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DISFOCALITY AND NONOSCILLATORY SOLUTIONS OF N-TH-ORDER DIFFERENTIAL EQUATIONS

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In this paper we shall study various disfocality properties and their consequences on solutions of the differential equation

$$\text{(E)} \qquad y^{(n)} + py = 0,$$

where p is continuous and of constant sign on $[a, \infty)$. Equation (E) is said to be *disfocal* on an interval I if, for every nontrivial solution y of (E), at least one of the functions $y, y', \ldots, y^{(n-1)}$ does not vanish on I. If equation (E) is not disfocal on I, then there exists an integer $k (1 \leqq k \leqq n - 1)$, a pair of points $b, c \in I, b < c$, and a nontrivial solution y of (E) such that k of the functions $y, y', \ldots, y^{(n-1)}$ vanish at b and the remaining $n - k$ functions at c, i.e.,

$$\text{(1)} \qquad \begin{aligned} y^{(j_i)}(b) &= 0, i = 0, 1, \ldots, k - 1, \\ y^{(j_i)}(c) &= 0, i = k, \ldots, n - 1, \end{aligned}$$

$$0 \leqq j_0 < j_1 < \cdots < j_{k-1} \leqq n - 1, 0 \leqq j_k < j_{k+1} < \cdots < j_{n-1} \leqq n - 1.$$

Here, $n - k$ is even or odd according as $p < 0$ or $p > 0$ [**10**], which is the well-known parity condition that every nontrivial solution of the problem (E)-(1) must satisfy. Equation (E) is said to be $(j_0, j_1, \ldots, j_{k-1}) - (j_k, \ldots, j_{n-1})$ *disfocal* on an interval I if for every pair of points b and c in I, $b < c$, the only solution satisfying the conditions in (1) is the trivial solution; furthermore, if $j_i = i$, $i = 0, 1, \ldots, n - 1$, it is said to be $k - (n - k)$ *disfocal*, and this special case has been investigated by Nehari [**10, 11, 12**] and Elias [**2, 3, 4**]. We shall say that equation (E) is *eventually* $(j_0, j_1, \ldots, j_{k-1}) - (j_k, \ldots, j_{n-1})$ *disfocal* on $[a, \infty)$ if there exists a point $b \geqq a$ such that (E) is $(j_0, j_1, \ldots, j_{k-1}) - (j_k, \ldots, j_{n-1})$ disfocal on $[b, \infty)$. The concept of eventual $(j_0, j_1, \ldots, j_{k-1}) - (j_k, \ldots, j_{n-1})$ disfocality is related to the existence of nonoscillatory solutions satisfying a set of sign conditions as shown in Lemma 2. On the other hand, Lemma 1 states that only certain sets of sign conditions are admissible for nonoscillatory solutions of (E). Since the admissible sign conditions strongly depend on the parity of n and the sign of p, it is convenient to consider the following four cases:

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(i) n even, $p > 0$,

(ii) n odd, $p > 0$,

(iii) n even, $p < 0$,

(iv) n odd, $p < 0$.

Equation (E) satisfying condition (i), for example, is denoted by (E_i); (E_{ii}), (E_{iii}), and (E_{iv}) are similarly defined.

LEMMA 1. [8]. *Let y be a nonoscillatory solution of* (E) *such that $y \geqq 0$ on $[b, \infty)$ for some $b \geqq a$, and let $p \not\equiv 0$ on $[b_1, \infty)$ for every $b_1 \geqq a$. Define $[C]$ to be the greatest integer less than or equal to C. If y is a solution of* (E_i) *or* (E_{iv}), *there exists an integer j, $0 \leqq j \leqq [(n-1)/2]$, such that*

$$y^{(i)} > 0, \quad i = 0, 1, \ldots, 2j, \tag{2}$$

on $[b_2, \infty)$ for some $b_2 \geqq b$, and

$$(-1)^{i+1} y^{(i)} > 0, \qquad i = 2j+1, \ldots, n-1, \tag{3}$$

on $[b, \infty)$. If y is a solution of (E_{ii}) *or* (E_{iii}), *there exists an integer j, $0 \leqq j \leqq [n/2]$, such that*

$$y^{(i)} > 0, \qquad i = 0, 1, \ldots, 2j-1, \tag{4}$$

on $[b_2, \infty)$ for some $b_2 \geqq b$, and

$$(-1)^{i} y^{(i)} > 0, \qquad i = 2j, \ldots, n-1, \tag{5}$$

on $[b, \infty)$.

Following Kiguradze [6], we say that a nonoscillatory solution y of (E_i) or (E_{iv}) belongs to *class A_j* if y or $-y$ satisfies the inequalities (2) and (3), $0 \leqq j \leqq [(n-1)/2]$. Similarly, a nonoscillatory solution y of (E_{ii}) or (E_{iii}) is said to belong to *class A_j* if y or $-y$ satisfies the inequalities (4) and (5), $0 \leqq j \leqq [n/2]$.

The parity condition [10] mentioned earlier for (E) is equivalent to the condition

$$(-1)^{n-k}\, p(x) < 0. \tag{P}$$

We shall henceforth assume (P) whenever $(j_0, \ldots, j_{k-1}) - (j_k, \ldots, j_{n-1})$ disfocality is discussed. Note that equation (E) is trivially $(j_0, \ldots, j_{k-1}) - (j_k, \ldots, j_{n-1})$ disfocal unless (P) holds.

Our main result is Theorem 5, in which we determine the number of solutions belonging to the class A_j. More specifically, we shall determine the number of solutions in A_j with the property that every nontrivial linear combination of them again belongs to A_j. In Theorem 2 we prove that if equation (E) is eventually $(j_0, \ldots, j_{k-1}) - (j_k, \ldots, j_{n-1})$ disfocal on $[a, \infty)$, then $j_i = i$, $i = 0, 1, \ldots, n-1$. Preliminary results required for our investigation are contained in Theorem 1 and Lemma 2.

THEOREM 1. *Assume that equation* (E) *is* $(j_0, \ldots, j_{k-1}) - (j_k, \ldots, j_{n-1})$ *disfocal on an interval* $[b, c)$ *for some* $b \geqq a$. *Let* $u_l = u_l(x, s)$, $l = 1, 2, \ldots, n$, $b < s < c$, *be a solution of* (E) *satisfying the* $n-1$ *boundary conditions obtained from*

$$(6) \qquad \begin{aligned} &u^{(j_0)}(b) = u^{(j_1)}(b) = \cdots = u^{(j_{k-1})}(b) = 0, \\ &u^{(j_k)}(s) = \cdots = u^{(j_{n-1})}(s) = 0, \end{aligned}$$

when the condition on $u^{(j_{l-1})}$ *is deleted, and normalized by*

$$(7) \qquad \begin{aligned} &u_l^{(j_{l-1})}(b, s) = \frac{\partial^{j_{l-1}}}{\partial x^{j_{l-1}}} u_l(x, s)|_{x=b} = 1 \qquad \text{if } l \leqq k, \\ &u_l^{(j_{l-1})}(s, s) = \frac{\partial^{j_{l-1}}}{\partial x^{j_{l-1}}} u_l(x, s)|_{x=s} = (-1)^{l-k-1} \qquad \text{if } l > k. \end{aligned}$$

Then u_l, $l = 1, 2, \ldots, n$, *has the following properties*:

$$(8) \qquad \begin{aligned} &u_l^{(i)} > 0 \text{ or } u_l^{(i)} < 0, i = 0, 1, \ldots, n-1, \\ &\operatorname{sgn} u_l^{(j_i)} = \operatorname{sgn} u_l^{(j_i+1)}, \quad i = 0, 1, \ldots, k-1, \\ &\operatorname{sgn} u_l^{(j_i)} = -\operatorname{sgn} u_l^{(j_i+1)}, i = k, \ldots, n-1 \end{aligned}$$

on (b, s) *for every* s, $b < s < c$.

