ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 10, Number 1, Winter 1980

ON SOME PROBLEMS RELATED TO THE EXTENDED DOMAIN OF THE FOURIER TRANSFORM

P. SZEPTYCKI*

This note could have been written by N. Aronszajn (to whom the credit is due for the result in Theorem 1), and it is only proper that it should be dedicated to him.

Our objective here is to prove that the extended domain of the Fourier transform as defined in [1], [2] is the largest solid F space of functions whose Fourier transforms (in the sense of distributions) are functions. To make this statement precise, we have to recall some facts from [1] and [2].

If (X, dx) is a σ-finite measure space then $\mathfrak{M}(X)$ denotes the space of all measurable finite a.e. functions on X with the metric topology of convergence in measure on all subsets of X of finite measure. If (X, dx), (Y, dy) are two σ-finite measure spaces and $K \in \mathfrak{M}(X \times Y)$, then the *integral transformation* with the kernel K is a mapping from $\mathfrak{M}(X)$ into $\mathfrak{M}(Y)$ defined by $Kf(y) = \int_X K(x, y)f(x)\, dx$. The *proper domain* of K is defined by $\mathscr{D}_K = \{f \in \mathfrak{M}(X) : \int_X |K(x, y)f(x)|\, dx < \infty \text{ a.e.}\}$. K is *nonsingular* if there is $f \in \mathscr{D}_K$ such that $f > 0$ a.e.

The symbol $\subset_c$ will denote the continuous inclusion.

If A is an F-space, $A \subset_c \mathfrak{M}(X)$ then K is *A-semi-regular* (A-s.r.) if 1) $A \cap \mathscr{D}_K$ is dense in A, 2) the restriction $K|_{A \cap \mathscr{D}_K}$ is continuous from $A \cap \mathscr{D}_K$ (with the topology of A) into $\mathfrak{M}(Y)$. If K is A-s.r. then K can be extended to a continuous linear transformation $K_A : A \to \mathfrak{M}(Y)$ (which may no longer be an integral transformation).

An F-space $A \subset \mathfrak{M}(X)$ is *solid* provided $f \in A$, $g \in \mathfrak{M}(X)$, $|g(x)| \leqq |f(x)|$ a.e. imply $g \in A$. The class of solid F-subspaces of $\mathfrak{M}(X)$ we denote by FL.

The following result is quoted from [1].

PROPOSITION 1. *If K is a nonsingular integral transformation, then there exists an FL-subspace of $\mathfrak{M}$, denoted by $\tilde{\mathscr{D}}_K$, with the following properties:*

1) *K is $\tilde{\mathscr{D}}_K$-s.r.; denote $\tilde{K} = K_{\tilde{\mathscr{D}}_K}$,*
2) *If $A \in FL$ and K is A-s.r. then $A \subset_c \tilde{\mathscr{D}}_K$ and $K_A = \tilde{K}|_A$.*

$\tilde{\mathscr{D}}_K$ is referred to as *the extended domain* of K.

We turn now to the case of the Fourier transform; here $X = Y = \mathbf{R}^1$, $\mathfrak{M}(X) = \mathfrak{M}(\mathbf{R}^1) = \mathfrak{M}$, $K(x, y) \equiv \mathfrak{F}(x, y) = (2\pi^{-1/2} e^{-ixy}$, dx is the Lebesgue measure and the corresponding integral transform we denote by $\mathfrak{F}$. We

*This research was supported by NSF Grant GP-16292.

Received by the editors on April 13, 1978.

will consider only the one dimensional case, the results extend without essential changes to several dimensions and in suitable formulation to Borel measures (see [2]).

It is obvious that $\mathscr{D}_{\mathfrak{F}} = L^1(\mathbf{R}^1)$; the following proposition from [2] describes the extended domain $\tilde{\mathscr{D}}_{\mathfrak{F}}$.

PROPOSITION 2. *$\tilde{\mathscr{D}}_{\mathfrak{F}} = \{f \in L^1_{\mathrm{loc}}(\mathbf{R}^1);\ \sum_{-\infty}^{\infty}(\int_K^{k+1}|f(x)|)^2 < \infty\}$ with natural norm. In particular $\tilde{\mathscr{D}}_{\mathfrak{F}}$ is a Banach space. Also $\mathfrak{F}(\tilde{\mathscr{D}}_{\mathfrak{F}}) \subset L^2_{\mathrm{loc}}(\mathbf{R}^1)$ and $\mathfrak{F}\colon \tilde{\mathscr{D}}_{\mathfrak{F}} \to L^2_{\mathrm{loc}}(\mathbf{R}^1) \cap \mathscr{S}'(\mathbf{R}^1)$ is continuous.*

By $\hat{}$ we denote the Fourier transform in $\mathscr{S}'(\mathbf{R}^1)$ (in the sense of Schwartz distributions), $\hat{}\ : \mathscr{S}'(\mathbf{R}^1) \to \mathscr{S}'(\mathbf{R}^1)$ where $\mathscr{S}'(\mathbf{R}^1)$ is the space of tempered distributions on $\mathbf{R}^1$.

