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FUNCTORIAL TOPOLOGIES ON ABELIAN GROUPS

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1. **Introduction.** B. Charles [3] introduced the concept of "functorial topology" which includes the ubiquitous p-adic and Z-adic topologies. He proposed a method for constructing such topologies which was slightly generalized by Fuchs [4], Vol I, p. 33. Closer inspection shows that this method amounts to specifying the class of discrete groups and furnishing all other groups with the coarsest topology required to make all homomorphisms continuous. The "discrete classes" satisfy certain closure properties and studying these classes enables us to partially solve Fuchs' Problem 2, [4], asking for a description of functorial linear topologies. There is no bijective correspondence between functorial topologies and discrete classes — examples are readily available (see 2.8) — but there is a bijective correspondence between the "minimal functorial topologies" obtained via the Charles-Fuchs construction and discrete classes (Theorem 2.5). We call a functorial topology "ideal" if in addition to having continuous homomorphisms all epimorphisms are open maps. This is true for the p-adic and Z-adic topologies, for example. We obtain a bijective correspondence between linear ideal functorial topologies and "ideal" discrete classes. The latter are satisfactorily characterized (3.5, 3.11, 3.19) except for one nasty case.

In describing the ideal discrete classes we adopt the methods of Balcerzyk's description [1] of "classes" (= Serre classes). A Serre class is a class of groups closed under subgroups, homomorphic images, and extensions; while an ideal discrete class is only closed under subgroups, homomorphic images, and finite direct sums. Because of this difference all of Balcerzyk's results had to be modified, and we present our own proofs and arrangement. The later theorems on mixed groups have no analogues in [1].

In §2 we discuss functorial topologies and discrete classes. In §3 we outline without proofs the steps involved in describing the ideal discrete classes, and in §4 we supply the proofs. In a final section we discuss possible further work and related literature.

We use the standard notation of Fuchs [4], and all unexplained terminology and symbols can be found there. We write maps on the right. P is the set of all primes; p always denotes a prime; N is the set of positive

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integers; Card is the class of all cardinal numbers; $\tau(A)$ is the type of the torsion-free rank 1 group A. If χ is a characteristic, $[\chi]$ denotes the type represented by χ. For a group M, M_t denotes the maximal torsion subgroup, and $M_{tf} = M/M_t$.

All groups considered are abelian and "group" stands for abelian group.

2. **Functorial topologies.**

DEFINITION 2.1. Let $\mathscr{B}$ be a full subcategory of the category $\mathscr{A}$ of abelian groups, and let $\mathscr{T}$ be the category of topological abelian groups. A *functorial topology* on $\mathscr{B}$ is a functor $T: \mathscr{B} \to \mathscr{T}$ such that for any A in $\mathscr{B}$, $T(A) = (A, T_A)$, where T_A denotes the topology on A, and for any morphism f in $\mathscr{A}$, $T(f) = f$. If, moreover, every epimorphism is an open map, then T is an *ideal functorial topology*. A functorial topology T is *linear* if T_A is a linear topology for every group A.

The concept of functorial topology is due to Charles [3]. The condition $T(f) = f$ implies that every homomorphism is continuous.

Our first theorem shows, among other things, that there is no need to look at functorial topologies on subcategories of $\mathscr{A}$ since any functorial topology defined on a full subcategory extends to $\mathscr{A}$. In particular, the functorial topology which assigns the discrete topology to all groups of a given class of groups extends to a functorial topology on $\mathscr{A}$. In the latter case the extension is the one suggested by Charles and Fuchs.

THEOREM 2.2. a) *Let $\mathscr{B}$ be a full subcategory of $\mathscr{A}$ and suppose T is a functorial topology on $\mathscr{B}$. Then T is the restriction of a functorial topology $\bar{T}$ defined on $\mathscr{A}$.*

b) *If $\mathscr{B}$ is closed under subgroups, isomorphic groups, and finite direct sums, then there is an extension $\bar{T}$ of T such that $\bar{T}_D$ is discrete if and only if $D \in \mathscr{B}$ and T_D is discrete.*

PROOF. a) For A is $\mathscr{A}$ let $\mathscr{U}_A$ be the family of finite intersections of sets of the form $U\phi^{-1}$ with $\phi \in \mathrm{Hom}(A, B)$, $B \in \mathscr{B}$, and U an open neighborhood of 0 in B. It is straightforward to check that $\mathscr{U}_A$ satisfies the conditions of Hewitt-Ross [7], (4.5), p. 18, and hence is a neighborhood basis for a group topology $\bar{T}_A$ on A. We check next that every homomorphism is continous with respect to the topologies $\bar{T}_A$. Let $\alpha \in \mathrm{Hom}(A, B)$, $\phi \in \mathrm{Hom}(B, C)$, $C \in \mathscr{B}$, and U an open neighborhood of 0 in (C, T_C). Then $U\phi^{-1}$ is a member of a subbasis of open neighborhoods of 0 in $(B, \bar{T}_B)$ and $(U\phi^{-1})\alpha^{-1} = U(\alpha\phi)^{-1}$ is open in $(A, \bar{T}_A)$. Hence, α is continuous. Finally, suppose A belongs to $\mathscr{B}$. If U is an open neighborhood of 0 in (A, T_A), then by definition of $\bar{T}_A$, $U\,\mathrm{id}_A^{-1} = U$ is open in $\bar{T}_A$, so $T_A \subset \bar{T}_A$. On the other hand, if U is an open neighborhood of 0 in (B, T_B), with B in $\mathscr{B}$, and if $\phi \in \mathrm{Hom}(A, B)$, then $U\phi^{-1}$ is open in (A, T_A) since T is a functorial topology. Thus $\bar{T}_A \subset T_A$.

b) Let $\bar{T}$ be as in a). The "if" part of the claim is clear by a). Now sup-suppose $\bar{T}_D$ is discrete. Then there are finitely many groups $B_i \in \mathscr{B}$ and maps $\phi_i \in \mathrm{Hom}(D, B_i)$ such that $\bigcap \mathrm{Ker}\, \phi_i = 0$. By hypothesis $\oplus B_i \in \mathscr{B}$. Define $\Delta: D \to \oplus B_i$ by $a\Delta = \sum a\phi_i$. Then Δ is a monomorphism. The closure properties of $\mathscr{B}$ imply that D is in $\mathscr{B}$ and $T_D = \bar{T}_D$ is discrete.