PROOF. Since (E) is $(j_0, \ldots, j_{k-1}) - (j_k, \ldots, j_{n-1})$ disfocal on $[b, s]$, u_l is uniquely determined by the boundary conditions (6) and (7). In fact,

$$(9) \qquad \begin{aligned} &u_l(x, s) = \frac{W_l(x, s)}{W_l^{(j_{l-1})}(b, s)} \qquad \text{if } l \leqq k, \\ &u_l(x, s) = (-1)^{l-k-1} \frac{W_l(x, s)}{W_l^{(j_{l-1})}(s, s)} \qquad \text{if } l > k, \end{aligned}$$

where $W_l(x, s)$ is the $n \times n$ determinant formed from the array

$$\begin{array}{cccc} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1^{(j_0)}(b) & y_2^{(j_0)}(b) & \cdots & y_n^{(j_0)}(b) \\ & \cdots\cdots & & \\ y_1^{(j_{k-1})}(b) & y_2^{(j_{k-1})}(b) & \cdots & y_n^{(j_{k-1})}(b) \\ y_1^{(j_k)}(s) & y_2^{(j_k)}(s) & \cdots & y_n^{(j_k)}(s) \\ & \cdots\cdots & & \\ y_1^{(j_{n-1})}(s) & y_2^{(j_{n-1})}(s) & \cdots & y_n^{(j_{n-1})}(s) \end{array}$$

after deleting the row involving the j_{l-1}-th derivative, i.e., the $(l+1)$-th row, and $y_1, y_2, \ldots, y_n$ are a fundamental set of solutions of (E). Since $W_l^{(j_l-1)}(b, s) \neq 0$ if $l \leqq k$ and $W_l^{(j_l-1)}(s, s) \neq 0$ if $l > k$ for $b < s < c$, the determinant $W_l(x, s)$ cannot vanish identically on the x-interval $[b, s]$, $s \in (b, c)$. Moreover, it is easily seen from (9) that $u_l^{(j)}(x, s)$ is a continuous function of $s, j = 0, 1, \ldots, n-1, l = 1, 2, \ldots, n$.

We assert that $u_l^{(j)}, j = 0, 1, \ldots, n-1$, cannot vanish on (b, s). If this were not the case, $u_l^{(m)}$ for some m, $0 \leqq m \leqq n-1$, would have a zero in (b, s). Recalling the boundary conditions satisfied by u_l and repeatedly applying Rolle's theorem, if necessary, we conclude that $u_l^{(j_l-1)}$ has an odd-order zero $\xi \in (b, s)$, i.e., $u_l^{(j_l-1)}(\xi, s) = 0$ and $u_l^{(j_l-1)}(\xi + \varepsilon, s)$ $u_l^{(j_l-1)}(\xi - \varepsilon, s) < 0$ for some sufficiently small $\varepsilon > 0$. Since $u_l^{(j_l-1)}(x, s)$ is a continuous function of s, its odd-order zero ξ is also a continuous function of s. Move s towards b in a continuous manner. The odd-order zero ξ cannot disappear from the interval (b, s) without crossing the boundary point b or s as s approaches b. However, it cannot cross the boundary point $s[b]$ if $l \leqq k[l > k]$, for otherwise it would imply the existence of a solution $u_l(x, s_1)$ for some s_1 which violates the parity condition mentioned earlier. On the other hand, the zero ξ cannot cross $b[s]$ if $l \leqq k[l > k]$ because Equation (E) is $(j_0, j_1, \ldots, j_{k-1}) - (j_k, \ldots, j_{n-1})$ disfocal on $[b, c)$. Therefore, the zero ξ of $u_l^{(j_l-1)}$ must remain in the interval (b, s) until s coincides with b. This means that we can construct a sequence of solutions $u_l(x, s_m)$, $m = 1, 2, \ldots$, with $s_m \to b$ as $m \to \infty$ and $u_l^{(j_l-1)}(\xi_m, s_m) = 0$ for some $\xi_m \in (b, s_m)$. Evidently, this sequence can be normalized in such a way as to guarantee a nontrivial limit $u_l(x) \equiv \lim_{m\to\infty} u_l(x, s_m)$ (e.g., $c_{m1}^2 + \cdots + c_{mn}^2 = 1$, $u_l(x, s_m) = c_{m1}y_1 + \cdots + c_{mn}y_n$) with $u_l(b) = u_l'(b) = \cdots = u_l^{(n-1)}(b) = 0$. But this is absurd. Hence, $u_l^{(j)}, j = 0, 1, \ldots, n-1$, cannot vanish on (b, s).

The relation between the signs of $u_l^{(j)}$, $j = 0, 1, \ldots, n-1$, can be determined from the boundary conditions satisfied by u_l; for example, the condition $u_l^{(m)}(b, s) = 0$ implies that $\operatorname{sgn} u_l^{(m)}(b + \varepsilon, s) = \operatorname{sgn} u_l^{(m+1)}(b + \varepsilon, s)$ for any sufficiently small $\varepsilon > 0$, while the condition $u_l^{(m)}(s, s) = 0$ requires that $\operatorname{sgn} u_l^{(m)}(s - \varepsilon, s) = -\operatorname{sgn} u_l^{(m+1)}(s - \varepsilon, s)$. Since $u_l^{(j)}$, $j = 0, 1, \ldots, n-1$, does not vanish on (b, s) the above relations must hold throughout the interval (b, s), that is,

$$(10) \qquad \begin{aligned} &\operatorname{sgn} u_l^{(j_i)}(x, s) = \operatorname{sgn} u_l^{(j_i+1)}(x,s), \qquad i = 0, 1, \ldots, k-1, \\ &\operatorname{sgn} u_l^{(j_i)}(x,s) = -\operatorname{sgn} u_l^{(j_i+1)}(x, s), \qquad i = k, \ldots, n-1, \end{aligned}$$

$x \in (b, s)$, provided $i \neq l-1$. Now it only remains to show that (10) holds even for $i = l-1$, i.e.,

$$\operatorname{sgn} u_l^{(j_l-1)}(x, s) = \operatorname{sgn} u_l^{(j_{l-1}+1)}(x, s) \text{ if } l \leqq k$$

and

$$\operatorname{sgn} u_l^{(j_{l-1})}(x, s) = -\operatorname{sgn} u_l^{(j_{l-1}+1)}(x, s) \text{ if } l > k.$$

For the case $l \leqq k$, consider the number of sign changes in the sequence of the $n + 1$ functions

$$(11) \qquad u_l^{(j_{l-1}+1)}, u_l^{(j_{l-1}+2)}, \ldots, u_l^{(n)}, u_l, u_l', \ldots, u_l^{(j_{l-1})}.$$

Since $u_l^{(n)} = -pu_l$, there are $n - k$ sign changes if $p < 0$ and $n - k + 1$ sign changes if $p > 0$. Recalling the parity condition that $n - k$ is even or odd according as $p < 0$ or $p > 0$, we deduce that the total number of sign changes in (11) is even regardless of the sign of coefficient p. Thus, $\operatorname{sgn} u_l^{(j_{l-1})} = \operatorname{sgn} u_l^{(j_{l-1}+1)}$ if $l \leqq k$. On the other hand, if $l > k$, the number of sign changes in sequence (11) is $n - k - 1$ if $p < 0$ and $n - k$ if $p > 0$. Again from the parity condition, we conclude that the total number of sign changes in sequence (11) is odd, i.e., $\operatorname{sgn} u_l^{(j_{l-1})} = -\operatorname{sgn} u_l^{(j_{l-1}+1)}$ if $l > k$. This establishes (8) and completes the proof.