The main result of this note is

THEOREM 1. *Let $f \in L^1_{\mathrm{loc}}(\mathbf{R}^1) \cap \mathscr{S}'(\mathbf{R}^1)$; then the following properties of f are equivalent.*

i) *$g \in \mathfrak{M}$, $|g| \leqq |f|$ a.e. implies $\hat{g} \in L^1_{\mathrm{loc}}(\mathbf{R}^1)$,*

ii) *$f \in \tilde{\mathscr{D}}_{\mathfrak{F}}$.*

PROOF. ii) $\Rightarrow$ i) is almost immediate. For $f \in \mathscr{D}_{\mathfrak{F}} = L^1(\mathbf{R}^1)$ we have $\mathfrak{F}f = \hat{f}$ and, by Proposition 2, this implies that for $f \in \tilde{\mathscr{D}}_{\mathfrak{F}}$, $\tilde{\mathfrak{F}}f = \hat{f} \in L^2_{\mathrm{loc}}$. Since $\tilde{\mathscr{D}}_{\mathfrak{F}}$ is solid, $f \in \tilde{\mathscr{D}}_{\mathfrak{F}}$, $|g| \leqq |f|$ imply $g \in \tilde{\mathscr{D}}_{\mathfrak{F}}$ and $\hat{g} \in L^2_{\mathrm{loc}}(\mathbf{R}^1) \cap \mathscr{S}'(\mathbf{R}^1) \subset L^1_{\mathrm{loc}} \cap \mathscr{S}'(\mathbf{R}^1)$.

i) $\Rightarrow$ ii). For any f satisfying i) define $A_f = \{g \in \mathfrak{M}\colon |g| \leqq \alpha\, |f| \text{ a.e. for some } \alpha > 0\}$ and for $g \in A_f$

$$\|g\|_f = \inf\{\alpha > 0;\ |g| \leqq \alpha\, |f| \text{ a.e.}\}.$$

It is clear that A_f is a solid vector space of functions and that $\|\ \|_f$ is a norm on A_f. It is also easy to verify that A_f with the norm $\|\ \|_f$ is a Banach space.

We also have $A_f \subset \mathfrak{M}$.

i) implies that $\hat{}\ : A_f \to L^1_{\mathrm{loc}}$; let us show now that this mapping is continuous. If $\{g_n\} \subset A_f$, $g_n \to g$ in A_f, $\hat{g}_n \to h$ in L^1_{loc}, then for every $\varphi \in C_0^\infty(\mathbf{R}^1)$ we have $(\hat{g}_n, \varphi) \to (h, \varphi)$ and $(\hat{g}_n, \varphi) = (g_n, \hat{\varphi}) \to_{n\to\infty} (g, \hat{\varphi}) = (\hat{g}, \varphi)$ by the dominated convergence theorem. It follows that $\hat{g} = h$ which establishes the claim by virtue of the closed graph theorem.

Denote by A_f' the closure of $L^1(\mathbf{R}^1) \cap A_f$ in A_f. On $L^1(\mathbf{R}^1) \cap A_f$ $\hat{}$ coincides with $\mathfrak{F}$ and the continuity of $\hat{}$ implies that $\mathfrak{F}$ is A_f'-s.r. By Proposition 1, which is applicable since A_f' is solid, we get $A_f' \subset \tilde{\mathscr{D}}_{\mathfrak{F}}$.

If $\{a_n\}_{n=-\infty}^{\infty}$ is any sequence such that $a_n \to 0$ as $|n| \to \infty$, χ_k is the characteristic function of $(k, k+1]$, then it is easily checked that $f' = \sum a_k\chi_k\, f$ belongs to A_f'. By Proposition 2 we can write

$$\sum_{-\infty}^{\infty} \int_k^{k+1} |f'|^2 = \sum |a_k|^2 \Big(\int_k^{k+1} |f|\Big)^2 < \infty,$$

the latter being true for every sequence $\{a_n\}$ with $a_n \to 0$ as $|n| \to \infty$. This implies that $\sum_{-\infty}^{\infty}(\int_k^{k+1} |f|)^2 < \infty$, i.e. $f \in \tilde{\mathscr{D}}_{\mathfrak{F}}$.

