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TIME DECAY AND THE BORN SERIES

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ABSTRACT. A time decay estimate from scattering theory always implies the convergence of the Born series at high energies. That is, if H_0 and V are selfadjoint operators in Hilbert space, and $V\exp(-itH_0)$ is integrable (in a certain sense), then the series expansion of $(H_0 + V - \lambda \pm iO)^{-1}$ in terms of $V(H_0 - \lambda \pm iO)^{-1}$ converges for sufficiently large λ. This abstract result is applied to Schrödinger operators $-\Delta + V$, generalizing work of Zemach and Klein.

1. **Introduction.** Let H_0 be a selfadjoint operator acting in a Hilbert space $\mathfrak{H}$. Consider its resolvent $(H_0 - z)^{-1}$, for z not real. If H_0 has absolutely continuous spectrum, its resolvent will have boundary values $(H_0 - \lambda \pm iO)^{-1}$ for λ real. The values of these operators will lie in a larger space.

In a perturbation problem we consider another selfadjoint operator $H_0 + V$. Then $(H_0 + V - z)^{-1}$ exists for z not real and we may ask about its boundary values. (They occur in expressions for the S operator and wave operators, as well as for spectral representations and spectral projections.) The most elementary approach is through the Born series

$$(H_0 + V - \lambda \pm iO)^{-1} = (H_0 - \lambda \pm iO)^{-1} \sum_{n=0}^{\infty} (-V(H_0 - \lambda \pm iO)^{-1})^n.$$

If this converges for some range of λ, then $H_0 + V$ must have absolutely continuous spectrum there.

One may often expect convergence for all λ when V is sufficiently small. However, this is a very special case, since in general $H_0 + V$ may have eigenvalues in addition to continuous spectrum. On the other hand, it is known that the Born series gives a useful approximation for sufficiently large energies λ. It will be shown here that whenever $V\exp(-itH_0)$ is integrable with respect to t, the Born series converges for sufficiently high energies, whatever the strength of the coupling. The advantage of this criterion is that estimates on the norm of $V\exp(-itH_0)$ for large t are available from time dependent scattering theory. However, the question of measur-

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ability is a new feature. (Recall that $\exp(-itH_0)$ is never measurable in the usual uniform operator norm, unless H_0 is bounded.)

This abstract result is applied to the case where $\mathfrak{H} = L^2(\boldsymbol{R}^{\mathfrak{l}})$, $\mathfrak{l} \geqq 3$, and the operator is the Schrödinger operator $-\Delta + V$. V is a real valued function on $\boldsymbol{R}^{\mathfrak{l}}$ which is required to be in certain L^p spaces, in order to ensure decrease at infinity. There is no other restriction on the size of V, nor is it required to be continuous or positive.

The result is interesting in that for $\mathfrak{l} > 3$ the operator $V(H_0 - \lambda \pm iO)^{-1}$ is never Hilbert-Schmidt and its kernel is not bounded uniformly in λ (in nontrivial cases). This generalizes and simplifies work of Zemach and Klein [5], who treat the case $\mathfrak{l} = 3$, and of Aaron and Klein [1], who deal with spherically symmetric V.

2. **The notion of a scale of Hilbert spaces.** We will refer to linear transformations as operators. If $\mathfrak{H}$ and $\mathfrak{H}'$ are Hilbert spaces, the Banach space of continuous operators defined on $\mathfrak{H}$ with values in $\mathfrak{H}'$ will be denoted $\mathfrak{L}(\mathfrak{H}, \mathfrak{H}')$. The norm of A in $\mathfrak{L}(\mathfrak{H}, \mathfrak{H}')$ is $\|A\| = \sup\{\|Af\|;\ \|f\| \leqq 1\}$. $\mathfrak{L}(\mathfrak{H}, \mathfrak{H})$ will be abbreviated as $\mathfrak{L}(\mathfrak{H})$. We will make the convention that if A is an operator defined on a dense subspace of $\mathfrak{H}$ with values in $\mathfrak{H}'$ and is continuous, then A will be identified with its extension by continuity to an operator in $\mathfrak{L}(\mathfrak{H}, \mathfrak{H}')$.

Let $\mathfrak{H}$ be a Hilbert space. The inner product of f and g in $\mathfrak{H}$ will be denoted $\langle f, g\rangle$. The norm of f is $\|f\| = \langle f, f\rangle^{1/2}$. Let T be a selfadjoint operator such that T^{-1} is in $\mathfrak{L}(\mathfrak{H})$. Let $\mathfrak{K}$ be the domain of T with the norm $\|g\|_{\mathfrak{K}} = \|Tg\|$. Let $\mathfrak{K}^*$ be the completion of $\mathfrak{H}$ with the norm $\|f\|_{\mathfrak{K}^*} = \|T^{-1}f\|$. Then $\mathfrak{K} \subset \mathfrak{H} \subset \mathfrak{K}^*$. For g in $\mathfrak{K}$ we have $|\langle f, g\rangle| = |\langle T^{-1}f, Tg\rangle| \leqq \|T^{-1}f\|\,\|Tg\| = \|f\|_{\mathfrak{K}^*}\|g\|_{\mathfrak{K}}$. Hence we may define $\langle f, g\rangle$ for f in $\mathfrak{K}^*$ and g in $\mathfrak{K}$ in such a way that $|\langle f, g\rangle| \leqq \|f\|_{\mathfrak{K}^*}\,\|g\|_{\mathfrak{K}}$. Such a triple $\mathfrak{K} \subset \mathfrak{H} \subset \mathfrak{K}^*$ will be called a scale.