From now on we always assume that functorial topologies are defined on all of $\mathscr{A}$. Theorem 2.2 says that a functorial topology can be constructed from a given class of discrete groups. On the other hand, a functorial topology T determines the class $\mathscr{C}(T)$ of all groups A such that T_A is discrete.

THEOREM. 2.3. *If T is a functorial topology, then $\mathscr{C}(T)$ is closed under subgroups, isomorphic groups, and finite direct sums. If T is ideal, then $\mathscr{C}(T)$ is also closed under epimorphic images.*

PROOF. If $\phi: A \to B$ is a monomorphism and B is discrete, then $0 = 0\phi^{-1}$ is open in A, so A is discrete. Hence, $\mathscr{C}(T)$ is closed under subgroups and isomorphic copies. Suppose $A = \oplus B_i$, a finite direct sum, B_i discrete. Let $\pi_i : A \to B_i$ be the projections. Then $0 = \bigcap 0\pi_i^{-1}$ is open in A, so A is discrete. Finally, if T is ideal and $\phi: A \to B$ is an epimorphism, hence continuous and open, then T_B is the quotient topology of T_A. Hence, if A is discrete, so is B.

The preceding results motivate the following definition.

DEFINITION 2.4. A class of groups is a *discrete class* if it is closed under subgroups, isomorphic groups and finite direct sums. A discrete class is an *ideal discrete class* if it is also closed under homomorphic images. A functorial topology T is *minimal* if for every A in $\mathscr{A}$, U is a neighborhood of 0 in A if and only if $\mathrm{Ker}\, \phi \subset U$ for some $\phi \in \mathrm{Hom}(A, D)$ with $D \in \mathscr{C}(T)$.

It is immediate that, starting with a discrete class $\mathscr{C}$, the construction used in 2.2 b) yields a linear minimal functorial topology, and it is trivial that for a minimal functorial topology $\mathscr{C}(T)$ determines T. This proves the following theorem.

THEOREM 2.5. *The map $T \to \mathscr{C}(T)$ is a bijective correspondence between minimal functorial topologies and discrete classes.*

We will see below that a functorial topology need not be minimal even if it is linear. The situation is nicer for ideal functors and classes, as we see in the next theorem.

THEOREM 2.6. *Every linear ideal functorial topology is minimal, and $T \to \mathscr{C}(T)$ is a bijective correspondence between linear ideal fnnctorial topologies and ideal discrete classes.*

PROOF. To show minimality let U be a neighborhood of 0 in A. Since T is linear, there is an open subgroup V contained in U. Now A/V is discrete since T is ideal and $V = \text{Ker}(A \to A/V)$, thus proving minimality, By 2.3, if T is ideal, so is $\mathscr{C}(T)$. Conversely, if $\mathscr{C}$ is an ideal discrete class, let T be the linear functorial topology constructed in 2.2 b). We must prove that T is ideal. Let $\alpha: A \to B$ be an epimorphism. Let U be a neighborhood of 0 in A. Then U contains an open subgroup V, and A/V is discrete. Now $B/V\alpha$ is a homomorphic image of A/V and since $\mathscr{C}(T)$ is ideal $B/V\alpha$ is discrete. So $V\alpha$ is an open subgroup of B and since $V\alpha \subset U\alpha$, $U\alpha$ is a neighborhood of 0 in B.

It is easy to list discrete and ideal discrete classes defining well-known and not so well-known minimal functorial topologies. The following are ideal discrete classes: the class of all bounded p-groups (p-adic topology); all bounded groups (Z-adic topology); all groups with minimum condition (Prüfer topology, [**4**], Vol. I, p. 31); all finite groups (the finite index topology); all finitely generated groups; all p-groups; all torsion groups; all groups of finite torsion free rank; all groups of cardinality $<\mathscr{M}$ where $\mathscr{M}$ is an infinite cardinal. The following are discrete but not ideal discrete classes: all p-groups of length $<\lambda$ (the "p^λ-topology"); all torsion free groups; all torsion free groups of finite rank; all free groups; all direct sums of cyclic groups.

Indeed, every class of groups is contained in a smallest discrete class and a smallest ideal discrete class. The following theorem describes these classes in detail.

THEOREM 2.7. *Let $\mathscr{S}$ be a non-void class of groups. The class of all groups isomorphic to some subgroup of some finite direct sum of groups of $\mathscr{S}$ is the smallest discrete class $\mathscr{C}$ containing $\mathscr{S}$. The class of all homomorphic images of groups in $\mathscr{C}$ is the smallest ideal discrete class containing $\mathscr{S}$.*

The proof is straightforward and simple.

We conclude this section with an example of a linear functorial topology which is not minimal.

EXAMPLE 2.8. Fix p and let $\mathscr{B}$ be the category of all p-groups. For G in $\mathscr{B}$ take the large subgroups defined by Pierce [**11**] as a neighborhood basis at 0 for G. The characterization of large subgroups, [**11**], Section 2, implies immediately that every homomorphism between p-groups topologized in this fashion is continuous. Let T be the extension to $\mathscr{A}$ of this functorial topology such that T satisfies 2.2 b). By 2.2 b) and [**11**], Examples 1., p. 218, $\mathscr{C}(T)$ is the class of all bounded p-groups. The minimal functorial topology associated with this class is the p-adic topology, and it is well-known and easily seen from Pierce's results that the p-adic topology on G and T_G usually do not coincide.