REMARK. The construction of spaces A_f goes back to Orlicz (oral communication by Iwo Labuda).

It is of interest to replace the space L^1_{loc} in i) by F-subspaces $V \subset_c L^1_{\text{loc}}(\mathbf{R}^1)$ and consider the following version of condition i) of Theorem 1:

ii′) If $g \in \mathfrak{M}$, $|g| \leqq |f|$ a.e. then $\hat{g} \in V$.
We denote by $V_{\mathfrak{F}}$ the (vector) space of all functions f satisfying ii′).

THEOREM 2. *Let V be an F-space, $V \subset_c L^1_{\text{loc}}(\mathbf{R}^1) \cap \mathscr{S}'(\mathbf{R}^1)$, with translation invariant metric $\| \ \|_V$ (which may be assumed to be bounded). Then:*
a) $\|f\|_V^{\mathfrak{F}}$ sup $= \{\|\hat{g}\|_V : |g| \leqq |f|\}$ *is a translation invariant metric on* $V_{\mathfrak{F}}$,
b) $V_{\mathfrak{F}}$ *with the metric* $\| \ \|_V^{\mathfrak{F}}$ *is an FL space,*
c) $V_{\mathfrak{F}} \subset_c \tilde{\mathscr{D}}_{\mathfrak{F}}$.
d) *In the case when V is a Banach space with the norm* $\| \ \|_V$, $\| \ \|_V^{\mathfrak{F}}$ *is a norm and* $V_{\mathfrak{F}}$ *with the norm* $\| \ \|_V^{\mathfrak{F}}$ *is a Banach space*

PROOF. a) The only item which is not obvious is the triangle inequality. It is clear that $\|f + g\|_V^{\mathfrak{F}} \leqq \| |f| + |g| \|_V^{\mathfrak{F}}$ and it suffices to consider the case when $f, g \geqq 0$. If $|h| \leqq f + g$ then $h = h' + h''$ where $h' = (f+g)^{-1}fh$, $h'' = (f+g)^{-1}gh$, if $f + g > 0$, $h' = h'' = 0$ for $f + g = 0$, and $|h'| \leqq f$, $|h''| \leqq g$. It follows that $\|\hat{h}\|_V \leqq \|\hat{h}'\|_V + \|\hat{h}''\|_V \leqq \|f\|_V^{\mathfrak{F}} + \|g\|_V^{\mathfrak{F}}$.

b) It is clear that the space $V_{\mathfrak{F}}$ is solid; to verify that is complete let $\{f_n\}_{n=1}^{\infty} \subset V_{\mathfrak{F}}$ be a Cauchy sequence. By choosing a subsequence we may assume that $\sum_{n=1}^{\infty} \|f_{n+1} - f_n\|_V^{\mathfrak{F}} < \infty$; we can also assume that $f_1 = 0$. Define $\tilde{f} = \sum |f_{n+1} - f_n|$ and let χ_R be the characteristic function of $[-R, R]$. By the definition of the metric $\| \ \|_V^{\mathfrak{F}}$ the series $\sum(\chi_R|f_{n+1} - f_n|)\hat{}$ is convergent in V, and since $V \subset_c L^1_{\text{loc}}$, it is also convergent in $\mathfrak{M}$. This implies that the series

$$\sum_1^{\infty} \int_{-R}^{R} |f_{n+1}(x) - f_n(x)| \cos xydy$$

is convergent for almost every y in $(-\pi/2R, \pi/2R)$, and thus $\tilde{f} \in L^1_{\text{loc}}(\mathbf{R}^1)$. (This part of the argument is taken from (**1**, Th. 9.3]). If $g \in \mathfrak{M}$, $|g| \leqq \tilde{f}$ a.e., then the functions $g_n(x) = g(x)|f_{n+1}(x) - f_n(x)|\tilde{f}(x)^{-1}$ when $\tilde{f}(x) \neq 0$, $g_n(x) = 0$ if $\tilde{f}(x) = 0$, satisfy $|g_n| \leqq |f_{n+1} - f_n|$ and hence $\sum_1^{\infty} \|\hat{g}_n\|_V < \infty$; in particular $\sum_1^{\infty} \hat{g}_n$ is convergent in V and thus in $L^1_{\text{loc}}(\mathbf{R}^1)$. Obviously $\sum g_n = g$, $\sum |g_n| = |g|$, both series convergent pointwise a.e. If follows that for every $\hat{\varphi} \in C_0^{\infty}(\mathbf{R}^1)$, $(g, \hat{\varphi}) = \sum(g_n, \hat{\varphi}) = \sum(\hat{g}_n, \varphi) = (\sum \hat{g}_n, \varphi)$ and $\hat{g} = \sum_1^{\infty} \hat{g}_n \in V$ (again we have used the hypothesis that $V \subset_c L^1_{\text{loc}}(\mathbf{R}^1) \cap \mathscr{S}'(\mathbf{R}^1)$ and the dominated convergence theorem). We conclude that