Suppose that equation (E) is $(j_0, \ldots, j_{k-1}) - (j_k, \ldots, j_{n-1})$ desfocal on $[b, \infty)$ for some $b \geqq a$. Let $\{s_m\}$ be a sequence of real numbers such that $s_m \to \infty$ as $m \to \infty$, and put

$$u_l(x, s_m) = \sum_{i=1}^{n} A_{lmi} y_i, \qquad l = 1, 2, \ldots, n,\ m = 1, 2, \ldots,$$

where $y_1, \ldots, y_n$ are a fundamental set of solutions of (E). Define

$$(12) \qquad v_l(x, s_m) = \frac{u_l(x, s_m)}{\left(\sum_{i=1}^{n} A_{lmi}^2\right)^{1/2}} = \sum_{i=1}^{n} B_{lmi}\, y_i, \qquad \sum_{i=1}^{n} B_{lmi}^2 = 1.$$

There exists a subsequence $\{v_l(x, s_{m_k})\}$ which converges to a nontrivial limit $v_l(x)$. If we denote the subsequence $\{v_l(x, s_{m_k})\}$ again by $\{v_l(x, s_m)\}$ for brevity,

$$(13) \qquad \begin{aligned} v_l(x) &= \lim_{m\to\infty} v_l(x, s_m) \\ &= \sum_{i=1}^{n} B_{li}\, y_i, \qquad \sum_{i=1}^{n} B_{li}^2 = 1, l = 1, 2, \ldots, n. \end{aligned}$$

Since $u_l^{(j)}(x, s_m)$ cannot vanish in (b, s_m) by Theorem 1, we have $v_l^{(j)}(x) \geqq 0$ or $v_l^{(j)}(x) \leqq 0$, $x \in [b, \infty)$, $j = 0, 1, \ldots, n - 1$, which implies that $v_l^{(j)}$ is monotone on $[b, \infty)$, $j = 0, 1, \ldots, n - 1$. Also note that $v_l^{(j)}$ cannot vanish identically in any subinterval of $[a, \infty)$ because v_l is a nontrivial solution of (E). Hence, $v_l^{(j)}$ cannot vanish at all in (b, ∞), i.e., $v_l^{(j)} > 0$ or $v_l^{(j)} < 0$ in (b, ∞), $j = 0, 1, \ldots, n - 1$, $l = 1, 2, \ldots, n$. Moreover, since $v_l(x, s_m)$, $m = 1, 2, \ldots$, satisfies the sign conditions in (8) in (b, s_m), the limit function $v_l(x)$ also satisfies the same sign conditions in(b, ∞). We summarize this result in the following lemma.

LEMMA 2. *If equation* (E) *is eventually* $(j_0, \ldots, j_{k-1}) - (j_k, \ldots, j_{n-1})$ *disfocal on* $[a, \infty)$, *the solution* v_l, $l = 1, 2, \ldots, n$, *defined in* (13) *has the following properties*:

$$v_l^{(j_i)}(b) = 0,\ i = 0, 1, \ldots, k-1,$$

where $v_l^{(j_l-1)}(b) = 0$ *is deleted when* $l \leqq k$,

$$v_l^{(i)} > 0 \text{ or } v_l^{(i)} < 0,\ i = 0, 1, \ldots, n-1,$$

$$\operatorname{sgn} v_l^{(j_i)} = \operatorname{sgn} v_l^{(j_i+1)},\ i = 0, 1, \ldots, k-1,$$

and

$$\operatorname{sgn} v_l^{(j_i)} = -\operatorname{sgn} v_l^{(j_i+1)},\ i = k, \ldots, n-1,$$

in the interval (b, ∞).

From Lemma 1 and Lemma 2, we easily obtain the following result.

THEOREM 2. *If equation* (E) *is eventually* $(j_0, \ldots, j_{k-1}) - (j_k, \ldots, j_{n-1})$ *disfocal on* $[a, \infty)$, *then* $j_i = i$, $i = 0, 1, \ldots, n-1$.

In view of the above theorem we only need to consider the case where equation (E) is eventually $k - (n-k)$ disfocal on $[a, \infty)$. For this case, we obtain the following statements from Lemma 2.

THEOREM 3. *If equation* (E) *is eventually* $k - (n-k)$ *disfocal on* $[a, \infty)$, *the solution* v_l, $l = 1, 2, \ldots, n$, *defined in* (13) *has the following properties*:

$$v_l^{(i)}(b) = 0,\ i = 0, 1, \ldots, k-1,$$

where $v_l^{(l-1)}(b) = 0$ is *deleted when* $l \leq k$,

$$v_l^{(i)} > 0,\ i = 0, 1, \ldots, k-1,$$

$$(-1)^{i-k} v_l^{(i)} > 0,\ i = k, \ldots, n-1,$$

on the interval (b, ∞).

We turn to the problem of determining the number of solutions belonging to class A_j. This problem has been studied by the author **[8]** under the assumption that (E) is nonoscillatory on $[a, \infty)$ (i.e., every nontrivial solution has a finite number of zeros on $[a, \infty)$). We shall determine the maximum number, $q(A_j)$, of solutions $y_1, \ldots, y_m$ belonging to A_j such that every nontrivial linear combination of $y_1, \ldots, y_m$ again belongs to A_j. In Theorem 5 we prove, among other results, that $q(A_j) = 0$ or 2, $j = 0, 1, \ldots, (n-2)/2$, for (E_i). For the proof of this result, it suffices to establish that $q(A_j) = 2$ if A_j is nonempty. But the nonemptiness of A_j is tied to disfocality: (E_i) is $k - (n-k)$ disfocal on $[b, \infty)$ if and only if $A_{[k/2]}$ is nonempty, provided (P) holds on $[b, \infty)$ (Cf. Theorem 3 and **[4,** Theorem 2]). Hence, all we need to prove is that $q(A_{[k/2]}) = 2$ if (E_i) is $k - (n-k)$ disfocal. This will be done by using the solutions v_k and v_{k+1} defined in (13).

Evidently, for constants A and B,

$$Av_k(x) + Bv_{k+1}(x) = \lim_{m\to\infty} (Av_k(x, s_m) + Bv_{k+1}(x, s_m)).$$

If (E) is $k - (n - k)$ disfocal on $[b, \infty)$, in view of (12) and recalling the definition of $u_l(x, s_m)$ in (6) with $j_i = i$, $i = 0, 1, \ldots, n - 1$, we see that $y \equiv Av_k(x, s_m) + Bv_{k+1}(x, s_m)$ satisfies $y(b) = y'(b) = \cdots = y^{(k-2)}(b) = 0 = y^{(k+1)}(s_m) = \cdots = y^{(n-1)}(s_m)$ for all m, A, and B. For the solution y we have the following theorem.

THEOREM 4. *If* (E) *is* $k - (n - k)$ *disfocal on* $[b, c)$, $b \geqq a$, *every nontrivial solution* y *satisfying the* $n - 2$ *boundary conditions with* $b < s < c$,

$$(14) \qquad \begin{aligned} &y(b) = y'(b) = \cdots = y^{(k-2)}(b) = 0, \\ &y^{(k+1)}(s) = \cdots = y^{(n-1)}(s) = 0, \end{aligned}$$

has the following properties:

(a) *If* $y^{(k-1)}(\alpha) = y^{(k)}(\beta) = 0$ *for some* $\alpha, \beta \in [b, s]$, *then* $\alpha > \beta$.

(b) *The functions* $y, y', \ldots, y^{(k-1)}, y^{(k+1)}, \ldots, y^{(n)}$ *can have at most one zero and* $y^{(k)}$ *at most two zeros on* (b, s), *counting multiplicities. Furthermore, if* $y > 0$ *on* (b, s), *then*

$$y^{(k+1)} < 0, \quad y^{(k+2)} > 0, \quad \ldots, \quad (-1)^{n-k-1}y^{(n-1)} > 0,$$

on (b, s).

PROOF. Using (14) and Rolle's theorem repeatedly, we can easily show that (a) cannot hold if (b) does not hold. Hence, it suffices to prove (a). When s is sufficiently close to b, $y^{(k-1)}$ and $y^{(k)}$ cannot both vanish in (b, s). If this were not the case, we could easily construct a nontrivial solution Y with $Y(b) = Y'(b) = \cdots = Y^{(n-1)}(b) = 0$, which is absurd. Thus, (a) is trivially satisfied if s is sufficiently close to b.