3. **Discrete classes.** Let $\mathscr{C}$ be a discrete class. Following Balcerzyk [**1**], we introduce certain subclasses of $\mathscr{C}$. We call a group *component bounded* if it is torsion and its primary components are bounded. Let k stand for either cb (= component bounded), dt (= divisible torsion), t (= torsion) or tf (= torsion free). A k-group is a group which is k. We let $\mathscr{C}_k$ be the class of all k-groups in $\mathscr{C}$. For $k = cb$, dt we call a class $\mathscr{C}'$ of k-groups *possible* if $\mathscr{C}' = \mathscr{C}_k$ for some discrete class $\mathscr{C}$. For $k = t$, tf we call a class $\mathscr{C}'$ of k-groups *possible* if $\mathscr{C}' = \mathscr{C}_k$ for some ideal discrete class $\mathscr{C}$.

It is clear that a class $\mathscr{C}$ of cb-groups is possible if and only if $\mathscr{C}$ is a discrete class. We begin by characterizing the possible classes $\mathscr{C}$ of cb-groups in terms of families of functions whose values are the invariants of the groups of $\mathscr{C}$. This accomplishes a classification of all minimal functorial topologies for which every discrete group is cb; in particular, it describes the possible minimal functorial topologies on the category of all cb-groups. It is interesting that these functorial topologies are necessarily ideal. Next we characterize the possible classes $\mathscr{C}$ of dt-groups again in terms of families of functions whose values are the invariants of the groups in $\mathscr{C}$. We then consider known the possible classes of cb-groups and of dt-groups, and characterize the possible classes of t-groups in terms of pairs $(\mathscr{C}', \mathscr{C}'')$ where $\mathscr{C}'$ is a possible class of cb-groups, and $\mathscr{C}''$ is a possible class of dt-groups. This is a departure from Balcerzyk's approach which uses complicated families of functions to do the same. At this point we accomplished a description of all minimal functorial topologies for which every discrete group is torsion; in particular, we have a classification of the possible minimal functorial topologies on the category of all t-groups. The characterization of possible classes of tf-groups breaks into three separate cases. Having characterized these Balcerzyk is finished since Serre classes are closed under extensions and, hence, a Serre class $\mathscr{C}$ consists of all groups M with $M_t \in \mathscr{C}_t$ and $M_{tf} \in \mathscr{C}_{tf}$. In the case of ideal discrete classes $\mathscr{C}$, a mixed group M with $M_t \in \mathscr{C}_t$ and $M_{tf} \in \mathscr{C}_{tf}$ may or may not belong to $\mathscr{C}$, and further work is needed.

We now state the results in detail.

The *invariant* of the cb-group G is the function $i_G : P \times N \to \text{Card}$ such that $i_G(p, n)$ is the cardinality of cyclic summands of order p^n in a decomposition of G into a direct sum of indecomposable cyclic groups. For a discrete class $\mathscr{C}$ of cb-groups we denote by $I(\mathscr{C})$ the family of invariants of the groups of $\mathscr{C}$. Our first theorem characterizes such classes $\mathscr{C}$ in terms of families of functions $P \times N \to \text{Card}$. Both the statement and the proof (see 4.1) of this theorem is much simpler than the corresponding Theorem 6, [**1**], p. 165, of Balcerzyk, mainly due to the fact that we need not worry about closure under extensions.

THEOREM 3.1. a) *Let I be a family of functions on $P \times N$ to* Card. *Then*

$I = I(\mathscr{C})$ for some discrete class $\mathscr{C}$ of cb-groups if and only if the following three conditions hold.

(3.2) *For each $i \in I$ there exists a positive integer $M(p)$ such that $i(p, n) = 0$ whenever $n > M(p)$.*

(3.3) *If $i, j \in I$, then $i + j \in I$.*

(3.4) *If $i \in I$, and j: $P \times N \to$* Card *satisfies*

$$\sum_{k \geqq n} j(p, k) \leqq \sum_{k \geqq n} i(p, k)$$

for all p and n, then $j \in I$.

b) *For two discrete classes $\mathscr{C}$, $\mathscr{C}'$ of cb-groups $\mathscr{C} = \mathscr{C}'$ if and only if $I(\mathscr{C}) = I(\mathscr{C}')$.*

COROLLARY 3.5. *The minimal functorial topologies for which every discrete group is component bounded are in bijective correspondence with the families I of functions on $P \times M$ to* Card *satisfying* (3.2)—(3.4). *Such functorial topoligies are necessarily ideal.*

The first part of the corollary is immediate. The second claim follows the fact that every homomorphic image of a *cb*-group is embedded in that group (Fuchs-Kertesz-Szele [**6**], corollary, p. 469).

The next theorem characterizes possible classes of *dt*-groups in terms of families of functions. Let $\mathscr{C}$ be a possible class of *dt*-groups. For D in $\mathscr{C}$ let r_D: $P \to$ Card be defined by $r_D(p) = \dim D[p]$. Let $R(\mathscr{C}) = \{r_D: D \in \mathscr{C}\}$.