Notice that any operator A in $\mathfrak{L}(\mathfrak{H})$ determines an operator in $\mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*)$, by restriction. We will continue to denote this by A.

EXAMPLE 1. Let $\mathfrak{H} = L^2(\boldsymbol{R}^{\mathfrak{l}}, dx)$. Let $\rho(x) = (1 + r^2)^{1/2}$, where $r = |x|$, for x in $\boldsymbol{R}^{\mathfrak{l}}$. (The most important properties of the function ρ are that $\rho(x) \geqq 1$ and $\rho(x) \sim r$ for x large.) Choose $s \geqq 0$. Define T to be the operator of multiplication by $\rho^{s/2}$. Then $\mathfrak{K} = L^2(\boldsymbol{R}^{\mathfrak{l}}, \rho(x)^s dx)$ and $\mathfrak{K}^* = L^2(\boldsymbol{R}^{\mathfrak{l}}, \rho(x)^{-s} dx)$.

If S is a subset of the real line, we write 1_S for the function defined on the line which is 1 on S and 0 elsewhere. Let $\mathfrak{H}$ be a Hilbert space and let H be a selfadjoint operator acting in $\mathfrak{H}$. If

S is a Borel measurable subset of the real line, then it is natural to denote the spectral projection of H associated with S by $1_S(H)$. $1_S(H)$ is of course in $\mathfrak{L}(\mathfrak{H})$. Notice however, that if $\mathfrak{K}$ is a dense subspace of $\mathfrak{H}$, $1_S(H)$ is uniquely determined by its matrix elements $\langle 1_S(H)f, g\rangle$, f, g in $\mathfrak{K}$. In particular, if $\mathfrak{K} \subset \mathfrak{H} \subset \mathfrak{K}^*$ is a scale, $1_S(H)$ in $\mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*)$ determines $1_S(H)$ in $\mathfrak{L}(\mathfrak{H})$.

PROPOSITION 1 (HOWLAND [2]). *Let $\mathfrak{K} \subset \mathfrak{H} \subset \mathfrak{K}^*$ be a scale. Let H be a selfadjoint operator acting in $\mathfrak{H}$. Assume that I is an open interval of real numbers such that $(H - z)^{-1}$ in $\mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*)$, $\operatorname{Im} z \neq 0$, has continuous boundary values $(H - \lambda \pm iO)^{-1}$ in $\mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*)$ for λ in I. Let*

$$\delta(H - \lambda) = (1/2\pi i)[H - \lambda - iO)^{-1} - (H - \lambda + iO)^{-1}] \quad in\ \mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*).$$

Then for any Borel set S contained in I,

$$1_S(H) = \int_S \delta(H - \lambda)d\lambda$$

as an equation in $\mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*)$. It follows that the spectrum of H must be absolutely continuous in the interval I.

PROOF. If $\delta_\epsilon(x) = (1/2\pi i)[(x - i\epsilon)^{-1} - (x + i\epsilon)^{-1}]$, then $\lim_{\epsilon \downarrow 0} \int_a^b \delta_\epsilon(x - \lambda)d\lambda = 1$ if $a < x < b$, 0 if $x < a$ or $b < x$. Furthermore, $\int_a^b \delta_\epsilon(x - \lambda)d\lambda$ is bounded by 1 for all x and all $\epsilon > 0$. The spectral theorem then implies (assuming a and b are not in the point spectrum of H) that for f in $\mathfrak{H}$, $\lim_{\epsilon\downarrow 0} \int_a^b \delta_\epsilon(H - \lambda)f d\lambda = 1_{[a,b]}(H)f$, with convergence in $\mathfrak{H}$.

It follows that for $[a, b]$ contained in I, $1_{[a,b]}(H) = \int_a^b \delta(H - \lambda)d\lambda$ as an equation in $\mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*)$. The conclusion is immediate.