Define $H = \{t \mid$ every nontrivial solution of (E) satisfying (14) with $s \leqq t$ has property (a)$\}$ and let $d = \sup H$. The proof of the first part will be complete if we can show that $d \geqq c$. Assume the contrary: $d < c$. Let $\{\tau_n\}$, $d < \tau_n$, $n = 1, 2, \ldots$, be a sequence of real numbers such that $\tau_n \to d$ as $n \to \infty$. Since $\tau_n \notin H$, there exists a nontrivial solution w_n of (E) such that

$$w_n(b) = w_n'(b) = \cdots = w_n^{(k-2)}(b) = 0 = w_n^{(k+1)}(\sigma_n) = \cdots = w_n^{(n-1)}(\sigma_n)$$

for some σ_n, $d < \sigma_n < \tau_n$, and

$$w_n^{(k-1)}(\alpha_n) = w_n^{(k)}(\beta_n) = 0$$

for some α_n and β_n, $b \leqq \alpha_n \leqq \beta_n \leqq \sigma_n$. Evidently, the sequence of solutions $\{w_n\}$ may be so normalized as to guarantee a subsequence converging to a nontrivial limit w. The limit w is a solution of (E) satisfying

$$w(b) = w'(b) = \cdots = w^{(k-2)}(b) = 0 = w^{(k+1)}(d) = \cdots = w^{(n-1)}(d),$$
$$w^{(k-1)}(\alpha) = w^{(k)}(\beta) = 0$$

for some α, β, $b \leqq \alpha \leqq \beta \leqq d$. If $b < \alpha < d$ [$b < \beta < d$], then $\beta \neq d$ [$b \neq \alpha$] by Theorem 1 since (E) is $k - (n - k)$ disfocal on $[b, d]$. Therefore, if $b < \alpha < d$ or $b < \beta < d$, then either (A) $b < \alpha < \beta < d$ or (B) $b < \alpha = \beta < d$. In case (A) we shall prove that $\beta[\alpha]$ cannot be an odd-order zero of $w^{(k)}[w^{(k-1)}]$. Define

$$(15) \qquad \omega(x; \alpha, d) \equiv \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1(b) & y_2(b) & \cdots & y_n(b) \\ & \cdots\cdots & & \\ y_1^{(k-2)}(b) & y_2^{(k-2)}(b) & \cdots & y_n^{(k-2)}(b) \\ y_1^{(k-1)}(\alpha) & y_2^{(k-1)}(\alpha) & \cdots & y_n^{(k-1)}(\alpha) \\ y_1^{(k+1)}(d) & y_2^{(k+1)}(d) & \cdots & y_n^{(k+1)}(d) \\ & \cdots\cdots & & \\ y_1^{(n-1)}(d) & y_2^{(n-1)}(d) & \cdots & y_n^{(n-1)}(d) \end{vmatrix},$$

where $y_1, \ldots, y_n$ are a fundamental system of solutions of (E). The determinant $\omega(x; \alpha, d)$ does not vanish identically because $d^k/dx^k\omega(x; \alpha, d)\,|_{x=d} \neq 0$ by Theorem 1 and $w(x) = K\omega(x; \alpha, d)$ for some constant K. If β were an odd-order zero of $w^{(k)}$, due to continuous dependence of $\omega(x; \alpha, d)$ and its derivatives on d, there would exist $\varepsilon > 0$ and β_1, $\alpha < \beta_1 < d - \varepsilon$, such that the solution $w_1(x) \equiv \omega(x; \alpha, d - \varepsilon)$ would satisfy

$$(16) \qquad \begin{gathered} w_1(b) = \cdots = w_1^{(k-2)}(b) = 0 = w_1^{(k+1)}(d - \varepsilon) = \cdots = w_1^{(n-1)}(d - \varepsilon), \\ w_1^{(k-1)}(\alpha) = w_1^{(k)}(\beta_1) = 0. \end{gathered}$$

But this contradicts the choice of d, and therefore, β cannot be an odd-order zero of $w^{(k)}$. Similarly, we may prove that α cannot be an odd-order zero of $w^{(k-1)}$. Hence, if $w^{(k-1)}(\alpha) = w^{(k)}(\beta) = 0, b < \alpha < \beta < d$, then α cannot be an odd-order zero of $w^{(k-1)}$ and β cannot be an odd-order zero of $w^{(k)}$. On the other hand, both α and β cannot be even-order zeros of $w^{(k-1)}$ and $w^{(k)}$, respectively. If $w^{(k-1)}(\alpha) = w^{(k)}(\alpha) = 0$ and $w^{(k)}(\beta) = w^{(k+1)}(\beta) = 0$, then $w^{(k+1)}(\gamma) = 0$ for some γ, $\alpha < \gamma < \beta$, that is, $w^{(k+1)}$ has three distinct zeros on $(b, d]$. Again by a repeated application of Rolle's theorem, we conclude that $w^{(n)} = -pw$ has two distinct zeros on (b, d) and eventually that $w^{(k)}$ has an odd-order zero between two distinct zeros of $w^{(k-1)}$. But we showed earlier that this is impossible. Consequently, case (A) $b < \alpha < \beta < d$ cannot hold.

In case (B) $b < \alpha = \beta < d$, we may assume that w has at most one zero on $(b, d]$. Furthermore, α as a zero of $w^{(k-1)}$ is a zero of order at most 3 and $w^{(k-1)}$ has no other zeros on $[b, d]$—for, otherwise, we would again be led to the conclusion that there exists an odd-order zero of $w^{(k)}$ between two distinct zeros of $w^{(k-1)}$. Hence, there are two possibilities:

(I) $w^{(k-1)}(\alpha) = w^{(k)}(\alpha) = 0,\ w^{(k+1)}(\alpha) \neq 0$

(II) $w^{(k-1)}(\alpha) = w^{(k)}(\alpha) = w^{(k+1)}(\alpha) = 0,\ w^{(k+2)}(\alpha) \neq 0.$

If (I) holds, the double zero α of $w^{(k-1)}$ must separate into two simple zeros of $w^{(k-1)}$ as d in $\omega(x; \alpha, d)$ defined by (15) moves toward b. But this requires the existence of a solution satisfying (16), again contradicting the choice of d. Assume that (II) holds. We shall prove that $w(\gamma) = 0$ for some $\gamma \in (b, d)$. If $k = n - 1$ (and $p > 0$), then $0 = w^{(k+1)}(\alpha) = -p(\alpha)w(\alpha)$ yields $w(\alpha) = 0$. If $k \neq n - 1, w > 0$ on (b, d), and $n - k$ is odd [even];

$$w^{(n-1)}(d) = w^{(n-1)}(\zeta) - \int_{\zeta}^{d} p(t)w(t)dt, \qquad \zeta \in [b, d),$$

$w^{(n-1)}(d) = 0$, and $p > 0\,[p < 0]$ require that $w^{(n-1)} > 0\,[w^{(n-1)} < 0]$ on $[b, d)$. Similarly,

$$\text{(17)} \qquad w^{(n-j)}(d) = w^{(n-j)}(\zeta) + \int_{\zeta}^{d} w^{(n-j+1)}(t)dt, \qquad \zeta \in [b, d),$$

and $w^{(n-j)}(d) = 0, j = 2, 3, \ldots, n - k - 1$, require that $(-1)^j w^{(n-j)} < 0$ $[(-1)^j w^{(n-j)} > 0]$ in $[b, d), j = 2, 3, \ldots, n - k - 1$. (This result proves the second part of the theorem when w and d are replaced by y and s, respectively.) In particular, for $j = n - k - 1$, $w^{(k+1)} < 0$ on $[b, d)$. Therefore, (II) and the assumption that $w > 0$ on (b, d) are incompatible.