THEOREM 3.6. a) *Let R be a family of functions on P to* Card. *Then $R = R(\mathscr{C})$ for some possible class $\mathscr{C}$ of dt-groups if and only if the following two conditions hold.*

(3.7) *If $r \in R$, s: $P \to$* Card, *and $s \leqq r$, then $s \in R$.*

(3.8) *If $r, s \in R$, then $r + s \in R$.*

b) *For two possible classes $\mathscr{C}$, $\mathscr{C}'$ of dt-groups, $\mathscr{C} = \mathscr{C}'$ if and only if $R(\mathscr{C}) = R(\mathscr{C}')$.*

This theorem is essentially Balcerzyk's Theorem 4, [**1**], p. 162. The proof is easy and we will only note that for a given possible class $\mathscr{C}'$ of *dt*-groups we take $\mathscr{C}$ to be the smallest discrete class containing $\mathscr{C}'$ (2.7). It follows trivially that $\mathscr{C}_{dt} = \mathscr{C}'$.

It is evident that a class of *t*-groups is possible if and only if it is itself an ideal discrete class.

THEOREM 3.9. a) *If $\mathscr{C}$ is an ideal discrete class of t-groups and G is any group, then $G \in \mathscr{C}$ if and only if G is embedded in a group $H \oplus D$ with $H \in \mathscr{C}_{cb}$ and $D \in \mathscr{C}_{dt}$.*

b) *Let $\mathscr{C}'$ be a possible class of cb-groups and let $\mathscr{C}''$ be a possible class of dt-groups. Then there is an ideal discrete class $\mathscr{C}$ of t-groups such that*

$\mathscr{C}_{cb} = \mathscr{C}'$ and $\mathscr{C}_{dt} = \mathscr{C}''$ if and only if the following condition holds.

(3.10) *Every cb-group embedded in a group of $\mathscr{C}''$ belongs to $\mathscr{C}'$.*

COROLLARY 3.11. *The ideal discrete classes $\mathscr{C}$ of t-groups are in bijective correspondence with the pairs $(\mathscr{C}', \mathscr{C}'')$ of possible classes $\mathscr{C}'$ of cb-groups and possible classes $\mathscr{C}''$ of dt-groups satisfying the compatibility condition* (3.10).

Our proof (4.2) of this theorem is considerably shorter than Balcerzyk's proof of his Theorem 7. We remark that for the purposes of topological applications the above results are quite satisfactory. A group A is topologized by means of the kernels of homomorphisms into members of a discrete class. The small kernels are relevant and hence the homomorphisms into large discrete groups are of principal interest. The groups $H \oplus D$ of 3.9a) are the "large" discrete groups since they contain all others.

The compatibility condition (3.10) can easily be stated in terms of the invariants $I(\mathscr{C}')$ and $R(\mathscr{C}'')$:

(3.12) If $i : P \times N \to \text{Card}$ is a function satisfying (3.2.) and $\sum_k i(p, k) \leqq r(p)$ for some $r \in R$, then $i \in I(\mathscr{C}')$.

We consider, next, possible classes of *tf*-groups. Let $\mathscr{C}$ be such a class. If $\mathscr{C}$ contains groups of arbitrary rank, we let $S(\mathscr{C}) = \infty$. In this case $\mathscr{C}$ contains all free groups, and hence, the only ideal discrete class containing $\mathscr{C}$ is $\mathscr{A}$. If $\mathscr{C} \neq \mathscr{A}$, but $\mathscr{C}$ contains groups of infinite rank, we put $S(\mathscr{C}) = \min\{r(A) \colon A \notin \mathscr{C}\}$. In this case it is easily seen that $\mathscr{C}$ contains exactly all *tf*-groups of rank less than $S(\mathscr{C})$. This leaves the case in which $\mathscr{C}$ contains only groups of finite rank. In this case we let $S(\mathscr{C}) = \{\tau(A) \colon A \in \mathscr{C}$ and $r(A) = 1\}$.

THEOREM 3.13 a) *For any cardinal $\mathscr{M} > \aleph_0$, there is a possible class $\mathscr{C}$ of tf-groups such that $S(\mathscr{C}) = \mathscr{M}$. In fact, if $\mathscr{C}$ is the ideal discrete class of all groups with cardinality $< \mathscr{M}$, then $S(\mathscr{C}_{tf}) = \mathscr{M}$. Also, $S(\mathscr{A}_{tf}) = \infty$.*

b) *If S is a set of types, then $S = S(\mathscr{C})$ for some possible class $\mathscr{C}$ of tf-groups if and only if the following two conditions hold.*

(3.14) *If $\tau \in S$ and σ is a type with $\sigma \leqq \tau$, then $\sigma \in S$.*

(3.15) *If $\sigma, \tau \in S$, then $\sigma \cup \tau \in S$.*

c) *For two possible classes $\mathscr{C}, \mathscr{C}'$ of tf-groups $S(\mathscr{C}) = S(\mathscr{C}')$ if and only if $\mathscr{C} = \mathscr{C}'$.*

This theorem owes much to Balcerzyk's Theorem 1, but there are by necessity essential differences and we give an independent proof below (4.5).

We are now left with the problem of mixed discrete groups. Our first theorem shows that the problem reduces to the case of mixed groups with component-bounded maximal torsion subgroups.

THEOREM 3.16. *Let $\mathscr{C}$ be an ideal discrete class and G any group. Then $G \in \mathscr{C}$ if and only if G is embedded in a group $D \oplus M$ where $D \in \mathscr{C}_{dt}$, $M \in \mathscr{C}$, $M_t \in \mathscr{C}_{cb}$, and $M_{tf} \in \mathscr{C}_{tf}$.*

The following theorem characterizes in a satisfactory way those ideal discrete classes $\mathscr{C}$ for which $S(\mathscr{C}_{tf})$ is a cardinal.