EXAMPLE 2. Let $\mathfrak{H} = L^2(\mathbf{R}^3, dx)$ and $\mathfrak{K} \subset \mathfrak{H} \subset \mathfrak{K}^*$ be as in Example 1, with $s > 2$. If $H_0 = -\Delta$, then $(H_0 - z)^{-1}$ is in $\mathfrak{L}(\mathfrak{H})$ for z off the positive real axis. For such z it is convolution by the integrable function $(4\pi r)^{-1} \exp(-(-z)^{1/2}r)$, where $r = |x|$. (The square root $(-z)^{1/2}$ is taken in the right half-plane.) $(H_0 - z)^{-1}$ may also be regarded as an operator in $\mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*)$. Then, as we shall see, it has continuous boundary values $(H_0 - \lambda \pm iO)^{-1}$ in $\mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*)$ on the positive real axis. For $\lambda \geqq 0$, $(H_0 - \lambda \mp iO)^{-1}$ is convolution by $(4\pi r)^{-1} \exp(\pm i\lambda^{1/2} r)$. $\delta(H_0 - \lambda)$ in $\mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*)$ is convolution by $(4\pi^2 r)^{-1} \sin(\lambda^{1/2} r)$.

3. **The Born series.** Let H_0 be a selfadjoint operator acting in the Hilbert space $\mathfrak{H}$. Then for t real, $\exp(-itH_0)$ is a unitary operator in $\mathfrak{L}(\mathfrak{H})$.

LEMMA 1. *Let $\mathfrak{H}$ be a Hilbert space. Let H_0 be a selfadjoint* *operator acting in* $\mathfrak{H}$. *Let* U *and* W *be selfadjoint operators acting in* $\mathfrak{H}$. *Assume that* $U \exp(-itH_0)W$ *is in* $\mathfrak{L}(\mathfrak{H})$ *for each* $t \neq 0$ *and that it is an integrable function of* t *with values in the Banach space* $\mathfrak{L}(\mathfrak{H})$. *Then for* z *not real,* $U(H_0 - z)^{-1}W$ *is in* $\mathfrak{L}(\mathfrak{H})$ *and has continuous boundary values as* z *approaches the real axis from either half-plane. Furthermore, the boundary values* $U(H_0 - \lambda \pm iO)^{-1}W \to 0$ *in* $\mathfrak{L}(\mathfrak{H})$ *as* $\lambda \to \pm\infty$.

Proof. For $\operatorname{Im} z > 0$,

$$U(H_0 - z)^{-1}W = i \int_0^\infty \exp(itz)\, U \exp(-itH_0)W\, dt.$$

Hence,

$$U(H_0 - \lambda - iO)^{-1}W = i \int_0^\infty \exp(it\lambda) U \exp(-itH_0)W\, dt.$$

The conclusion follows from the Riemann-Lebesgue lemma (see Appendix).

The other case is similar.

Let $\mathfrak{H}$ be a Hilbert space. Let H_0 and V be selfadjoint operators acting in $\mathfrak{H}$. If $V(H_0 - z)^{-1}$ is in $\mathfrak{L}(\mathfrak{H})$ with $\|V(H_0 - z)^{-1}\| < 1$ for z sufficiently imaginary, then V will be said to be a relatively small perturbation of H_0. For example, if V is in $\mathfrak{L}(\mathfrak{H})$, then $\|V(H_0 - z)^{-1}\| \leqq \|V\| / |\operatorname{Im} z|$, so V is a relatively small perturbation of H_0.

If V is a relatively small perturbation of H_0, then $(H_0 + V - z)^{-1}$ in $\mathfrak{L}(\mathfrak{H})$ is given by

$$(H_0 + V - z)^{-1} = (H_0 - z)^{-1} \sum_{n=0}^{\infty} (-V(H_0 - z)^{-1})^n$$

for z sufficiently imaginary. It follows by analytic continuation that $(H_0 + V - z)^{-1}$ is in $\mathfrak{L}(\mathfrak{H})$ for all z not real. Hence $H_0 + V$ is selfadjoint with the same domain as H_0.

Theorem 1. *Let* $\mathfrak{H}$ *be a Hilbert space. Let* H_0 *and* V *be selfadjoint operators acting in* $\mathfrak{H}$. *Assume that* V *is a relatively small perturbation of* H_0. *Let* $\mathfrak{K} \subset \mathfrak{H} \subset \mathfrak{K}^*$ *be a scale. Assume that* $\exp(-itH_0)$ *is an integrable function of* t *with values in the space* $\mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*)$. *Assume also that for* $t \neq 0$, $V \exp(-itH_0)$ *is in* $\mathfrak{L}(\mathfrak{K})$ *and that it is an integrable function of* t *with values in the space* $\mathfrak{L}(\mathfrak{K})$. *Then* $(H_0 - z)^{-1}$, *for* z *not real, has continuous boundary values* $(H_0 - \lambda \pm iO)^{-1}$ *in* $\mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*)$. $V(H_0 - z)^{-1}$ *is in* $\mathfrak{L}(\mathfrak{K})$ *for* z *not* *real and has continuous boundary values* $V(H_0 - \lambda \pm iO)^{-1}$ *in* $\mathfrak{L}(\mathfrak{K})$. *Furthermore,* $V(H_0 - \lambda \pm iO)^{-1} \to 0$ *in* $\mathfrak{L}(\mathfrak{K})$ *as* $\lambda \to \pm\infty$. *If* λ *is such that the norm of* $V(H_0 - \lambda \pm iO)^{-1}$ *in* $\mathfrak{L}(\mathfrak{K})$ *is strictly less than* 1, *then* $(H_0 + V - z)^{-1}$ *in* $\mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*)$ *has continuous boundary values* $(H_0 + V - \lambda \pm iO)^{-1}$ *in* $\mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*)$ *given by*

$$(H_0 + V - \lambda \pm iO)^{-1} = (H_0 - \lambda \pm iO)^{-1} \sum_{n=0}^{\infty} (-V(H_0 - \lambda \pm iO)^{-1})^n.$$

The spectrum of $H_0 + V$ *is absolutely continuous in this range of* λ.