The only remaining possibility is that w has exactly one zero $\gamma \in (b, d)$ such that $w > 0$ on (b, γ) and $w < 0$ on (γ, d) (take $-w$ if necessary). We shall show that this too is impossible. If $n - k$ is odd [even], then $p > 0$ $[p < 0]$, $w^{(n)} = -pw < 0\,[> 0]$ on (b, γ) and $w^{(n)} > 0\,[< 0]$ on (γ, d). If $k = n - 1$, then $p > 0$ and $w^{(n-2)}(\alpha) = w^{(n-1)}(\alpha) = w^{(n)}(\alpha) = 0$ by (II); thus, $w(\alpha) = 0$, i.e., $\alpha = \gamma$ and $w^{(n-1)} > 0$ on $(b, \gamma) \cup (\gamma, d)$. Since $w^{(n-2)}(\gamma) = 0, b < \gamma < d$, we must have $w^{(n-2)} < 0$ on $[b, \gamma)$, which in view of $w(b) = w'(b) = \cdots = w^{(n-3)}(b) = 0$ implies that $w < 0$ on $(b, b + \varepsilon)$ for some $\varepsilon > 0$. But this contradicts our assumption that $w > 0$ on (b, γ). If $k \neq n - 1$, then $w^{(n-1)}(d) = 0$ and either $w^{(n-1)} < 0\,[> 0]$ on (b, d) or else there exists $c_{n-1} \in (b, d)$ such that $w^{(n-1)} > 0\,[< 0]$ on (b, c_{n-1}) and $w^{(n-1)} < 0$ $[> 0]$ on (c_{n-1}, d). The first alternative is impossible for it would again lead to the conclusion that $w^{(k+1)}$ does not vanish on $[b, d)$, contradicting (II), when (17) is used successively. If the second alternative holds, we may repeat a similar argument using $w^{(k+1)}(d) = \cdots = w^{(n-2)}(d) = 0$ and prove successively that there exists $c_{n-j} \in (b, d)$ such that $(-1)^j w^{(n-j)} < 0$ $[> 0]$ on (b, c_{n-j}) and $(-1)^j w^{(n-j)} > 0\,[< 0]$ on $(c_{n-j}, d), j = 2, 3, \ldots,$ $n - k - 1$. In particular, for $j = n - k - 1$, we have $w^{(k+1)} < 0$ on (b, α) and $w^{(k+1)} > 0$ on (α, d). Hence, $w^{(k)} \geqq 0$ on $[b, d]$, $w^{(k)}(\alpha) = w^{(k+1)}(\alpha) = 0$, $w^{(k+2)}(\alpha) \neq 0$, and $w^{(k)}$ has no other zeros on $[b, d]$, and this in turn implies that $w^{(k-1)} < 0$ on $[b, \alpha)$, $w^{(k-1)}(\alpha) = 0$, and $w^{(k-1)} > 0$ on $(\alpha, d]$. Since $w(b) = w'(b) = \cdots = w^{(k-2)}(b) = 0$, the inequality $w^{(k-1)} < 0$ on $[b, \alpha)$ requires that $w < 0$ on $(b, b + \varepsilon)$ for some $\varepsilon > 0$, contrary to our assumption that $w > 0$ on (b, γ). Consequently, (II) cannot hold and therefore (B) cannot hold. Hence, $\alpha \notin (b, d)$ and $\beta \notin (b, d)$. But $b \leqq \alpha \leqq \beta \leqq d$, and we must have $\alpha = b$ or $\alpha = d$, and $\beta = b$ or $\beta = d$. Obviously, $b = \alpha = \beta$ and $\alpha = \beta = d$ are impossible because they violate the parity condition [10]. Since $\alpha \leqq \beta$, we see that $b = \alpha$ and $\beta = d$. But this means that (E) is not $k - (n - k)$ disfocal on $[b, d]$, contrary to the assumption $d < c$. This completes the proof.

We are now ready to determine the number of solutions belonging to class $A_{[k/2]}$; in fact, we shall prove that $q(A_{[k/2]}) = 2$ if (E) is $k - (n - k)$ disfocal on $[b, \infty)$ for some $b \geqq a$. Take the two solutions v_k and v_{k+1} defined in (13). These solutions belong to $A_{[k/2]}$ by Theorem 3. However, v_k and v_{k+1} may or may not be linearly independent. First assuming that v_k and v_{k+1} are linearly independent, we shall establish that every nontrivial linear combination belongs to $A_{[k/2]}$. Every nontrivial linear combination of v_k and v_{k+1} is nonoscillatory on $[b, \infty)$. This is because for constants A and B,

$$Av_k(x) + Bv_{k+1}(x) = \lim_{m\to\infty} [Av_k(x, s_m) + Bv_{k+1}(x, s_m)],$$

and $w_m \equiv Av_k(x, s_m) + Bv_{k+1}(x, s_m)$ is subject to (b) of Theorem 4 in the interval (b, s_m), $m = 1, 2, \ldots$ Furthermore, no linear combination can belong to A_j, $j > [k/2]$ (Cf. Remark following Theorem 5). In view of Lemma 1, it suffices to prove that $w \equiv Av_k + Bv_{k+1}$ cannot belong to A_j, $j < [k/2]$, for any constants $A \neq 0$ and $B \neq 0$. Suppose that $w \in A_j$ for some $j < [k/2]$ and that $w > 0$ on $[b_1, \infty)$ for some $b_1 \geqq b$. Then $w^{(k-2)} > 0$, $w^{(k-1)} < 0$, $w^{(k)} > 0$, on $[b_1, \infty)$ by Lemma 1. Since $w^{(i)} = \lim_{m\to\infty} w_m^{(i)}$, $i = 0, 1, \ldots, n - 1$, we have for sufficiently large l, $w_l^{(k-2)} > 0$, $w_l^{(k-1)} < 0$, $w_l^{(k)} > 0$ in some subinterval (ξ, η) of $(b, s_l) \cap [b_1, \infty)$. We also note that $w_l^{(k-1)}(b) = Av_k^{(k-1)}(b, s_l) \neq 0$. If $w_l^{(k-1)}(b) > 0$, it is incompatible with the inequalities $w_l^{(k-1)} < 0$ and $w_l^{(k)} > 0$ in (ξ, η); it is easily seen in this case that $w_l^{(k-1)}(\alpha) = w_l^{(k)}(\beta) = 0$ for some α, $\beta \in (b, s_l)$, $\alpha < \beta$, contrary to Theorem 4. On the other hand, if $w_l^{(k-1)}(b) < 0$, we again obtain a contradiction. Since $w_l^{(k-2)}(b) = 0$ and $w_l^{(k-2)} > 0$ and $w_l^{(k-1)} < 0$ on (ξ, η), $w_l^{(k-1)}$ must have at least two zeros on (b, s_l). But this contradicts Theorem 4 and completes the proof that $w \in A_{[k/2]}$ if v_k and v_{k+1} are linearly independent.

If v_k and v_{k+1} and linearly dependent, $v_k = Cv_{k+1}$ for some constant C. Since $v_k > 0$ and $v_{k+1} > 0$ on (b, ∞) by Theorem 3, the constant C must be positive; hence it follows from (12) and (13) that $C = 1$. Consequently,

$$(18)\qquad \begin{aligned} B_{ki} &= B_{k+1,i}, \qquad i = 1, 2, \ldots, n, \\ B_{li} &= \lim_{m\to\infty} B_{lmi}, \qquad l = k, k+1. \end{aligned}$$

Define a sequence $\{g_m\}$ of solutions by

$$(19)\qquad g_m = \frac{v_k(x, s_m) - v_{k+1}(x, s_m)}{\left[\sum_{i=1}^{n} (B_{kmi} - B_{k+1,mi})^2\right]^{1/2}}, \; m = 1, 2, \ldots,$$

and let w be the nontrivial limit of a converging subsequence $\{g_{m_i}\}$ which we again denote by $\{g_m\}$ for brevity, i.e., $g = \lim_{m\to\infty} g_m$. We assert that g and $v_{k+1} = \lim_{m\to\infty} v_{k+1}(x, s_m)$ are linearly independent. In view of (12), (13) and (19),

$$g_m(x) = \sum_{i=1}^{n} c_{mi} y_i, \qquad c_{mi} = \frac{B_{kmi} - B_{k+1,mi}}{\left[\sum_{i=1}^{n} (B_{kmi} - B_{k+1,mi})^2\right]^{1/2}}, \qquad \begin{aligned} m &= 1, 2, \ldots, \\ i &= 1, 2, \ldots, n, \end{aligned}$$

and it suffices to show that

$$\sum_{i=1}^{n} \left(\lim_{m\to\infty} c_{mi}\right)\left(\lim_{m\to\infty} B_{k+1,mi}\right) \neq \pm 1.$$

Indeed,

$$\sum_{i=1}^{n} \left(\lim_{m\to\infty} c_{mi}\right)\left(\lim_{m\to\infty} B_{k+1,mi}\right) = \lim_{m\to\infty} \sum_{i=1}^{n} c_{mi} B_{k+1,mi}$$

$$= \lim_{m\to\infty} \frac{\sum_{i=1}^{n} B_{kmi} B_{k+1,mi} - 1}{\left[2 - 2\sum_{i=1}^{n} B_{kmi} B_{k+1,mi}\right]^{1/2}} = -\lim_{m\to\infty} \left[\frac{1 - \sum_{i=1}^{n} B_{kmi} B_{k+1,mi}}{2}\right]^{1/2} = 0$$

due to (18), i.e., g and v_{k+1} are "orthogonal." In any case, g and v_{k+1} are linearly independent.