THEOREM 3.17. a) *Let $\mathscr{C}$ be an ideal discrete class for which $S(\mathscr{C}_{tf})$ is a cardinal, and let G be any group. Then $G \in \mathscr{C}$ if any only if G is embedded in a group $T \oplus N$ with $T \in \mathscr{C}_t$ and $|N| < S(\mathscr{C}_{tf})$.*

b) *Let $\mathscr{M}$ be an uncountable cardinal, and $\mathscr{C}'$ an ideal discrete class of torsion groups. Then there is an ideal discrete class $\mathscr{C}$ with $S(\mathscr{C}_{tf}) = \mathscr{M}$ and $\mathscr{C}_t = \mathscr{C}'$ if and only if the following condition holds.*

(3.18) *$\mathscr{C}'$ contains all torsion groups of cardinality $< \mathscr{M}$.*

COROLLARY 3.19. *The ideal discrete classes $\mathscr{C}$ for which $S(\mathscr{C}_{tf})$ is a cardinal are in bijective correspondence with the pairs $(\mathscr{C}', \mathscr{M})$ where $\mathscr{M}$ is an uncountable cardinal and $\mathscr{C}'$ is an ideal discrete class of torsion groups satisfying the compatibility condition* (3.18).

Finally we consider the involved case in which $S(\mathscr{C}_{tf})$ is a set of types, i.e., the ideal discrete class $\mathscr{C}$ contains *tf*-groups of finite rank only. We are not able to give a satisfactory solution in this case and restrict ourselves to giving an example which illustrates the difficulties which arise even in the simplest situations.

LEMMA 3.20. *Let S be a set of types satisfying* (3.14) *and* (3.15). *Then there exists a smallest ideal discrete class $\mathscr{C}$ such that $S(\mathscr{C}_{tf}) = S$. Moreover, $\mathscr{C}_t$ consists of all groups*

$$\oplus_{i=1}^{n} \oplus_p Z(p^{\chi_i(p)})$$

where χ_i is a characteristic with $[\chi_i] \in S$.

EXAMPLE 3.21. Let χ be the characteristic with $\chi(p) = 1$ for all p, and let $S = \{\tau : \tau \leqq [\chi]\}$. Let $\mathscr{C}$ be the smallest ideal discrete class with $S(\mathscr{C}_{tf}) = S$. It follows easily from 3.20 that $\mathscr{C}_t$ consists of all groups $\oplus_p T_p$ such that each T_p is a finite p-group and $pT_p = 0$ for almost all p. Let $R = \langle l/p : p \in P \rangle \leqq Q$. Then $R \in \mathscr{C}_{tf}$ and $R/\mathbf{Z} \cong \oplus_p Z(p)$. There is an obvious exact sequence $0 \to \oplus_p Z(p) \to \oplus_p Z(p^2) \to R/\mathbf{Z} \to 0$ which can be embedded in a commutative diagram with exact rows and right square a pullback as follows.

$$\begin{array}{ccccccccc} 0 & \to & \oplus_p Z(p) & \longrightarrow & M & \longrightarrow & R & \to & 0 \\ & & \| & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \to & \oplus_p Z(p) & \to & \oplus_p Z(p^2) & \to & R/\mathbf{Z} & \to & 0 \end{array}$$

Since γ is surjective, so is β. Since $\oplus_p Z(p^2) \notin \mathscr{C}_t$, the following three observations hold.

1) M is an extension of a group in $\mathscr{C}_t$ by a group in $\mathscr{C}_{tf}$ which does not belong to $\mathscr{C}$.

2) If $\mathscr{C}'$ is the smallest ideal discrete class containing $\mathscr{C}$ and M, then $\mathscr{C}'_t$ is strictly larger than $\mathscr{C}_t$.

3) $\mathscr{C}'$ is an ideal discrete class with $\mathscr{C}'_t = \mathscr{C}'_{cb}$, $S(\mathscr{C}'_{tf})$ a set of types and $\mathscr{C}'$ contains non-split mixed groups.

4. PROOFS.

PROOF 4.1. (proof of 3.1) a) Assume first that $I = I(\mathscr{C})$ for some discrete class $\mathscr{C}$ of *cb*-groups. Then (3.2) holds since i is the invariant of a *cb*-group, and (3.3) holds since $\mathscr{C}$ is closed under direct sums of two groups. Finally, let i, j be as in (3.4). Then $i = i_G$ for some *cb*-group $G \in \mathscr{C}$, and j is the invariant for some *cb*-group H. An easy induction on the exponent of G shows that H is embedded in G. Hence $H \in \mathscr{C}$ and $j \in I(\mathscr{C})$.

Now suppose I satisfies conditions (3.2) – (3.4). Let $\mathscr{C}$ be the class of direct sums of cyclic groups whose invariants are in I. By (3.2) every group of $\mathscr{C}$ is *cb*. By definition $\mathscr{C}$ is closed under isomorphic copies. By (3.3) $\mathscr{C}$ is closed under finite direct sums. Suppose $G \in \mathscr{C}$, G has invariant $i \in I$, and $H \leqq G$. Let j be the invariant of H. Then $\sum_{k \geqq n} j(p, k) = \dim(p^{n-1}H)[P] \leqq \dim(p^{n-1}G)[p] = \sum_{k \geqq n} i(p, k)$. By (3.4) $j \in I$, and hence $H \in \mathscr{C}$. Hence, $\mathscr{C}$ is also closed under subgroups, and is a discrete class. It is clear that $I = I(\mathscr{C})$.

b) Obvious.

PROOF 4.2. (proof of 3.9) a) We will use the following lemma (Fuchs [5], Lemma 1, p. 302): Any p-group G has a decomposition $G = H \oplus K$, where H is bounded and $r(K) = \text{fin } r(G) = \text{fin } r(K)$.