PROOF. Let the scale $\mathfrak{K} \subset \mathfrak{H} \subset \mathfrak{K}^*$ be defined by the operator T. Then T is an isomorphism from $\mathfrak{K}$ to $\mathfrak{H}$ and from $\mathfrak{H}$ to $\mathfrak{K}^*$. Thus we may apply Lemma 1 to the operators $T^{-1}\exp(-itH_0)T^{-1}$ and $TV\exp(-itH_0)T^{-1}$.

The absolute continuity of the spectrum follows from Proposition 1. Notice that

$$\|V(H_0 - \lambda - iO)^{-1}\|_{\mathfrak{L}(\mathfrak{K})} \leqq \int_0^\infty \|V\exp(-itH_0)\|_{\mathfrak{L}(\mathfrak{K})}dt$$

and analogously for $V(H_0 - \lambda + iO)^{-1}$. The integral on the right-hand side of this inequality is independent of λ. Thus if it is strictly less than one, the Born series converges for all real λ. (In this case, the spectrum of $H_0 + V$ consists *only* of absolutely continuous spectrum.)

4. **Schrödinger operators.** Let $\mathfrak{H} = L^2(\boldsymbol{R}^l, dx)$. For t real, $\exp(it\Delta)$ is a unitary operator on $\mathfrak{H}$. If $t \neq 0$, and f is an integrable function in $\mathfrak{H}$, then $\exp(it\Delta)$ is the convolution of the bounded function $(4\pi it)^{-l/2}\exp(ir^2/4t)$ with f. Thus $\exp(it\Delta)$, $t \neq 0$, is determined on a dense subspace of $\mathfrak{H}$ as an integral operator with bounded kernel.

LEMMA 2. *Let* $\mathfrak{H} = L^2(\boldsymbol{R}^l, dx)$. *Let* U *and* W *be in* $L^p(\boldsymbol{R}^l, dx)$ *for some* p, $2 \leqq p \leqq \infty$. *Then* $U\exp(it\Delta)W$ *is in* $\mathfrak{L}(\mathfrak{H})$ *for* $t \neq 0$, *and* $\|U\exp(it\Delta)W\| \leqq C\|U\|_p\,\|W\|_p\, t^{-l/p}$ *for some constant* C.

PROOF. Choose r and s so that $1/2 + 1/p = 1/r$, $1/r + 1/s = 1$ and $1/s + 1/p = 1/2$. Then the result follows immediately from the estimate $\|\exp(it\Delta)g\|_s \leqq Ct^{-l/p}\|g\|_r$. This may be proved by writing the integral operator $\exp(it\Delta)$ in the form of a Fourier transform and applying the Hausdorff-Young theorem (Kato [4, p. 277]).

A function F with values in a Banach space is integrable if and only if $\int\|F(t)\|dt < \infty$ and F is measurable. If the Banach space is nonseparable, it is quite easy to construct examples of nonmeasurable functions.

EXAMPLE 3. Let H be an unbounded selfadjoint operator acting in $\mathfrak{H}$. Regard $\exp(-itH)$ as a function having values in the Banach space $\mathfrak{L}(\mathfrak{H})$. This function is not measurable. (However, for each f in $\mathfrak{H}$, $\exp(-itH)f$ is a continuous function with values in $\mathfrak{H}$.)

To see this, observe that

$$\exp(-itH) - 1 = \int_0^1 \exp(isH)[\exp(-i(s+t)H) - \exp(-isH)]\, ds,$$

so that

$$\|\exp(-itH) - 1\| \leqq \int_0^1 \|\exp(-i(s+t)H) - \exp(-isH)\| ds.$$

If $\exp(-isH)$ were measurable, then by Proposition 4 of the Appendix the right hand would approach zero as $t \to 0$. We would have as a consequence that $\|\exp(-itH) - 1\| < 1$ for t sufficiently small. Then $-itH = \log(1 + (\exp(-itH) - 1))$ could be expressed as a power series which converges in norm. So H would be bounded.

The Riemann-Lebesgue lemma states that the Fourier transform (or Fourier coefficients) of an integrable function goes to zero at infinity. The conclusion of the Riemann-Lebesgue lemma may fail when the function is nonmeasurable. Again it is easy to construct simple examples when the space is nonseparable.