Evidently, g is nonoscillatory on $[b, \infty)$ because $g = \lim_{m\to\infty} g_m$ and g_m is subject to the conditions in (*b*) of Theorem 4 on the interval (b, s_m). Furthermore, we shall show that $g \in A_{[k/2]}$. Suppose that $g \in A_l$ for some l, $\lambda \geqq l > [k/2]$, where $\lambda = [(n-1)/2]$ for (E_i) and (E_{iv}) and $\lambda = [n/2]$ for (E_{ii}) and (E_{iii}). Then it follows from Lemma 1 that $g > 0, g' > 0, \ldots, g^{(k+2)} > 0$, or $g < 0, g' < 0, \ldots, g^{(k+2)} < 0$, on $[\gamma, \infty)$ for some $\gamma > b$. If I is a finite subinterval of $[\gamma, \infty)$, there exists N such that $m > N$ implies

$$(20)\qquad g_m > 0, g_m' > 0, \ldots, g_m^{(k+2)} > 0,$$

or

$$(21)\qquad g_m < 0, \; g_m' < 0, \; \ldots, \; g_m^{(k+2)} < 0,$$

in $I \cap (b, s_m)$, since $g^{(i)} = \lim_{m\to\infty} g_m^{(i)}$, $i = 0, 1, \ldots, n-1$, uniformly in any finite subinterval of $[b, \infty)$. In view of (6), (7), (12), and (19), we have

$$g_m(b) = g_m'(b) = \cdots = g_m^{(k-2)}(b) = 0, g_m^{(k-1)}(b) > 0, \tag{22}$$

$$g_m^{(k)}(s_m) < 0, g_m^{(k+1)}(s_m) = \cdots = g_m^{(n-1)}(s_m) = 0; \tag{23}$$

hence by Theorem 4, g_m can have at most one zero in (b, s_m). Due to (22) there exists $\varepsilon > 0$ such that $g_m > 0$ in $(b, b+\varepsilon)$, and if g_m does not vanish in (b, s_m), then $g_m^{(i)}$, $i = k+1, \ldots, n-1$, cannot have a zero in (b, s_m) (for otherwise a repeated application of Rolle's theorem leads to the contradiction that g_m vanishes at some point of (b, s_m)). In addition we deduce from (E) and (23) that

$$g_m^{(k+1)} < 0, g_m^{(k+2)} > 0, \ldots, (\operatorname{sgn} p) g_m^{(n)} < 0 \tag{24}$$

in (b, s_m). However, the first two inequalities are incompatible with (20) and with (21). If g_m has a zero in (b, s_m), $g_m < 0$ in $(s_m - \varepsilon_1, s_m)$ for some $\varepsilon_1 > 0$ because $g_m > 0$ in $(b, b+\varepsilon)$ by (22) and g_m can have at most one zero (counting multiplicities) in (b, s_m). Thus, in $(s_m - \varepsilon_2, s_m)$ for some $\varepsilon_2 > 0$,

$$g_m^{(k)} < 0, g_m^{(k+1)}, > 0, g_m^{(k+2)} < 0, \ldots, (\operatorname{sgn} p) g_m^{(n)} > 0, \tag{25}$$

where the first inequality follows from $g_m^{(k)}(s_m) < 0$ in (23). If (20) holds, $g_m^{(k)} > 0$, $g_m^{(k+1)} > 0$ in $I \cap (b, s_m)$, while $g_m^{(k)} < 0$, $g_m^{(k+1)} > 0$ in $(s_m - \varepsilon_2, s_m)$ by (25). These four inequalities together imply that $g_m^{(k+1)}$ has at least two zeros in (b, s_m), contradicting Theorem 4. On the other hand, if (21) holds we take $g_m^{(k+1)} < 0, g_m^{(k+2)} < 0$ from (21) and $g_m^{(k+1)} > 0, g_m^{(k+2)} < 0$ from (25), and similarly conclude that $g_m^{(k+2)}$ has at least two zeros in (b, s_m), again contradicting Theorem 4. Consequently, $g \notin A_l$, $\lambda \geqq l > [k/2]$.

Next we prove that $g \notin A_l$, $0 \leqq l < [k/2]$. Assume that $g \in A_l$ for some l, $0 \leqq l < [k/2]$. Then Lemma 1 requires that $g^{(k-2)} > 0$, $g^{(k-1)} < 0$, $g^{(k)} > 0$, or $g^{(k-2)} < 0$, $g^{(k-1)} > 0$, $g^{(k)} < 0$, on $[\eta, \infty)$ for some $\eta > b$, according as $g > 0$ or $g < 0$ on $[\eta, \infty)$. Hence, as before, for any finite subinterval J of $[\eta, \infty)$ there exists N_1 such that $m > N_1$ implies

$$g_m^{(k-2)} > 0, g_m^{(k-1)} < 0, g_m^{(k)} > 0, \tag{26}$$

or

$$g_m^{(k-2)} < 0, g_m^{(k-1)} > 0, g_m^{(k)} < 0, \tag{27}$$

in $J \cap (b, s_m)$. Choose $m > N_1$. Due to (22), (23), and Theorem 4, $g_m^{(k-1)}$ can have at most one zero in (b, s_m). If $g_m^{(k-1)}$ does not vanish in (b, s_m), then $g_m^{(i)}$, $i = 0, 1, \ldots, k-1$, cannot vanish in (b, s_m) and

$$g_m > 0, g_m' > 0, \ldots, g_m^{(k-1)} > 0 \tag{28}$$

in (b, s_m) by (22). But (26) as well as (27) is incompatible with (28). If $g_m^{(k-1)}$ has a zero at $\zeta \in (b, s_m)$, $g_m^{(k-1)}$ cannot vanish in (b, ζ) and

$$(29) \qquad g_m > 0, g_m' > 0, \ldots, g_m^{(k-1)} > 0$$

in (b, ζ). If (26) holds, $g_m^{(k-1)} < 0, g_m^{(k)} > 0$ in $J \cap (b, s_m)$; while $g_m^{(k-1)}(b) > 0$ by (22). These inequalities, however, require that $g_m^{(k-1)}(\alpha) = g_m^{(k)}(\beta) = 0$ for some $\alpha, \beta \in (b, s_m)$ with $\alpha < \beta$, contradicting Theorem 4. If, on the other hand, (27) holds, we take two inequalities $g_m^{(k-2)} < 0$ and $g_m^{(k-1)} > 0$ valid in $J \cap (b, s_m)$ and two inequalities $g_m^{(k-2)} > 0$ and $g_m^{(k-1)} > 0$ from (29) which are valid in (b, ζ), and similarly conclude that $g_m^{(k-1)}$ must have at least two zeros in (b, s_m). This also contradicts Theorem 4, and completes the proof that $g \notin A_l$, $0 \leqq l < [k/2]$. Since Lemma 1 states that $g \in A_l$ for some l, $0 \leqq l \leqq \lambda$, where $\lambda = [(n-1)/2]$ for (E_i) and (E_{iv}) and $\lambda = [n/2]$ for (E_{ii}) and (E_{iii}), we deduce that $g \in A_{[k/2]}$.

Every nontrivial linear combination of g and v_{k+1} belongs to $A_{[k/2]}$. The proof of this assertion is obtained when v_k is replaced by g in the earlier proof that every nontrivial linear combination of v_k and v_{k+1} belongs to $A_{[k/2]}$ if v_k and v_{k+1} are linearly independent. Summarizing the results so far obtained, we have that $q(A_{[k/2]}) \geqq 2$ if equation (E) is $k-(n-k)$ disfocal on $[b, \infty)$ for some $b \geqq a$.