Let $G \in \mathscr{C}$. Using the above lemma we write $G = H \oplus K$ with H *cb* and $r(K_p) = \text{fin } r(K_p)$ for each prime p. By Fuchs [4], Theorem 3.56, p. 150, K can be mapped homomorphically onto a torsion divisible group D with $r_D(p) = \text{fin } r(K_p)$. The closure propertes of $\mathscr{C}$ imply that $H \in \mathscr{C}_{cb}$ and $D \in \mathscr{C}_{dt}$. Since $r(K_p) = \text{fin } r(K_p) = r_D(p)$, K can be embedded in D, and hence, G is embedded in $H \oplus D$ as desired. The converse is trivial.

b) Clearly, condition (3.10) is necessary. If $\mathscr{C}'$ and $\mathscr{C}''$ satisfy (3.10), let $\mathscr{C}$ be the class of all subgroups of groups $H \oplus D$ with $H \in \mathscr{C}'$ and $D \in \mathscr{C}''$. Clearly, $\mathscr{C}$ is closed under isomorphic copies and subgroups. Suppose $G \in \mathscr{C}$, ϕ is a homomorphism on G, and $G \subset H \oplus D$ as in the definition of $\mathscr{C}$. Then ϕ extends to a homomorphism ψ of $H \oplus D$ into the divisible hull of $G\phi$. Since $D\psi$ is divisible, $\text{Im}\psi = K \oplus D\psi$. Now $K \cong \text{Im}\psi / D\psi$, and the latter is obviously a homomorphic image of H, hence K is in $\mathscr{C}'$.

Also $D\phi \in \mathscr{C}''$, so $G\phi \leqq \operatorname{Im}\phi = K \oplus D\phi$, $K \in \mathscr{C}'$, $D\phi \in \mathscr{C}''$, and thus $G\phi \in \mathscr{C}$. This shows that $\mathscr{C}$ is an ideal discrete class. By definition of $\mathscr{C}$, $\mathscr{C}' \subset \mathscr{C}_{cb}$, and $\mathscr{C}'' \subset \mathscr{C}_{dt}$. If $A \in \mathscr{C}_{dt}$, $A \subset H \oplus D$, $H \in \mathscr{C}'$, $D \in \mathscr{C}''$, then $A \subset D$, and hence $A \in \mathscr{C}''$. So $\mathscr{C}_{dt} = \mathscr{C}''$. If $K \in \mathscr{C}_{cb}$, $K \subset H \oplus D$, $H \in \mathscr{C}'$, $D \in \mathscr{C}''$, let $\pi: H \oplus D \to H$ and $\rho: H \oplus D \to D$ be the projections. Then $K \subset K\pi \oplus K\rho$. Here $K\pi \subset H \in \mathscr{C}'$, so $K\pi \in \mathscr{C}'$; while $K\rho$ is *cb* and embedded in $H \in \mathscr{C}''$, so $K\rho \in \mathscr{C}'$ by (3.10). It follows that $K \in \mathscr{C}'$ by the closure properties of $\mathscr{C}'$.

Corollary 3.11 is an immediate consequence of 3.9.

The proof of 3.13 requires two facts which we will prove first.

LEMMA 4.3. *Let $R_i (1 \leqq i \leqq n)$ be rank one tf-groups with corresponding types τ_i.*

a) *If $\phi \in \operatorname{Hom}(R_1 \oplus \cdots \oplus R_n, Q)$, then $\tau(\operatorname{Im}\phi) = \bigcup\{\tau_i : \phi | R_i \neq 0\}$.*

b) *There is $\phi \in \operatorname{Hom}(R_i \oplus \cdots \oplus R_n, Q)$ such that $\tau(\operatorname{Im}\phi) = \tau_1 \cup \tau_2 \cup \cdots \cup \tau_n$.*

PROOF. a) Without loss of generality we may assume that $\phi|R_i \neq 0$ for all i. It is then obvious that $\tau(\operatorname{Im}\phi) \geqq \tau_1 \cup \cdots \cup \tau_n$. To prove the reverse inequality choose $\chi_i \in \tau_i$. If $\chi_i(p) = \infty$ for some i, then there is nothing to prove. Hence, suppose the $\chi_i(p) < \infty$ for all i. Let $a_i \in R_i$ have characteristic χ_i. Then $R_i = \langle a_i/q^{k(i,q)} : q \in P,\ 0 \leqq k(i, q) \leqq \chi_i(q)\rangle$ and $\operatorname{Im}\phi = \langle a_i\phi/q^{k(i,q)} : 1 \leqq i \leqq n,\ q \in P,\ 0 \leqq k(i, q) \leqq \chi_i(q)\rangle$. Let v_p be the additive p-adic valuation on Q. Then $0 \neq a \in \operatorname{Im}\phi$ implies $v_p(a) \leqq \max_i\{v_p(a_i\phi) + \chi_i(p)\}$. Since $v_p(a_i\phi) = 0$ except for finitely many pairs (i, p), we have $v_p(a) \leqq \max_i\{\chi_i(p)\}$ for almost all p. This implies that $\tau(\operatorname{Im}\phi) \leqq \tau_1 \cup \tau_2 \cup \cdots \cup \tau_n$, and the proof is complete.

b) Let ϕ be the direct sum of embeddings $R_i \to Q$. Then $\phi | R_i \neq 0$, and a) proves the claim.

The second fact is a lemma due to Balcerzyk, [1], p. 159.

LEMMA 4.4. *Let $\mathscr{C}$ be a possible class of tf-groups containing only groups of finite rank. Then $G \in \mathscr{C}$ if and only if G is embedded in a direct sum $R_1 \oplus \cdots \oplus R_n$ of tf rank one groups with $\tau(R_i) \in S(\mathscr{C})$.*

PROOF. If $G \in \mathscr{C}$, then G has finite rank, and hence G is embedded in a finite direct sum Q^n. Let R_i be the component of G in the ith summand Q. Then $R_i \in \mathscr{C}$, so $\tau(R_i) \in S(\mathscr{C})$, and G is embedded in $R_1 \oplus \cdots \oplus R_n$. The converse follows immediately from the closure properties of $\mathscr{C}$.