EXAMPLE 4. Let S^1 be the unit circle and let $\mathfrak{H} = L^2(S^1)$. Note that

$$\int_0^{2\pi} \exp(-int) \exp\left(t\frac{d}{d\theta}\right) f(\theta)dt = \int_0^{2\pi} \exp(-int) f(\theta + t)dt$$

$$= \exp(in\theta) \int_0^{2\pi} \exp(-int) f(t)dt.$$

Thus the nth Fourier coefficient of $\exp(t(d/d\theta))$ is the projection onto the subspace spanned by $\exp(in\theta)$.

We may regard $\exp(t(d/d\theta))$ as a function of t with values which are unitary operators in the Banach space $\mathfrak{L}(\mathfrak{H})$ with the uniform norm. We have $\int_0^{2\pi} \|\exp(t(d/d\theta))\| dt < \infty$, and yet the projections each have norm one. Thus they do not converge to zero in the norm of $\mathfrak{L}(\mathfrak{H})$ as $n \to \pm\infty$. The conclusion of the Riemann-Lebesgue lemma fails.

The explanation for this is that $\exp(t(d/d\theta))$ is not a measurable function of t. In fact, $\|\exp(t(d/d\theta)) - \exp(s(d/d\theta))\| \geqq \sqrt{2}$ for $0 < |t-s| < 2\pi$, so the values of this function cannot lie in a separable subspace of $\mathfrak{L}(\mathfrak{H})$. (Thus it cannot possibly be approximated by continuous functions.)

Let $\mathfrak{H}$ be a separable Hilbert space. Then the compact operators in $\mathfrak{L}(\mathfrak{H})$ form a separable subspace of the Banach space $\mathfrak{L}(\mathfrak{H})$. Thus it is not so easy to find an example of a function whose values are compact operators which is not measurable.

LEMMA 3. *Let $\mathfrak{H} = L^2(\mathbf{R}^l, dx)$. Let U and W be in $L^p(\mathbf{R}^l, dx)$ for some p, $2 \leqq p < \infty$, or in the closure of $L^2 \cap L^\infty$ in $L^\infty(\mathbf{R}^l, dx)$. Then $U \exp(it\Delta)W$ is a measurable function of t with values in $\mathfrak{L}(\mathfrak{H})$. In fact for $t \neq 0$ it is continuous and its values are compact operators.*

PROOF. Let $F(t) = U \exp(it\Delta)W$. Then Lemma 2 shows that for $t \neq 0$, $F(t)$ is in $\mathfrak{L}(\mathfrak{H})$.

Consider first the case $p = 2$. For $t \neq 0$, $F(t)$ is an intergral operator with kernel

$$(4\pi it)^{-l/2}U(x) \exp(i|x-y|^2/4t)W(y).$$

This is dominated (locally in t) by a constant times $|U(x)V(y)|$ which is in $L^2(\mathbf{R}^{2l}, dxdy)$. Hence the kernel varies continuously with t in $L^2(\mathbf{R}^{2l}, dxdy)$, by the dominated convergence theorem. That is, $F(t)$ is continuous for $t \neq 0$ in the Hilbert-Schmidt norm. It follows that for $t \neq 0$, $F(t)$ is continuous in the uniform norm and its values are compact operators.

Now consider the general case of U, W in L^p. Let U_n, W_n be a sequence in $L^2 \cap L^p$ which converges to U, W in L^p. Let $F_n(t) = U_n \exp(it\Delta)W_n$. It follows from Lemma 2 that $F_n(t)$ converges to $F(t)$ in the $\mathfrak{L}(\mathfrak{H})$ norm, uniformly on compact sets of $t \neq 0$. Hence for $t \neq 0$ each $F(t)$ is a compact operator and $F(t)$ is continuous in t.

PROPOSITION 2. Let $\mathfrak{H} = L^2(\mathbf{R}^l, dx)$. *Let V be a real valued function in some space L^p, $p > l/2$, $p \geqq 2$. Then V is a relatively small perturbation of $-\Delta$ and $-\Delta + V$ is selfadjoint with the same domain as $-\Delta$.*

PROOF. Let $1/p + 1/2 = 1/r$, $1/r + 1/s = 1$, $1/s + 1/p = 1/2$. Then if $p \geqq 2$, we have

$$\begin{aligned}\|(-\Delta - z)^{-1}f\|_s &\leqq \|(k^2 - z)^{-1}\hat{f}\|_r \leqq \|(k^2-z)^{-1}\|_p \|f\|_2 \\ &= \|(k^2-z)^{-1}\|_p \|\hat{f}\|,\end{aligned}$$

by the Hausdorff-Young and Plancherel theorems. But if $p > l/2$, then $(k^2 - z)^{-1}$ is in L^p and its L^p norm approaches zero as z becomes imaginary or negative.

If V is in L^p, then

$$\begin{aligned}\|V(-\Delta - z)^{-1}f\|_2 &\leqq \|V\|_p \, \|(-\Delta - z)^{-1}f\|_s \\ &\leqq \|V\|_p \, \|(k^2 - z)^{-1}\|_p \, \|f\|_2.\end{aligned}$$

Thus $V(-\Delta - z)^{-1} \to 0$ in $\mathfrak{L}(\mathfrak{H})$.