Now it only remains to show that $A_{[k/2]}$ cannot contain more than two solutions of which every nontrivial linear combination again belongs to $A_{[k/2]}$. The required proof is essentially the same as the proof of the Theorem in [**8**]. For the sake of completeness, however, it will be presented here. Assume to the contrary that $A_{[k/2]}$ contains three solutions Y_1, Y_2, and Y_3 such that every nontrivial linear combination of Y_1, Y_2, and Y_3 belongs to $A_{[k/2]}$. According to Lemma 2 in [**8**], we may assume that $Y_3 > Y_2 > Y_1 > 0$ on $[c, \infty)$ for some $c \geqq b$ and

$$\lim_{x\to\infty} \frac{Y_k(x)}{Y_j(x)} = \infty, 1 \leqq j < k \leqq 3.$$

Let $\{\eta_i\}$ be an increasing sequence of numbers such that $\eta_i \geqq c$ and $\eta_i \to \infty$ as $i \to \infty$. By virtue of Lemma 3 in [**8**] there exists for each i, a solution

$$V_i = \alpha_i Y_1 + \beta_i Y_2 + \gamma_i Y_3, \qquad \alpha_i^2 + \beta_i^2 + \gamma_i^2 = 1,$$

such that $V_i \geqq 0$ on $[c, \infty)$ and $V_i(\zeta_i) = V_i'(\zeta_i) = 0$ for some $\zeta_i \in (\eta_i, \infty)$. Put

$$\lim_{i\to\infty} \alpha_i = \alpha, \lim_{i\to\infty} \beta_i = \beta, \lim_{i\to\infty} \gamma_i = \gamma$$

(take subsequences if necessary). Then $W(x) \equiv \alpha Y_1(x) + \beta Y_2(x) + \gamma Y_3(x)$ is a nonoscillatory solution belonging to class $A_{[k/2]}$. Since $W \geqq 0$ in $[c, \infty)$, we have

$$W > 0,\ W' > 0,\ \ldots,\ W^{(k-1)} > 0 \tag{30}$$

on $[c_1, \infty)$ for some $c_1 \geqq c$ by Lemma 1. We remark that k is odd for (E_i) and (E_{iv}) and even for (E_{ii}) and (E_{iii}). Since $\lim_{i\to\infty} V_i^{(j)} = W^{(j)}, j = 0, 1, \ldots, n$, uniformly in any finite subinterval of $[c, \infty)$, there exists a number N such that $i > N$ implies

$$V_i^{(j)}(c_1) > \frac{W^{(j)}(c_1)}{2} > 0, \qquad j = 0, 1, \ldots, k-1. \tag{31}$$

We may assume that $\eta_i > c_1$ for $i > N$. Since $V_i \in A_{[k/2]}$ and $V_i \geqq 0$ in $[c, \infty]$ for all i, $V_i^{(k)} > 0$ on $[c, \infty)$ by Lemma 1. This means that

$$V_i^{(k-1)}(c_1) \leqq V_i^{(k-1)}(\tau), \qquad \tau \in [c_1, \infty),$$

which may be combined with (31) to get

$$V_i^{(k-1)}(\tau) > \frac{W^{(k-1)}(c_1)}{2}, \qquad \tau \in [c_1, \infty). \tag{32}$$

When this inequality is integrated from c_1 to $x \in [c_1, \infty)$ and (31) with $j = k - 2$ substituted in the resulting expression, we obtain

$$V_i^{(k-2)}(x) > \frac{W^{(k-1)}(c_1)}{2}(x - c_1) + \frac{W^{(k-2)}(c_1)}{2}.$$

If we repeat a similar procedure $k - 2$ times, we finally arrive at the inequality

$$\begin{aligned} V_i(x) > \frac{W^{(k-1)}(c_1)}{2(k-1)!}(x-c_1)^{k-1} + \frac{W^{(k-2)}(c_1)}{2(k-2)!}(x-c_1)^{k-2} + \cdots \\ + \frac{W(c_1)}{2}, \qquad x \in [c_1, \infty). \end{aligned} \tag{33}$$

However, this inequality cannot hold throughout the interval $[c_1, \infty)$. In fact, for $x = \zeta_i > \eta_i > c_1 (i > N)$, the left-hand side $V_i(\zeta_i) = 0$, while the right-hand side is positive by (30). This contradiction proves that $q(A_{[k/2]}) = 2$ if (E) is $k - (n - k)$ disfocal and (P) holds on $[b, \infty)$, $b \geqq a$. On the other hand, if (E) is not $k - (n - k)$ disfocal, then $A_{[k/2]}$ is empty [**4**, Theorem 2], i.e., $q(A_{[k/2]}) = 0$.

Since k is even for (E_{ii}) and (E_{iii}), the class A_0 of (E_{ii}) and (E_{iii}) is not included in the above consideration. Likewise, the class $A_{[n/2]}$ of (E_{iii}) and (E_{iv}) has to be considered separately. In this connection we have $q(A_0) = 1$ for (E_{ii}) and (E_{iii}) and $q(A_{[n/2]}) \geqq 1$ for (E_{iii}) and (E_{iv}) [**5**, **7**]. Furthermore, employing a procedure similar to the one used to establish the inequality $q(A_{[k/2]}) \leqq 2$, we can prove that $q(A_{[n/2]}) \leqq 1$ [**8**]. Consequently, $q(A_{[n/2]}) = 1$ for (E_{iii}) and (E_{iv}). Thus we have proved the following statements.

THEOREM 5. *For the equation* (E), *we have*
$q(A_j) = 0$ *or* $2, j = 0, 1, \ldots, (n-2)/2,$ *for* (E_i);
$q(A_0) = 1$ *and* $q(A_j) = 0$ *or* $2, j = 1, 2, \ldots, (n-1)/2,$ *for* (E_{ii});
$q(A_0) = 1$, $q(A_j) = 0$ *or* 2, $j = 1, 2, \ldots, (n-2)/2$, *and* $q(A_{n/2}) = 1$ *for* (E_{iii}); *and*
$q(A_j) = 0$ *or* 2, $j = 0, 1, \ldots, (n-3)/2$, *and* $q(A_{(n-1)/2}) = 1$ *for* (E_{iv}).

REMARK. If $u \in A_i$ and $v \in A_{i+k}$, $k \geqq 1$, then $w \equiv v + Cu \in A_{i+k}$ for any constant C.

For definiteness we consider (E_{iii}); proofs for the other cases are similar. We may assume that $u > 0$ and $v > 0$ on $[b, \infty)$ for some $b \geqq a$, in which case we have by Lemma 1 $u > 0$, $u' > 0, \ldots, u^{(2i-1)} > 0$ on $[b_2, \infty)$ for some $b_2 \geqq b$ and $u^{(2i)} > 0$, $u^{(2i+1)} < 0$, $u^{(2i+2)} > 0, \ldots, u^{(n-1)} < 0$, on $[b, \infty)$, and $v > 0$, $v' > 0, \ldots, v^{(2i+2k-1)} > 0$, on $[b_2, \infty)$ and $v^{(2i+2k)} > 0$, $v^{(2i+2k+1)} < 0$, $v^{(2i+2k+2)} > 0, \ldots, v^{(n-1)} < 0$, on $[b, \infty)$. If $C \geqq 0$, then $w > 0$ on $[b_2, \infty)$. If $C < 0$, then $w^{(2i+2k-1)} > 0$ on $[b_2, \infty)$ and w cannot be oscillatory. Hence, w is nonoscillatory for any constant C and $w \in A_l$ for some l, $0 \leqq l \leqq n/2$, by Lemma 1. For $2i \leqq l \leqq 2i + 2k - 2$, $v^{(l)}(x) \to \infty$ as $x \to \infty$ while $|u^{(l)}|$ is bounded on $[b, \infty)$; thus eventually $w^{(l)}(x) > 0$ as $x \to \infty$. Similarly, $v^{(2i+2k-1)} > 0$ and monotonically increasing on $[b_2, \infty)$ while $u^{(2i+2k-1)} < 0$ and monotonically increasing. In fact, $u^{(2i+2k-1)}(x) \to 0$ as $x \to \infty$. If this were not the case, we could find a positive constant k such that $u^{(2i+2k-1)} < -k$ on $[b, \infty)$ and conclude by integration that $u^{(2i+2k-2)}$ is eventually negative. However, this is impossible since $u^{(2i+2k-2)} > 0$. Consequently, $w^{(2i+2k-1)}(x) > 0$ for sufficiently large x, and therefore $w \in A_j, j \geqq i + k$. To complete the proof, it suffices to show that $w \in A_j$, $j < i + k + 1$. Evidently, $v^{(2i+2k+1)} < 0$ and $u^{(2i+2k+1)} < 0$ are monotonically increasing, and we conclude as in the case of $u^{(2i+2k-1)}$ that $v^{(2i+2k+1)}(x) \to 0$ and $u^{(2i+2k+1)}(x) \to 0$ as $x \to \infty$. This means that $w^{(2i+2k+1)}(x) \to 0$ as $x \to \infty$. Moreover, $w^{(2i+2k+1)}$ and $w^{(2i+2k+2)}$ are eventually of constant sign because w is a nonoscillatory solution of (E_{iii}). Hence sgn $w^{(2i+2k+1)}$ $\neq$ sgn $w^{(2i+2k+2)}$ eventually, which implies $w \in A_j, j < i + k + 1$.