PROOF 4.5. (proof of 3.13) a) This is immediate.

b) Suppose first that $S = S(\mathscr{C})$ for some possible class $\mathscr{C}$ of *tf*-groups. If $\tau \in S$ and $\sigma \leqq \tau$, let R be a rank one *tf*-group with type τ, and R' a rank one group with type σ. Since $\tau \in S$, $R \in \mathscr{C}$; and, since $\sigma \leqq \tau$, R' is embedded in R. Hence, $R' \in \mathscr{C}$ and $\sigma \in S$. This proves (3.14). Let $\sigma, \tau \in S$ and let R and R' be rank one groups with types σ and τ. By Lemma 4.3 there is a rank one homomorphic image R'' of $R \oplus R'$ with type $\sigma \cup \tau$. Since $R'' \in \mathscr{C}$, $\sigma \cup \tau \in S$. This proves (3.15).

To prove the converse let S be given satisfying (3.14) and (3.15). Let $\mathscr{C}$ be the smallest ideal discrete class containing all *tf* rank one groups R with $\tau(R) \in S$. It follows from 2.7 that $\mathscr{C}_{tf}$ contains only groups of finite rank, and clearly $S(\mathscr{C}_{tf}) \supset S$. Suppose $\tau \in S(\mathscr{C}_{tf})$. Then there is a rank one group $R \in \mathscr{C}_{tf}$ with $\tau(R) = \tau$. By 2.7 there exist rank one groups R_i with $\tau(R_i) \in S$ and a homomorphism $\phi: R_1 \oplus \cdots \oplus R_n \to Q$ such that $\mathrm{Im}\phi = R$. By 4.3, $\tau = \tau(R) \leqq \tau(R_1) \cup \cdots \cup \tau(R_n)$; and hence $\tau \in S$. Thus $S(\mathscr{C}_{tf}) = S$, and this completes the proof.

c) This is an immediate consequence of 4.4.

Proof 4.6. (proof of 3.16). Suppose $G \in \mathscr{C}$. By 3.9 a) there is a monomorphism $\alpha: G_t \to H \oplus D$ with $H \in \mathscr{C}_{cb}$ and $D \in \mathscr{C}_{dt}$. Hence, we get the following commutative diagram with exact rows.

$$\begin{array}{rcccccccc} e: 0 & \to & G_t & \to & G & \to & G_{tf} & \to & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| & & \\ \alpha e: 0 & \to & H \oplus D & \to & N & \to & G_{tf} & \to & 0 \end{array}$$

The squares of the diagram are push-outs; hence, N is a factor group of $H \oplus D \oplus G$, and thus belongs to $\mathscr{C}$. Since α is injective, so is β, and thus G is embedded in N. Since D is divisible, $N = D \oplus M$, and it is easily seen that $M \in \mathscr{C}$, $M_t = H \in \mathscr{C}_{cb}$, and $M_{tf} = G_{tf} \in \mathscr{C}_{tf}$. The converse is immediate.

For the proof of 3.17 we need the following theorem (unpublished).

Theorem 4.7. *Suppose that every subgroup S of the group U can be embedded in a direct summand of U of cardinality $\leqq |S|\aleph_0$, and suppose $U \leqq V$. Then $V = U' \oplus V'$ with U' a direct summand of U and $|V'| \leqq |V/U|\,\aleph_0$.*

Note that U satisfies the hypothesis of 4.7 if U is a direct sum of cyclic groups.

Proof 4.8. (proof of 3.17) a) Suppose $G \in \mathscr{C}$. By 3.16, G is embedded in $D \oplus M$, $D \in \mathscr{C}_{dt}$, $M \in \mathscr{C}$, $M_t \in \mathscr{C}_{cb}$, $M_{tf} \in \mathscr{C}_{tf}$. By 4.7, $M = T \oplus N$ where $T \in \mathscr{C}_{cb}$ and $|N| \leqq |M_{tf}| < S(\mathscr{C}_{tf})$. Hence, G is embedded in $T' \oplus N$ with $T' = D \oplus T \in \mathscr{C}_t$ and $|N| < S(\mathscr{C}_{tf})$. Conversely, if G is embedded in $T \oplus N$, $T \in \mathscr{C}_t$, $|N| < S(\mathscr{C}_{tf})$, then $N \in \mathscr{C}$ since $\mathscr{C}_{tf}$ contains all free groups of rank $< S(\mathscr{C}_{tf})$ and thus $G \in \mathscr{C}$.