LEMMA 4. *Let* $\mathfrak{H} = L^2(\boldsymbol{R}^l, dx)$, $l \geqq 3$. *Let* U *and* W *be in* $L^p(\boldsymbol{R}^l)$ *and in* $L^q(\boldsymbol{R}^l)$ *for some* $p > l$, $q < l$. *Then the boundary values* $U(-\Delta - \lambda + iO)^{-1}W$ *exist as operators in* $\mathfrak{L}(\mathfrak{H})$ *and are continuous in* λ. *For each* λ, $U(-\Delta - \lambda \pm iO)^{-1}W$ *is a compact operator. As* $\lambda \to +\infty$, $\|U(-\Delta - \lambda \pm iO)^{-1}W\| \to 0$.

PROOF. Since $l \geqq 3$ we can take $q \geqq 2$. From Lemma 2 we see that we can estimate $\|U \exp(it\Delta)W\|$ by $t^{-l/p}$ near 0 and by $t^{-l/q}$ near infinity. On the other hand, Lemma 3 implies that $U \exp(it\Delta)W$ is continuous for $t \neq 0$. Therefore, Lemma 1 shows that $U(-\Delta - \lambda \pm iO)^{-1}W$ is in $\mathfrak{L}(\mathfrak{H})$ and approaches zero as $\lambda \to +\infty$.

Lemma 3 also implies that $U \exp(it\Delta)W$ is a compact operator for $t \neq 0$. But the compact operators form a closed subspace of $\mathfrak{L}(\mathfrak{H})$. The integral of a function with values in the Banach space of compact operators lies in the same space. Thus $U(-\Delta - \lambda \pm iO)^{-1}W$ is compact.

Notice that this last argument does not show that $U(-\Delta - \lambda \pm iO)^{-1}W$ is Hilbert-Schmidt. Even if $U \exp(it\Delta)W$ is Hilbert-Schmidt for all $t \neq 0$, it may not be integrable as a Hilbert-Schmidt operator valued function.

For the next theorem we will assume that for some $s > 2$, $\rho^{s/2}V$ is in L^p and in L^q for some $p > l$, $q < l$. It follows that V is in L^p and in L^q for some $p > l$, $q < l/2$.

The condition that $\rho^{s/2}V$ is in L^q for some $q < l$ says roughly that V decreases faster than r^{-2} at infinity.

THEOREM 2. *Let* $\mathfrak{H} = L^2(\boldsymbol{R}^l, dx)$, $l \geqq 3$. *Let* $\mathfrak{K} \subset \mathfrak{H} \subset \mathfrak{K}^*$ *be a scale* $L^2(\boldsymbol{R}^l, \rho(x)^s dx) \subset L^2(\boldsymbol{R}^l, dx) \subset L^2(\boldsymbol{R}^l, \rho(x)^{-s}dx)$ *with* $s > 2$.

Let V *be a real valued function on* $\boldsymbol{R}^l$ *such that* $\rho^{s/2}V$ *is in* L^p *and in* L^q *for some* $p > l$, $q < l$.

Then $(-\Delta - \lambda \pm iO)^{-1}$ *is in* $\mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*)$ *and* $V(-\Delta - \lambda \pm iO)^{-1}$ *is in* $\mathfrak{L}(\mathfrak{K})$ *for all real* λ. *Further,* $V(-\Delta - \lambda \pm iO)^{-1} \to 0$ *as* $\lambda \to +\infty$. *Thus if* λ *is sufficiently large,* $V(-\Delta - \lambda \pm iO)^{-1}$ *is in* $\mathfrak{L}(\mathfrak{K})$ *and has norm* < 1, *so that*

$$(-\Delta + V - \lambda \pm iO)^{-1}$$

$$= \sum_{n=0}^{\infty} (-\Delta - \lambda \pm iO)^{-1} (-V(-\Delta - \lambda \pm iO)^{-1})^n$$

in $\mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*)$. *In particular, the spectrum of* $-\Delta + V$ *in this range of* λ *is absolutely continuous.*

PROOF. First of all, note that since $\mathfrak{l} \geqq 3$, and V is in L^p for some $p > \mathfrak{l}$, the hypothesis of Proposition 2 is satisfied and V is a relatively small perturbation of $-\Delta$. Thus $-\Delta + V$ is selfadjoint.

Let $U(x) = \rho(x)^{s/2}V(x)$ and $W(x) = \rho(x)^{-s/2}$. Then U and W are in L^p and in L^q for $p > \mathfrak{l}$ and $q < \mathfrak{l}$. Thus Lemma 4 applies to show $V(-\Delta - \lambda \pm iO)^{-1}$ is in $\mathfrak{L}(\mathfrak{K})$ and approaches zero as $\lambda \to +\infty$. Thus we are in the situation of the abstract Theorem 1.