It is well-known that equation (E) is $k - (n-k)$ disfocal if and only if its adjoint equation is $(n-k) - k$ disfocal **[10, 12]**. Therefore, the self-adjoint equation

$$y^{(2m)} + py = 0 \tag{34}$$

is $k - (n-k)$ disfocal if and only if it is $(n-k) - k$ disfocal. Recalling that $A_{[k/2]}$ is nonempty if and only if equation (E) is eventually $k - (n-k)$ disfocal on $[a, \infty]$ (provided (P) is assumed), we conclude from Theorem 5 that

(35) $$q(A_{[k/2]}) = q(A_{[(n-k)/2]})$$

for (34), provided (P) holds on $[a, \infty)$. If all the classes A_j are nonempty for (E), then $\sum q(A_k) = n$ by Theorem 5 and (E) has a fundamental system of solutions $F = \{y_1, y_2, \ldots, y_n\}$ such that every linear combination of them is nonoscillatory, i.e., (E) is nonoscillatory. For example, we choose for (E_{iii}), $y_1 \in A_0$, y_{2j}, $y_{2j+1} \in A_j$ such that every nontrivial linear combination of y_{2j} and y_{2j+1} belongs to A_j, $j = 1, 2, \ldots, (n-2)/2$, and $y_n \in A_{n/2}$. This choice yields a desired system for (E_{iii}) by the earlier remark. Thus, one of the classes A_j must be empty if (E) is oscillatory. In particular, if the equation

(36) $$y^{iv} + py = 0, p > 0,$$

is oscillatory on $[a, \infty)$, either A_0 or A_1 must be empty. In view of (35), we further deduce that both A_0 and A_1 are empty. But every nonoscillatory solution of (36) belongs to $A_0 \cup A_1$ by Lemma 1. Therefore, every solution of (36) is oscillatory if (36) is oscillatory. This result was obtained earlier by Leighton and Nehari [**9**].

It is also known that if $y^{iv} + py = 0$, $p < 0$, is oscillatory on $[a, \infty)$, it has three linearly independent oscillatory solutions [**1**]. The present method enables us to extend the above results on the fourth-order equations to the higher-order equation (34) with $m \geqq 2$. Consider the case $p > 0$. If (34) is oscillatory, $A_{[k/2]}$ is empty for some odd k (recalling the parity condition for (E_i)). If, in addition, m is even, then $[k/2] \neq [(2m-k)/2]$ for all odd k and $q(A_{[k/2]}) = q(A_{[(2m-k)/2]}) = 0$ for at least one k by (35), i.e, there are at least two distinct classes that are empty. Suppose that $A_{j_1}, \ldots, A_{j_r}, j_1 < j_2 < \cdots < j_r$, are nonempty while the other classes are empty. Then $q(A_{j_i}) = 2$, $i = 1, 2, \ldots, r$, by Theorem 5; let $y_{2i-1}, y_{2i} \in A_{j_i}$ be such that every nontrivial linear combination again belongs to A_{j_i}, $i = 1, 2, \ldots, r$. Evidently, $y_1, \ldots, y_{2r}$ are linearly independent and can be extended to a fundamental system $y_1, \ldots, y_{2r}, y_{2r+1}, \ldots, y_n$. We may assume that $y_{2r+1}, \ldots, y_n$ are oscillatory solutions: If y_i is nonoscillatory for some i, $2r + 1 \leqq i \leqq n$, then $y_i \in A_{j_s}$ for some s, $1 \leqq s \leqq r$. Due to Theorem 5, there exist constants c_{2s-1} and c_{2s} such that $y_i - c_{2s-1}y_{2s-1} - c_{2s}y_{2s}$ either is oscillatory or else belongs to A_{j_l} for some $l, l < s$. If it is oscillatory, we replace y_i by $y_i - c_{2s-1}\, y_{2s-1} - c_{2s}y_{2s}$ in the fundamental system. If it is nonoscillatory, we may repeat a similar argument as many times as necessary and conclude that

$$w_i \equiv y_i - \sum_{j=1}^{2r} c_j y_j$$

is oscillatory for some constants $c_1, \ldots, c_{2r}$. Again, we may replace y_i by w_i in the fundamental system. Since $A_{[k/2]}$ and $A_{[(2m-k)/2]}$ are empty,

$n - 2r \geqq 4$ and (34) has at least four linearly independent oscillatory solutions. The proof is similar if m is odd but $m \neq k$. However, if $m = k$, then $A_{[k/2]} = A_{[(2m-k)/2]}$ and the preceding argument only shows that (34) has at least two linearly independent oscillatory solutions w_1 and w_2. If in addition A_0 is empty, then there are at least two empty classes and we may again conclude that (34) has at least four oscillatory solutions. If A_0 is nonempty, let $y_1, y_2 \in A_0$ be such that every nontrivial linear combination of y_1 and y_2 again belongs to A_0. We assert that there exists a nonzero constant K_1 such that $y_1 - K_1 w_1$ is oscillatory on $[a, \infty)$. If this were not true, $f_\kappa \equiv y_1 - \kappa w_1$ would be nonoscillatory for any constant κ. Assume that $y_1 > 0$ on $[b, \infty)$. Since y_1 is a solution of (34) with $p > 0$ and belongs to the class A_0, $y_1' > 0$, $y_1'' < 0$, $y_1''' > 0, \ldots, y_1^{(2m-1)} > 0$ on $[b, \infty)$. For $\kappa > 0$ $[< 0]$ we can find a sequence $\{\rho_i\}$ of real numbers with $\rho_i \to \infty$ as $i \to \infty$ such that $w_1''(\rho_i) > 0$ $[< 0]$, $i = 1, 2, \ldots$, because w_1 is an oscillatory solution of (34). Hence, $f_\kappa''(\rho_i) < 0$ for all i and f_κ'' cannot be positive throughout any interval of the form $[c, \infty)$; thus $f_\kappa \in A_0$ for any constant κ. Choose $K > 0$ such that $f_K(\xi) = f_K(\eta) = 0$ for some ξ and η, $b \leqq \xi < \eta < \infty$, and $f_K > 0$ on (η, ∞). Then $f_K \in A_0$ and $f_K'(\eta) \neq 0$ by Lemma 1. Since $f_K > 0$ on (η, ∞), $f_K'(\eta) \neq 0$ implies $f_K'(\eta) > 0$; for this reason we may assume that $f_K < 0$ on (ξ, η). Let $\bar{K} = \sup G$, where $G = \{\kappa \mid f_\kappa \geqq 0 \text{ on } [\xi, \eta]\}$. Evidently, G is nonempty, $K > \bar{K} > 0$, and there exists a point $\tau \in (\xi, \eta)$ such that $f_{\bar{K}}(\tau) = f_{\bar{K}}'(\tau) = 0$ and $f_{\bar{K}} > 0$ on $(\tau, \eta]$. Moreover, $f_{\bar{K}} > 0$ on $[\eta, \infty)$ because $K > \bar{K} > 0$. Consequently, $f_{\bar{K}}(\tau) = f_{\bar{K}}'(\tau) = 0$, $f_{\bar{K}} \geqq 0$ on $[\tau, \infty)$ and $f_{\bar{K}} \in A_0$. But this is contrary to Lemma 1 and proves that $f_{K_1} = y_1 - K_1 w_1$ is oscillatory for some constant K_1. In a similar manner we may prove that $y_2 - K_2 w_1$ is oscillatory for some constant K_2. We thus have four linearly independent oscillatory solutions w_1, w_2, $y_1 - K_1 w_1$, and $y_2 - K_2 w_1$.

Using essentially the same argument, we can prove that if (34) with $p < 0$ and $m \geqq 2$ is oscillatory, it has at least 3 or 5 linearly independent oscillatory solutions according as m is even or odd. Also if the odd-order equation $y^{(2m+1)} + py = 0$, $m \geqq 1$, is oscillatory on $[a, \infty)$, it has at least 2 or 3 linearly independent oscillatory solutions according as $p < 0$ or $p > 0$.

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