b) It is clear that the compatibility condition (3.18) is necessary. Conversely, let $\mathcal{M}$ and $\mathscr{C}'$ be given such that (3.18) holds. Let $\mathscr{C}$ be the smallest ideal discrete class containing $\mathscr{C}'$ and all *tf*-groups of cardinality $< \mathcal{M}$. Then $S(\mathscr{C}_{tf}) \geqq \mathcal{M}$. Suppose $K \in \mathscr{C}_{tf}$. Then, by 2.7, there is $T \in \mathscr{C}'$, a *tf*-group L with $|L| < \mathcal{M}$, a subgroup G of $T \oplus L$ and an epimorphism $G \twoheadrightarrow K$. Since K is *tf*, $G_{tf} \twoheadrightarrow K$, and since G_{tf} is embedded in $(T \oplus L)_{tf} \cong L$, we have $|K| \leqq |G_{tf}| \leqq |L| < \mathcal{M}$. Hence, $S(\mathscr{C}_{tf}) \leqq \mathcal{M}$, so that $S(\mathscr{C}_{tf}) = \mathcal{M}$ as claimed. Now suppose $T \in \mathscr{C}_t$. By 3.9 a) T is embedded in $H \oplus D$ with $H \in \mathscr{C}_{cb}$ and $D \in \mathscr{C}_{dt}$. Thus, $\mathscr{C}_{cb} \subset \mathscr{C}'$ and $\mathscr{C}_{dt} \subset \mathscr{C}'$ will imply that $\mathscr{C}_t \subset \mathscr{C}'$. Since trivially $\mathscr{C}' \subset \mathscr{C}_t$, we will have $\mathscr{C}_t = \mathscr{C}'$ and the proof will be finished. Suppose first that $D \in \mathscr{C}_{dt}$. Then there is an epimorphism ϕ: $T \oplus L \twoheadrightarrow D$ where $T \in \mathscr{C}'$, L is *tf*, and $|L| < \mathcal{M}$. Since $|L\phi| \leqq |L| < \mathcal{M}$ and $L\phi$ is torsion, (3.18) implies that $L\phi \in \mathscr{C}'$. Also $T\phi \in \mathscr{C}'$ since $\mathscr{C}'$ is closed under homomorphic images. Finally, $L\phi \oplus T\phi \in \mathscr{C}'$, and there is a natural epimorphism $L\phi \oplus T\phi \twoheadrightarrow D$, hence $D \in \mathscr{C}'$.

Finally, suppose $H \in \mathscr{C}_{cb}$. As before we have $G \leqq T \oplus L$, $T \in \mathscr{C}'$, L *tf*, $|L| < \mathcal{M}$, and an epimorphism ϕ: $G \twoheadrightarrow H$. Note that $|G_{tf}| \leqq |L| < \mathcal{M}$. Clearly, ϕ induces an epimorphism $G_{tf} \twoheadrightarrow H/G_t\phi$. Since H is *cb*, $G_t\phi$ is a direct sum of cyclic groups, so 4.7 applies and gives $H = H_1 \oplus H_2$, with H_1 a subgroup of $G_t\phi$ and $|H_2| \leqq |H/G_t\phi| \leqq |G_{tf}| < \mathcal{M}$. Hence, $H_1 \in \mathscr{C}'$ since $G_t \in \mathscr{C}'$, and $H_2 \in \mathscr{C}'$ by (3.18); therefore, $H \in \mathscr{C}'$ as required.

Corollary 3.19 follows immediately from 3.17.

PROOF 4.9. (proof of 3.20) Let $\mathscr{C}$ be the smallest ideal discrete class containing all *tf* rank one groups R with $\tau(R) \in S$. By 4.5 b) $S(\mathscr{C}_{tf}) = S$ and by 4.4 and 2.7 $\mathscr{C}$ consists of all homomorphic images of subgroups of direct sums $R_1 \oplus \cdots \oplus R_n$ of *tf* rank one groups with $\tau(R_i) \in S$. If $[\chi] \in S$, then $R = \langle 1/p^{n(p)}: n(p) \leqq \chi(p)\rangle \in \mathscr{C}$ and $R/\mathbf{Z} \cong \oplus_p Z(p^{\chi(p)})$. Hence, $\mathscr{C}_t$ contains all groups $\oplus_i \oplus_p Z(p^{\chi_i(p)})$ with $[\chi_i] \in S$. Conversely, suppose $T \in \mathscr{C}_t$. Then there exist groups $H = R_1 \oplus \cdots \oplus R_n$, $K \leqq G \leqq H$, such that $G/K \cong T$. Let L be G-high in H. Then $H/(G \oplus L)$ is torsion and $(G \oplus L)/(K \oplus L)$ is torsion; hence, $H/(K \oplus L)$ is torsion. Moreover, G/K is embedded in $H/(K \oplus L)$. Hence, $\mathscr{C}_t$ consists of all subgroups of torsion homomorphic images of groups $H = R_1 \oplus \cdots \oplus R_n$. Suppose $K \leqq H$ and H/K is torsion. Then $K \cap R_i \neq 0$ for all i, and $\oplus_i R_i/(K \cap R_i)$ maps homorphically onto H/K. But $R_i/(K \cap R_i) \cong \oplus_p Z(P^{\chi(p)})$ for some χ with $[\chi] \in S$. Since homomorphic images and subgroups of groups of type $\oplus_i \oplus_p Z(p^{\chi_i(p)})$ are of the same type, the lemma is proved.

5. **Remarks.** Bowman [**2**] describes the closure properties of discrete and ideal discrete classes ([**2**], Propositions 2.6 and 2.7). He discusses the large subgroup topology (the topology described in 2.8), and observes that it is not a minimal functorial topology. For the rest he develops a subfunctor

E_T of Ext for any ideal functorial topology T: $E_T(C, A)$ contains those exact sequences $0 \to A \to B \to C \to 0$ for which T_A is the topology induced by T_B. His main result ([2], Theorem 2.2) states that E_T is a proper subfunctor of Ext. Janvier [8] develops in detail the categorical properties of the category of linear topological abelian groups. He also discusses completions and linearly compact and locally linearly compact groups. Zelger [12] describes explicitly the completions for the ideal functorial topology corresponding to the discrete class of all finite abelian groups (finite index topology). He proves a necessary and sufficient condition for the completion topology to coincide with the functorial topology. The topologies coincide in special cases only.

There is some hope that properties of the Z-adic and p-adic topologies can be generalized to ideal functorial topologies. Specifically, it would be interesting to study completions and to determine whether the completions have special injective properties as is the case with complete Z-adic and p-adic groups. A major problem is the determination of those minimal functorial topologies which are "completable" in the sense that all completions carry the functorial topology. The best results in this direction are due to Mines-Oxford [10]. It may also be worthwhile to investigate specific functorial topologies.

A characterization of all discrete classes seems very difficult because of the great variety of such classes.

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