5. **Schrödinger operators with local singularities.** In this section we will impose only the condition that V is in L^p and in L^q for some $p > \mathfrak{l}/2$ and $q < \mathfrak{l}/2$. The first condition limits the amount of local singularity, but less than before. The second condition is that of decrease at infinity. The improvement over the previous result is due to the device of factoring the perturbation (Kato [4]).

Let $\mathfrak{H}$ be a Hilbert space. Let H_0 and V be selfadjoint operators acting in $\mathfrak{H}$. If $(H_0 - z)^{-1/2}V(H_0 - z)^{-1/2}$ is in $\mathfrak{L}(\mathfrak{H})$ with norm < 1 for z sufficiently imaginary, then V will be said to be a relatively small Friedrichs perturbation of H_0. (Kato [3, p. 341] discusses this type of perturbation under the name of pseudo-Friedrichs extension.)

If V is a relatively small Friedrichs perturbation of H_0, then we may write

$$(H_0 + V - z)^{-1}$$

$$= \sum_{n=0}^{\infty} (H_0 - z)^{-1/2} (-(H_0 - z)^{-1/2}V(H_0 - z)^{-1/2})^n (H_0 - z)^{-1/2}$$

for z sufficiently imaginary. This defines $H_0 + V$ as a selfadjoint operator. Notice that all we can say about the domain of $H_0 + V$ is that it is contained in the range of $(H_0 - z)^{-1/2}$, that is, the domain of $H_0{}^{1/2}$.

PROPOSITION 3. *Let* $\mathfrak{H} = L^2(\mathbf{R}^{\mathfrak{l}}, dx)$. *Let* V *be a real valued function in some space* L^p, $p > \mathfrak{l}/2$, $p \geqq 1$. *Let* $V^{1/2}$ *be any square root of* V. *Then* $\|V^{1/2}(-\Delta - z)^{-1/2}\| < 1$ *for* z *sufficiently imaginary. Thus* V *is a relatively small Friedrichs perturbation of* $-\Delta$.

Notice that if V is not in L^2 locally, one cannot expect that $V(-\Delta - z)^{-1}$ has values in $\mathfrak{H}$.

PROOF. Let $1/2p + 1/2 = 1/r$, $1/r + 1/s = 1$, $1/s + 1/2p = 1/2$. Then if $p \geqq 1$, we have

$$\begin{aligned}\|(-\Delta - z)^{-1/2}f\|_s &\leqq \|(k^2 - z)^{-1/2}\hat{f}\|_r \\ &\leqq \|(k^2 - z)^{-1/2}\|_{2p}\|\hat{f}\|_2 \\ &= \|(k^2 - z)^{-1/2}\|_{2p}\|f\|_2\end{aligned}$$

by the Hausdorff-Young and Plancherel theorems. But if $p > \mathfrak{l}/2$, then $(k^2 - z)^{-1/2}$ is in L^{2p} and its L^{2p} norm approaches zero as z becomes imaginary or negative.

If V is in L^p, then

$$\begin{aligned}\|V^{1/2}(-\Delta - z)^{-1/2}f\|_2 &\leqq \|V^{1/2}\|_{2p}\ \|(-\Delta - z)^{-1/2}f\|_s \\ &\leqq \|V\|_p{}^{1/2}\|(k^2 - z)^{-1/2}\|_{2p}\ \|f\|_2.\end{aligned}$$

Thus $V^{1/2}(-\Delta - z)^{-1/2} \rightarrow 0$ in $\mathfrak{L}(\mathfrak{H})$.

From Proposition 3 we see that the geometric series expansion for $(-\Delta + V - z)^{-1}$ may either be regarded as an expansion in powers of $(-\Delta - z)^{-1/2}V(-\Delta - z)^{-1/2}$ or as one in $V^{1/2}(-\Delta - z)^{-1}V^{1/2}$. (Both of these operators are in $\mathfrak{L}(\mathfrak{H})$ with norm < 1.) The second alternative is especially convenient for scattering theory.

THEOREM 3. *Let* $\mathfrak{H} = L^2(\boldsymbol{R}^{\mathfrak{l}}, dx)$, $\mathfrak{l} \geqq 3$. *Let* $\mathfrak{K} \subset \mathfrak{H} \subset \mathfrak{K}^*$ *be as in Theorem* 2. *Let* V *be a real valued function which is in* L^p *and* L^q *for some* $p > \mathfrak{l}/2$ *and* $q < \mathfrak{l}/2$.

Then if λ *is sufficiently large,*

$$\begin{aligned}&(-\Delta + V - \lambda \pm iO)^{-1} \\ &\qquad = \sum_{n=0}^{\infty} (-\Delta - \lambda \pm iO)^{-1}(-V(-\Delta - \lambda \pm iO)^{-1})^n\end{aligned}$$

converges in the norm of $\mathfrak{L}(\mathfrak{K}, \mathfrak{K}^*)$.

PROOF. First of all, note that if $\mathfrak{l} \geqq 4$, V is in L^p for some $p > \mathfrak{l}/2 \geqq 2$. Hence Proposition 2 implies that V is a relatively small perturbation of $-\Delta$ and $-\Delta + V$ is selfadjoint with the same domain as $-\Delta$.

