I & *\\ 0 & 1\end{pmatrix}$ where the four corner matrices $*$ sum to zero. Since

$$\begin{pmatrix} I & U \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} A & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} I & U \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & (A + b)U \\ 0 & b \end{pmatrix}$$

it therefore follows that $\mathrm{tr}_P\mathrm{tr}_T(R) = 0$ and hence that $\mathrm{tr}_G(R) = 0$. On the other hand, $\mathrm{tr}_P(R) \neq 0$ since, for matrix units e_{jj}, one calculates that $\mathrm{tr}_P(e_{31}) = e_{23}$.

REFERENCES

1. M. Cohen and S. Montgomery, *Trace functions for finite automorphism groups of rings*, Arch. Math. **35** (1981), 516–527.

2. W. Feit and J. G. Thompson, *Solvability of groups of odd order*, Pacific J. Math. **13** (1963) 775–1029.

UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA 90007.
UNIVERSITY OF WISCONSIN-MADISON, MADISON, WI 53706.
UNIVERSITY OF WISCONSIN-MADISON, MADISON, WI 53706.