$\tilde{f} \in V_{\mathfrak{F}}$; if $f = \sum(f_{n+1} - f_n)$ then $|f| \leqq \tilde{f}$ and $f \in V_{\mathfrak{F}}$. It is now a routine matter to verify that $f_n \to f$ in $V_{\mathfrak{F}}$ as $n \to \infty$.

c) follows from Theorem 1.

d) The only point to check is that $\|f\|_V^{\mathfrak{F}} < \infty$ for every $f \in V_{\mathfrak{F}}$. To this effect consider again the space A_f as defined in the proof of Theorem 1. ii′) implies that $\hat{}: A_f \to V$ and by closed graph theorem the mapping is continuous (here we use the hypothesis that $V \subset_c L^1_{\text{loc}}(\mathbf{R}^1) \cap \mathscr{S}'(\mathbf{R}^1)$). The set $\{g: |g| \leqq |f| \text{ a.e.}\}$ coincides with the unit ball in A_f and by the continuity of $\hat{}$, $\|\hat{g}\|_V$ is bounded on this set.

K. Bichteller conjectured that in the case when $V = L^q(\mathbf{R}^1)$, $2 \leqq p \leqq \infty$, $V_{\mathfrak{F}} = L^{p'}(\mathbf{R}^1)$ where $(1/p) + (1/p') = 1$.

For $q = 2$ the conjecture is obviously true; it is also true for $q = \infty$. For the following argument we are indebted to W. F. Donoghue Jr.

Suppose that $f \in L^\infty_{\mathfrak{F}}$; without loss of generality we can assume that $f \geqq 0$. Let $\varphi \in \mathscr{S}(\mathbf{R}^1)$ be such that $\varphi \geqq 0$, $\hat{\varphi} \geqq 0$, $\varphi \not\equiv 0$ and $\varphi(-x) = \varphi(x)$. Then with $\varepsilon > 0$ we can write

$$\int_{\mathbf{R}^1} \hat{\varphi}(\varepsilon x) f(x)\, dx = \frac{1}{\varepsilon} \int_{\mathbf{R}^1} \varphi(y/\varepsilon) \hat{f}(y)\, dy \leqq \|\hat{f}\|_\infty \frac{1}{\varepsilon} \int_{\mathbf{R}^1} \varphi(y/\varepsilon)\, dy$$
$$= (2\pi)^{1/2}\, \hat{\varphi}(0)\, \|\hat{f}\|_\infty.$$

Letting $\varepsilon \to 0$ we conclude, using Fatou's lemma, that $\int_{\mathbf{R}^1} f \leqq (2\pi^{1/2} \|\hat{f}\|_\infty$ as announced.

However, for $2 < q < \infty$ the problem remains open.

The problems of the same nature can be raised for other integral transformations, e.g., those connected with Fourier series. In this case $X = [0, 2\pi]$, $Y = \mathbf{Z}$, $K(x, y) = (1/2\pi)\, e^{-ixy}$ and

$$(Kf)(y) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ixy}\, dx,\ y = 0, \pm 1, \pm 2,$$

or

$$X = \mathbf{Z},\ Y = [0, 2\pi],\ K(x, y) = e^{ixy}$$

and

$$(Kf)(y) = \sum_{x=-\infty}^{\infty} f(x) e^{ixy}.$$

In the first case $\mathscr{D}_K = \tilde{\mathscr{D}}_K = L^1(0, 2\pi)$, and in the second $\mathscr{D}_K = l^1(\mathbf{Z})$, $\tilde{\mathscr{D}}_K = l^2(\mathbf{Z})$.

For a subspace V of $\mathfrak{M}(Y)$ one can define $V_K = \{f \in \mathfrak{M}(X)\ ;\ \tilde{K}f \in V\}$ and try to characterize V_K directly. As in the case of $\mathfrak{F}$ the problem seems to be open except for obvious extreme cases.

REFERENCES

1. N. Aronszajn and P. Szeptycki, *On general integral transformations*, Math. Annalen **163** (1966), 127–154.

2. P. Szeptycki, *On functions and measures whose Fourier transforms are functions*, Math. Ann. **179** (1968), 31–41.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KS 66044