In the case $\mathfrak{l} = 3$ the situation is different. If we assume V is in L^p for some $p \geqq 2$, then we have the above result. However in general all that we can say is that V is a relatively small Friedrichs perturbation of $-\Delta$ and $-\Delta + V$ is selfadjoint with domain contained in the domain of $(-\Delta)^{1/2}$.

Let $V^{1/2}$ stand for any square root of V. Then $V^{1/2}$ is in L^p and in L^q for $p > \mathfrak{l}$ and $q < \mathfrak{l}$. Thus Lemma 4 establishes that $V^{1/2}(-\Delta - \lambda \pm iO)^{-1}$ is in $\mathfrak{L}(\mathfrak{K}, \mathfrak{H})$, $V^{1/2}(-\Delta - \lambda \pm iO)^{-1}V^{1/2}$ is in $\mathfrak{L}(\mathfrak{H})$, and $(-\Delta - \lambda \pm iO)V^{1/2}$ is in $\mathfrak{L}(\mathfrak{H}, \mathfrak{K}^*)$. It shows also that the norm of $V^{1/2}(-\Delta - \lambda \pm iO)^{-1}V^{1/2} \to 0$ as $\lambda \to +\infty$. Thus at least for sufficiently large λ, this has norm less than one, so the Born series must converge.

In the case $\mathfrak{l} = 3$ one can improve this result by requiring only that V be in $L^{3/2}$. (In this case V is still a small Friedrichs perturbation.) The idea is to approximate V by functions for which the previous result holds, and use the estimate given by Kato [4, p. 276].

This result is stronger than that of Zemach and Klein [5] in their case $\mathfrak{l} = 3$. They prove essentially that $(V^{1/2}(-\Delta - \lambda \pm iO)^{-1}V^{1/2})^2$ approaches zero in Hilbert-Schmidt norm. When $\mathfrak{l} = 3$ and V is in $L^{3/2}(\boldsymbol{R}^3)$, the Hilbert-Schmidt norm of $V^{1/2}(-\Delta - \lambda \pm iO)^{-1}V^{1/2}$ is bounded uniformly in λ (in fact constant). Hence its square must approach zero in Hilbert-Schmidt norm as $\lambda \to +\infty$. Thus their result is recovered.

The cases $\mathfrak{l} = 1$ and $\mathfrak{l} = 2$ are anomalous and are best studied by other methods.

Appendix.

PROPOSITION 4. *Let F be an integrable function from the real line to a Banach space $\mathfrak{X}$. Then*

$$\int_{-\infty}^{\infty} \|F(t) - F(t-a)\| dt \to 0 \quad \text{as } a \to 0.$$

PROOF. Let $\epsilon > 0$. Since F is integrable there is a continuous function G with compact support such that

$$\int_{-\infty}^{\infty} \|F(t) - G(t)\| dt \leqq \epsilon.$$

Then

$$\int_{-\infty}^{\infty} \|F(t) - F(t-a)\| dt \leqq 2\epsilon + \int_{-\infty}^{\infty} \|G(t) - G(t-a)\| dt.$$

Let G have support in an interval of length L. Then for a sufficiently small a, $\|G(t) - G(t-a)\| \leqq \epsilon/L$ for all t.

PROPOSITION 5 (RIEMANN-LEBESGUE LEMMA). *Let $\mathfrak{X}$ be a Banach space. Let F be an integrable function with values in $\mathfrak{X}$. Then*

$$\int_{-\infty}^{\infty} \exp(it\lambda)F(t)dt \to 0 \qquad as\ \lambda \to \pm\infty.$$

PROOF.

$$\int_{-\infty}^{\infty} \exp(it\lambda)F(t)dt = \frac{1}{2} \int_{-\infty}^{\infty} \exp(it\lambda)[F(t) - F(t - \pi/\lambda)]\,dt.$$

But

$$\int_{-\infty}^{\infty} \|F(t) - F(t - \pi/\lambda)\| dt \to 0 \qquad \text{as } \lambda \to \pm\infty$$

by Proposition 4.

REFERENCES

1. R. Aaron and A. Klein, *Convergence of the Born expansion*, J. Mathematical Phys. **1** (1960), 131–138. MR **22** #3457.
2. J. S. Howland, *Banach space techniques in the perturbation theory of self-adjoint operators with continuous spectra*, J. Math. Anal. Appl. **20** (1967), 22–47. MR **36** #2011.
3. T. Kato, *Perturbation theory for linear operators*, Die Grundlehren der math. Wissenschaften, Band 132, Springer-Verlag, New York, 1966. MR **34** #3324.
4. ———, *Wave operators and similarity for some non-self-adjoint operators*, Math. Ann. **162** (1965/66), 258–279. MR **32** #8211.
5. Ch. Zemach and A. Klein, *The Born expansion in non-relativistic quantum theory*, Nuovo Cimento (10) **10** (1958), 1078–1087. MR **21** #580.

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